Some of the problems on the second midterm will be chosen from the following list [up to a change in notation]. The actual exam problems may have multiple parts and be a combination of some of the questions below.

1. State and prove the triangle inequality. [p12]

2. Prove that a subset $D$ of $\mathbb{C}$ is open if and only if it contains no point of its boundary. [p25]

3. Assume that $\{z_n\}$ and $\{w_n\}$ are convergent sequences of complex numbers. Let $\lambda \in \mathbb{C}$. Prove that $\{z_n + \lambda w_n\}$ is convergent and that
$$
\lim_{n \to \infty} (z_n + \lambda w_n) = \left( \lim_{n \to \infty} z_n \right) + \lambda \left( \lim_{n \to \infty} w_n \right).
$$
Prove that $\{z_n w_n\}$ is convergent and that
$$
\lim_{n \to \infty} z_n w_n = \left( \lim_{n \to \infty} z_n \right) \left( \lim_{n \to \infty} w_n \right).
$$
[p34 and your lecture notes]

4. Assume that $\{z_n\}$ and $\{w_n\}$ are convergent sequences of complex numbers such that
$$
\lim_{n \to \infty} w_n \neq 0.
$$
Prove that the sequence $\left\{ \frac{z_n}{w_n} \right\}$ is convergent and that
$$
\lim_{n \to \infty} \frac{z_n}{w_n} = \frac{\lim_{n \to \infty} z_n}{\lim_{n \to \infty} w_n}
$$
[lecture notes]

5. Let $g$ be a complex-valued continuous function on $[a, b]$. Prove that
$$
\left| \int_a^b g(t) \, dt \right| \leq \int_a^b |g(t)| \, dt.
$$
*Hint:* There exists $r \geq 0$ and $\theta \in \mathbb{R}$ such that $\int_a^b g(t) \, dt = re^{i\theta}$. Use the fact that $\text{Re} \, z \leq |z|$. [p60-61]

6. Assume that $\gamma$ is a piecewise $C^1$ curve\(^1\) and that $u$ is a continuous function on the range of $\gamma$. Obtain an upper bound for $\left| \int_{\gamma} u(z) \, dz \right|$ which holds without any additional assumptions on $\gamma$ and $u$. [p61-62 and lecture notes].

7. Let $\Omega$ be a domain whose boundary $\Gamma$ consists of a finite number of disjoint, piecewise smooth simple closed curves. Assume that $f$ is a real-valued harmonic function on an open set which contains $\Omega$ and its boundary. Prove that $f = 0$ on $\Gamma$ if and only if $f = 0$ on $\Omega \cup \Gamma$.
*Hint:* You may want to consider
$$
v = f \frac{\partial f}{\partial x} \quad \text{and} \quad u = -f \frac{\partial f}{\partial y}.
$$
[p73]

\(^1\)A piecewise $C^1$ curve is called piecewise smooth in the textbook.
8. Prove that a \( C \)-differentiable function on an open set satisfies the Cauchy-Riemann equations.

[p80]

9. Let \( f : U \longrightarrow \mathbb{C} \) be a \( C \)-differentiable function. Prove that \( \text{Re} f \) is harmonic. You may use, without proving it, the fact that any \( C \)-differentiable function has complex derivatives of all orders. [p80-81]

10. Suppose that \( f = u + iv \) is \( C \)-differentiable on a domain \( D \), where \( u \) and \( v \) are real-valued. If either \( u \) is constant on \( D \) or \( u^2 + v^2 \) is constant on \( D \), then \( f \) is constant on \( D \).

\textit{Hint:} For the second part, you may begin by proving the claim for the case when \( |f| = 1 \) or you could compute the first order partial derivatives of \( u^2 + v^2 \).

[p82]

11. Suppose that \( f = u + iv \), where \( u \) and \( v \) are real-valued. Assume that \( u, v \), and their first order partial derivatives are continuous in an open disc centered at \( z_0 \). Prove that if \( u \) and \( v \) satisfy the Cauchy-Riemann equations at \( z_0 \) then \( f \) is \( C \)-differentiable at \( z_0 \) and obtain a formula for \( f'(z_0) \) in terms of first order partial derivatives of \( u \) and \( v \). [p97-99]

12. Suppose there is some \( z_1 \neq z_0 \) such that \( \sum a_n (z_1 - z_0)^n \) converges. Prove that for each \( z \) with \( |z - z_0| < |z_1 - z_0| \), the series \( \sum a_n (z - z_0)^n \) is absolutely convergent. [p93]

