Solutions to Practice Midterm Exam # 1

Note: Re-read the conventions on the website please.

1. Suppose that \( f = u + iv \), where \( u \) and \( v \) are real-valued. Assume that \( u, v, \) and their first order partial derivatives are continuous in an open disc centered at \( z_0 \). Prove that if \( u \) and \( v \) satisfy the Cauchy-Riemann equations at \( z_0 \) then \( f \) is \( \mathbb{C} \)-differentiable at \( z_0 \) and obtain a formula for \( f'(z_0) \) in terms of first order partial derivatives of \( u \) and \( v \).

See p82-83.

2. (a) Let \( f : U \to \mathbb{C} \) be a \( \mathbb{C} \)-differentiable function. Prove that \( \text{Re} f \) is harmonic. You may use, without proving it, the fact that any \( \mathbb{C} \)-differentiable function has complex derivatives of all orders.

(b) Let \( u \) be a real-valued harmonic function on a disc \( \{z : |z - z_0| < r\} \). Prove that there exists a \( \mathbb{C} \)-differentiable function on this disc whose real part is \( u \).

\[ f(x + iy) = u(x, y) - i \int_{x_0}^{x} \frac{\partial u}{\partial y}(t, y) dt + i \int_{y_0}^{y} \frac{\partial u}{\partial x}(x_0, s) ds. \]

For (a), see p80-81.
For (b), see p246.
3. Evaluate the integrals:

(a) \( \int_{|z|=2} \frac{z^2}{(z-3)^2} \, dz; \)
(b) \( \int_{|z-1|=3} \frac{z}{z^2-9} \, dz. \)

Solution.

(a) The function \( \frac{z^2}{(z-3)^2} \) is \( C \)-differentiable on \( \mathbb{C} - \{3\} \). The curve \( \{ z \in \mathbb{C} : |z|=2 \} \) and its inside \( \{ z \in \mathbb{C} : |z|<2 \} \) are included in \( \mathbb{C} - \{3\} \).

By Cauchy’s theorem,
\[
\int_{|z|=2} \frac{z^2}{(z-3)^2} \, dz = 0.
\]

(b) We assume that the circle \( \{ z \in \mathbb{C} : |z-1|=3 \} \) is positively oriented.

Note that
\[
\frac{z}{z^2-9} = \frac{z + \frac{3}{z}}{z - 3} \quad \forall \quad z \in \mathbb{C} - \{-3, 3\}.
\]

The function \( f(z) \equiv \frac{z}{z^2-9} \) is \( C \)-differentiable on \( \mathbb{C} - \{-3\} \). The curve \( \{ z \in \mathbb{C} : |z-1|=3 \} \) and its inside \( \Omega \equiv \{ z \in \mathbb{C} : |z-1|<3 \} \) are included in \( \mathbb{C} - \{-3\} \).

By Cauchy’s Integral Formula,
\[
\int_{|z-1|=3} \frac{z}{z^2-9} \, dz \equiv \int_{|z-1|=3} \frac{f(\zeta)}{\zeta - 3} \, d\zeta = 2\pi i f(3) = 2\pi i \cdot \frac{1}{2} = \pi i,
\]

since \( 3 \in \Omega \).
4. Find the power series about the origin for the given function:

(a) \( \frac{1 + z}{1 - z} \), \(|z| < 1\);

(b) \( \frac{z^2}{(5 - z)^2} \), \(|z| < 5\); 

*Hint:* \( \frac{1}{(a - z)^2} = \frac{d}{dz} \left[ (a - z)^{-1} \right] \).

*Solution.*

(a) Note that

\[
\frac{1 + z}{1 - z} = -1 + \frac{2}{1 - z}
\]

and

\[
\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n \quad \forall \, z \text{ with } |z| < 1.
\]

Therefore, for all \( z \) with \(|z| < 1\),

\[
\frac{1 + z}{1 - z} = 1 + 2 \sum_{n=1}^{\infty} z^n.
\]

(b) For all \( z \) with \(|z| < 5\),

\[
\frac{1}{5 - z} = \frac{1}{5} \sum_{n=0}^{\infty} \frac{1}{5^n} z^n.
\]

Differentiating this with respect to \( z \), we obtain that

\[
\frac{1}{(5 - z)^2} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{5^n} n z^{n-1} \quad \forall \, z \text{ with } |z| < 5.
\]

Therefore,

\[
\frac{z^2}{(5 - z)^2} = \sum_{n=1}^{\infty} \frac{n}{5^{n+1}} z^{n+1} \quad \forall \, z \text{ with } |z| < 5.
\]
5. Prove that $h(z) = \overline{z}$ is not $\mathbb{C}$-differentiable on any domain.

