Math 403, Homework 3 Solutions

1.5(15). It follows from the definitions of complex trigonometric functions that
$\cos(x+iy) = \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) = \frac{1}{2}(e^{ix-y} + e^{-ix+y})$
$= \frac{1}{2}e^{-y}(\cos(x) + i\sin(x)) + \frac{1}{2}e^{y}(\cos(x) - i\sin(x))$
$= \cos(x)\frac{1}{2}(e^{y} + e^{-y}) - i\sin(x)\frac{1}{2}(e^{y} - e^{-y}) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$
and
$\sin(x+iy) = \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}) = \frac{1}{2i}(e^{ix-y} - e^{-ix+y})$
$= \frac{1}{2i}e^{-y}(\cos(x) + i\sin(x)) - \frac{1}{2i}e^{y}(\cos(x) - i\sin(x))$
$= \sin(x)\frac{1}{2}(e^{y} + e^{-y}) + i\cos(x)\frac{1}{2}(e^{y} - e^{-y}) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$.

1.5(26). We know from class that $\sin(z)$ maps the set $\{z \in \mathbb{C} \mid -\pi/2 < \mathrm{Re}(z) < \pi/2\}$ bijectively onto the set $D = \{w \in \mathbb{C} \mid w \notin \mathbb{R} \text{ or } |w| < 1\}$. Since $\cos(z) = \sin(\pi/2 - z)$, it follows that $\cos(z)$ maps the set $\{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < \pi\}$ bijectively onto D.

1.5(28). Write $g(z) = e^{f(z)}$ where $f(z) = z^2$. We know that the exponential function maps any interval $\{it \mid t_0 \le t < t_0 + 2\pi\}$ of the imaginary axis onto the unit circle $\{|w| = 1\}$. We also know that the exponential function maps the right half-plane $\{\mathrm{Re}(w) > 0\}$ onto $\{|w| > 1\}$, and it maps the left half-plane $\{\mathrm{Re}(w) < 0\}$ onto $\{0 < |w| < 1\}$. Write $z = x + iy$. It is enough to show:

(a) f maps the line $\{y = x\}$ onto the positive imaginary axis $\{it \mid t \ge 0\}$. This follows because $(x + ix)^2 = 2ix^2$.

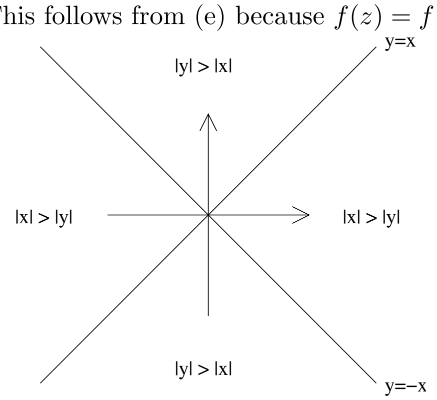
(b) f maps the line $\{y = -x\}$ onto the negative imaginary axis $\{it \mid t \le 0\}$. This follows because $(x - ix)^2 = -2ix^2$.

(c) f maps the region $A_1 = \{x+iy : |x| > |y| \text{ and } x > 0\}$ onto the right half-plane $\{\mathrm{Re}(w) > 0\}$. Notice first that if $|x| > |y|$ then $\mathrm{Re}((x+iy)^2) = x^2 - y^2 > 0$. On the other hand, if w is any point with $\mathrm{Re}(w) > 0$, then $\theta = \mathrm{Arg}(w) \in (-\pi/2, \pi/2)$, which implies that $z = \sqrt{|w|}(\cos(\theta/2) + i\sin(\theta/2))$ belongs to A_1. And we have $f(z) = w$.

(d) f maps the region $A_2 = \{x + iy : |x| > |y| \text{ and } x < 0\}$ onto the right half-plane. This follows from (c) because $f(z) = f(-z)$.

(e) f maps the region $B_1 = \{x+iy : |y| > |x| \text{ and } y > 0\}$ onto the left half-plane $\{\mathrm{Re}(w) < 0\}$. If $|x| < |y|$ then $\mathrm{Re}((x+iy)^2) = x^2 - y^2 < 0$. On the other hand, if $\mathrm{Re}(w) < 0$, then we can write $w = |w|(\cos(\theta) + i\sin(\theta))$ where $\pi/2 < \theta < 3\pi/2$. This implies that $z = \sqrt{|w|}(\cos(\theta/2) + i\sin(\theta/2))$ belongs to B_1, and $f(z) = w$.

(f) f maps the region $B_2 = \{x+iy : |y| > |x| \text{ and } y < 0\}$ onto the left half-plane. This follows from (e) because $f(z) = f(-z)$.



1.6(2). The curve γ is given by $\gamma(t) = tz_0$, $t \in [0,1]$. $\int_\gamma e^z dz = \int_0^1 e^{\gamma(t)}\gamma'(t)dt = \int_0^1 e^{tz_0} z_0 dt = \int_0^1 (\frac{d}{dt}e^{tz_0})dt = e^{z_0} - 1$. To be very careful, one should check that $\frac{d}{dt}e^{tz_0} = z_0 e^{tz_0}$ by differentiating the real and imaginary parts of e^{tz_0}.

1.6(4). The curve γ is given by $\gamma(t) = -4 + e^{it}$, $t \in [0, 2\pi]$. $\int_\gamma (z+4)^{-1}dz = \int_0^{2\pi}(\gamma(t)+4)^{-1}\gamma'(t)dt = \int_0^{2\pi} e^{-it} ie^{it}dt = \int_0^{2\pi} idt = 2\pi i$.

1.6(10). We have a continuous function $f(z) = u(z)+iv(z)$ and a piecewise smooth curve $\gamma(t) = x(t) + iy(t)$. Let $\gamma(t)$ be defined for $t \in [a, b]$. Choose real numbes $a = t_0 < t_1 < \cdots < t_n = b$ so that $\gamma : [t_{i-1}, t_i] \to \mathbb{C}$ is smooth for each i. Then we get:

$$\begin{aligned}
&\int_\gamma f(z)dz \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma(t))\gamma'(t)dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (u(\gamma(t)) + iv(\gamma(t)))(x'(t) + iy'(t))dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(u(\gamma(t))x'(t) - v(\gamma(t))y'(t) + iu(\gamma(t))y'(t) + iv(\gamma(t))x'(t)\big)dt \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(u(\gamma(t))x'(t) - v(\gamma(t))y'(t)\big)dt + i\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(v(\gamma(t))x'(t) + u(\gamma(t))y'(t)\big)dt \\
&= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)
\end{aligned}$$

This shows that the real and imaginary parts of $\int_\gamma f(z)dz$ are as specified in the problem.

1.6(12). Example 11 shows that if γ is any curve from p to q and $m \geq 0$ a non-negative integer, then $\int_\gamma z^m dz = \frac{1}{m+1}(q^{m+1} - p^{m+1})$.

a) If γ is a curve from $-1+i$ to 1, then $\int_\gamma (z^3 - 6z^2 + 4)dz = \int_\gamma z^3 dz - 6\int_\gamma z^2 dz + \int_\gamma 4dz = \frac{1}{4}(1^4 - (-1+i)^4) - 6\frac{1}{3}(1^3 - (-1+i)^3) + 4(1 - (-1+i)) = 45/4$.

b) If γ is a curve from $-i$ to $2+i$, then $\int_\gamma (z^4 + z^2)dz = \int_\gamma z^4 dz + \int_\gamma z^2 dz = \frac{1}{5}((2+i)^5 - (-i)^5) + \frac{1}{3}((2+i)^3 - (-i)^3) = -104/15 + i\,176/15$.