13. Assume that \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \) has a positive or infinite radius of convergence \( R \). Prove that within the disc \( |z - z_0| < R \), \( f \) is infinitely \( C \)-differentiable. Obtain a general formula for the \( k \)-th derivative of \( f \) and for \( a_n \). [p107]

14. State and prove Cauchy’s theorem. For the proof you may assume that \( f \) is of class \( C^1 \).

[p109-110]

15. Let \( f \) be a \( C \)-differentiable function on a simply-connected domain \( D \) and let \( \gamma \) be a piecewise \( C^1 \) closed curve in \( D \). Prove that

\[ \int_{\gamma} f(z) \, dz = 0. \]

[p110]

16. Let \( u \) be a real-valued harmonic function on a disc \( \{ z : |z - z_0| < r \} \). Prove that there exists a \( C \)-differentiable function on this disc whose real part is \( u \). [p246]

17. Prove that if \( f \) is \( C \)-differentiable in a simply-connected domain \( D \), then there exists a \( C \)-differentiable function \( F \) on \( D \) with \( F' = f \) on \( D \).

[p109-110]

18. State and prove Cauchy’s Integral Formula.

[p111 or your lecture notes]

\textit{Note: The textbook is quoting Example 10 in Section 6, Chapter 1 for the last part of the proof. The content of that example should be part of your proof.}

19. Let \( D \) be an open connected subset of \( \mathbb{C} \) and let \( f : D \longrightarrow \mathbb{C} \) be a continuous function.

Assume that

\[ \int_{\gamma} f(z) \, dz = 0. \]
for every triangle $\gamma$ that lies, together with its interior, in $D$.

Prove that $f$ is $C$-differentiable on $D$.

*Hint:* Fix an arbitrary point $z_0 \in D$ and choose an open disc centered at $z_0$ which is included in $D$. Prove that $f$ has a complex antiderivative on this disc.

[p129-130]

20. State and prove Liouville’s theorem regarding the bounded $C$-differentiable functions on the entire plane $\mathbb{C}$.

*Hint:* Consider the function $g(z) \equiv \frac{F(z) - F(0)}{z}$ where $F$ is $C$-differentiable on $\mathbb{C}$. Show that the function $g$ can be extended to a $C$-differentiable function $\tilde{g}$ on $\mathbb{C}$ and apply Cauchy’s Integral formula for $\tilde{g}$ and a circle centered at the origin of “big enough radius”. Show that $\tilde{g} = 0$.

[p130,131 and your lectures notes]

21. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a curve of class $C^1$. Let $f : D \rightarrow \mathbb{C}$ be a $C$-differentiable function on the open set $D$, $D \subseteq \mathbb{C}$. Assume that $\gamma([a, b]) \subseteq D$.

Prove that

$$(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t) \quad \forall t \in [a, b].$$

[lecture notes or Lemma 1 within the solution to Problem H on Homework 4[Due Feb.20]]

22. Assume that $\gamma : [a, b] \rightarrow \mathbb{C}$ is a piecewise $C^1$ curve. Let $f : D \rightarrow \mathbb{C}$ be a $C$-differentiable function on the open set $D$, $D \subseteq \mathbb{C}$. Assume that $\gamma([a, b]) \subseteq D$.

Prove that

$$\int_{\gamma} f'(z)dz = f(\text{endpoint}) - f(\text{initial point}),$$

where

initial point $\equiv \gamma(a) \quad \text{and} \quad \text{endpoint} \equiv \gamma(b)$.

[see your lecture notes for the $C^1$ case and below for the general case].

There exists a partition $a = t_0 < t_1 < \ldots < t_m = b$ of $[a, b]$ such that $\gamma_j \equiv \gamma \big|_{[t_j, t_{j+1}]}$ is of class $C^1$ for all $j \in \{0, 1, \ldots, m-1\}$. Using the $C^1$ case of the statement,

$$\int_{\gamma} f(z)dz \equiv \sum_{j=0}^{m-1} \int_{\gamma_j} f(z)dz = \sum_{j=0}^{m-1} [f(\gamma(t_{j+1})) - f(\gamma(t_j))] = f(\gamma(b)) - f(\gamma(a)).$$
23. Prove the Cauchy estimates:

\[ |f^{(n)}(z_0)| \leq \frac{n!}{r^n} \max_{|z-z_0|=r} |f(z)| \]

whenever \( f \) is \( \mathbb{C} \)-differentiable on a domain containing \( D_r(z_0) \).

Use the Cauchy estimates for \( n = 1 \) to prove Liouville’s theorem by showing that the derivative of a bounded entire function is identically zero.

[see Exercises 18,19 on p133 and p123]