The limit of a function

**Definition 1.** Let \((X, \tau)\) be a topological space and \(A \subseteq X\). Let \(x \in X\). We say that \(x\) is a **limit point of** \(A\) if and only if every open set \(D\) containing \(x\) satisfies \(D \cap (A - \{x\}) \neq \emptyset\).

**Remark 2.** Let \((X, \tau)\) be a topological space and \(A \subseteq X\). If \(x\) is a limit point of \(A\), then \(x \in A \cup \partial A\). If \(y \in A \cup \partial A\), then \(y\) is either a limit point of \(A\) or an isolated point of \(A\) (we say that \(y\) is an isolated point of \(A\) if and only if there exists an open set \(D\) such that \(D \cap A = \{y\}\)).

**Examples:**
1. The set of limit points of \(D_r(z_0) \equiv \{z \in \mathbb{C} : |z - z_0| < r\}\) is \(\overline{D}_r(z_0) \equiv \{z \in \mathbb{C} : |z - z_0| \leq r\}\).
2. The set of limit points of \(\{z \in \mathbb{C} : \text{Im}z \geq 2\} \cup \{0, -i\}\) is \(\{z \in \mathbb{C} : \text{Im}z \geq 2\}\).

**Definition 3.** Let \(S \subseteq \mathbb{C}\), let \(z_0\) be a limit point of \(S\), \(f : S \rightarrow \mathbb{C}\) a function, and \(L \in \mathbb{C}\). Then,

\[
\lim_{z \to z_0} f(z) = L \iff \forall \epsilon > 0 \exists \delta > 0 \text{ such that } |f(z) - L| < \epsilon \text{ for all } z \in (S - \{z_0\}) \cap D_\delta(z_0).
\]

Definition 3 is equivalent to:

\[
\lim_{z \to z_0} f(z) = L
\]

if and only if for every open set \(V\) containing \(L\), there exists an open set \(U\) containing \(z_0\) such that \(f(U \cap S - \{z_0\}) \subseteq V\).