Proof. Let $u \equiv \text{Re } h$ and $v \equiv \text{Im } h$. So

$$u(x, y) = x \quad \text{and} \quad v(x, y) = -y.$$ 

It follows that

$$\frac{\partial u(x, y)}{\partial x} = 1 \neq \frac{\partial v(x, y)}{\partial y} = -1 \quad \forall (x, y) \in \mathbb{C}.$$ 

So for any $(x, y) \in \mathbb{C}$, $h$ does not satisfy the Cauchy-Riemann equations at $(x, y)$. Therefore, $h$ is not $\mathbb{C}$-differentiable on any domain.
6. Assume that $f = u + iv$ is $C$-differentiable on $\mathbb{C}$, where $u = x^2 - y^2$, and $v$ is real-valued. Find $v$.

Solution.

Since $f$ is $C$-differentiable on $\mathbb{C}$, $f$ satisfies the Cauchy-Riemann equations on $\mathbb{C}$:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (1)$$

Note that

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y. \quad (2)$$

By (1) and (2),

$$2x = \frac{\partial v}{\partial y}$$

This implies that

$$v(x, y) = 2xy + C(x),$$

where $C$ is a function of $x$ only. So

$$\frac{\partial v}{\partial x} = 2y + C'(x).$$

This together with (1) and (2), shows that

$$-2y = -2y - C'(x)$$

and so $C$ is constant.

In conclusion,

$$v(x, y) = 2xy + C, \quad \text{where} \ C \ \text{is any constant.}$$
7. Let $y_0$ be a fixed real number. Prove that the map $F(z) = e^z$ is one-to-one on the strip

$$S_1 \equiv \{ z \in \mathbb{C} : y_0 \leq \text{Im} \, z < y_0 + 2\pi \}.$$ 

What is $F(S_1)$?

Let

$$S_2 \equiv \{ z \in \mathbb{C} : \frac{\pi}{2} \leq \text{Im} \, z \leq \frac{\pi}{2} \}.$$ 

What is $F(S_2)$?

**Solution.**

Let $z_1, z_2 \in S_1$ be such that $F(z_1) = F(z_2)$. Let $x_j = \text{Re} \, z_j$ and $y_j = \text{Im} \, z_j$ for $j \in \{1, 2\}$. It follows that

$$e^{x_1} (\cos y_1 + i \sin y_1) = e^{x_2} (\cos y_2 + i \sin y_2),$$

which implies that $x_1 = x_2$ and $y_1 - y_2 = 2m\pi$ for some $m \in \mathbb{Z}$. On the other hand, since $y_1, y_2 \in [y_0, y_0 + 2\pi)$,

$$2|m|\pi = |y_1 - y_2| < 2\pi$$

and so $m = 0$. In conclusion, $z_1 = z_2$.

This shows that $F$ is one-to-one on $S_1$.

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**Claim:** $F(S_1) = \mathbb{C}^*$. 

**Proof.** Since $|e^z| = e^{\text{Re} \, z} > 0$ for all $z$, $F(S_1) \subseteq \mathbb{C}^*$.

In order to show the reversed inclusion, let $w \in \mathbb{C}^*$. It follows that $w = |w|e^{i\theta}$ for some $\theta \in [-\pi, \pi)$. We next show that

$$\exists k \in \mathbb{Z} \quad \text{such that} \quad \theta + 2k\pi \in [y_0, y_0 + 2\pi). \quad (3)$$

**Proof of (3).** Note that

$$\theta + 2k\pi \in [y_0, y_0 + 2\pi) \iff \frac{y_0 - \theta}{2\pi} \leq k < \frac{y_0 + 2\pi - \theta}{2\pi}. \quad (4)$$

Since $\frac{y_0 + 2\pi - \theta}{2\pi} - \frac{y_0 - \theta}{2\pi} = 1$, there exists an integer $k$ satisfying (4), which concludes the proof of (3).

Let $k \in \mathbb{Z}$ be such that $\psi \equiv \theta + 2k\pi \in [y_0, y_0 + 2\pi)$. It follows that

$$w = |w|e^{i\psi} = F(\ln |w| + i\psi), \quad \ln |w| + i\psi \in S_1.$$ 

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Let $z \in S_2$. Let $x = \text{Re} \, z$ and $y = \text{Im} \, z$. It follows that $y \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$ and so $\cos y \geq 0$. This shows that $\text{Re} \, F(z) = e^x \cos y \geq 0$.

Therefore,

$$F(S_2) \subseteq \{ z \in \mathbb{C}^* : \text{Re} \, z \geq 0 \}.$$ 

(5)

We next let $w \in \{ z \in \mathbb{C}^* : \text{Re} \, z \geq 0 \}$. It follows that $w = |w|e^{i\theta}$ with $\theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$. Therefore,

$$w = F(\ln |w| + i\theta), \quad \ln |w| + i\theta \in S_2.$$ 

This shows that

$$\{ z \in \mathbb{C}^* : \text{Re} \, z \geq 0 \} \subseteq F(S_2). \quad (6)$$

By (5) and (6),

$$F(S_2) = \{ z \in \mathbb{C}^* : \text{Re} \, z \geq 0 \}.$$