CURVE NEIGHBORHOODS AND THE QUANTUM CHEVALLEY FORMULA

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Abstract. A previous result of the authors with Chaput and Perrin states that the union of all rational curves of fixed degree passing through a Schubert variety in a homogeneous space G/P is again a Schubert variety. In this paper we identify this Schubert variety explicitly in terms of the Demazure product, and we use combinatorics of the Weyl group to obtain bounds on its dimension. This provides an explicit formula for any two-point Gromov-Witten invariant. We apply this to give a new proof of the quantum Chevalley formula for G/P as well as its equivariant generalization.

1. Introduction

Let $X = G/P$ be a homogeneous space defined by a semisimple complex Lie group G and a parabolic subgroup P. Each element w in the Weyl group W defines a Schubert variety $X(w)$ in X. Given an effective degree $d \in H_2(X;\mathbb{Z})$, the *degree d neighborhood* of $X(w)$ is the variety $\Gamma_d(X(w)) \subset X$ defined as the closure of the union of all rational curves of degree d that meet $X(w)$. It was proved in [3] that the variety $\Gamma_d(X(w))$ is also a Schubert variety in X. When X is a Grassmannian of type A, the partition describing $\Gamma_d(X(w))$ can be obtained by removing the first d rows and columns from the partition describing $X(w)$. This operation on partitions has appeared in several references, possibly starting with [8]. More generally, the variety $\Gamma_d(X(w))$ was identified in [3] whenever X is a cominuscule variety. The main new result of this paper is an explicit description of the Weyl group element corresponding to $\Gamma_d(X(w))$ when X is an arbitrary homogeneous space.

Our description of $\Gamma_d(X(w))$ is formulated using the *Demazure product* on W [6], which by definition is the unique associative monoid product such that, for any simple reflection s_β and $w \in W$ we have

$$w \cdot s_\beta = \begin{cases} ws_\beta & \text{if } \ell(ws_\beta) > \ell(w)\ ; \\ w & \text{otherwise.} \end{cases}$$

Let $W_P \subset W$ be the Weyl group of P, and let $W^P \subset W$ denote the subset of minimal length representatives for the cosets in W/W_P. Then $\dim X(w) = \ell(w)$ for each $w \in W^P$. Let $z_d \in W^P$ be the unique minimal representative for the degree d neighborhood of a point, i.e. $\Gamma_d(X(1)) = X(z_d)$.

Theorem 1. *For every $w \in W$ we have $\Gamma_d(X(w)) = X(w \cdot z_d)$.*

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Given a positive root α with $s_\alpha \notin W_P$, let $C_\alpha \subset X$ be the unique T-stable curve that contains the T-fixed points $1.P$ and $s_\alpha.P$, where $T \subset P$ is the maximal torus. The element z_d is easy to obtain using the following result.

Theorem 2. *If $[C_\alpha] \leq d$, then $z_d W_P \geq z_{d'} \cdot s_\alpha W_P$ in the Bruhat order of W/W_P, where $d' = d - [C_\alpha]$. It is possible to choose α such that $z_d W_P = z_{d'} \cdot s_\alpha W_P$.*

When $X = \mathrm{GL}(n)/B$, one can always choose α to be the highest root for which $[C_\alpha] \leq d$, which makes the description of z_d completely explicit.

The degree d neighborhood $\Gamma_d(X(w))$ is related to Fulton and Woodward's work [8] on determining the smallest degree of the quantum parameter that appears in a product of Schubert classes in the (small) quantum ring $\mathrm{QH}(X)$. Let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps to X of degree d, with evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X,d) \to X^n$. The degree d neighborhood of $X(w)$ can then be defined by $\Gamma_d(X(w)) = \mathrm{ev}_1(\mathrm{ev}_2^{-1}(X(w))$. Let $X(w)^{\mathrm{op}} = w_0.X(w)$ be the opposite Schubert variety of $X(w)$, obtained by translating with the longest element $w_0 \in W$. It is proved in [8] that the quantum product $[X(u)] \star [X(w)]$ contains a term $q^{d'}[X(v)]$ with $d' \leq d$ if and only if the Gromov-Witten variety $\mathrm{ev}_1^{-1}(X(u)^{\mathrm{op}}) \cap \mathrm{ev}_2^{-1}(X(w))$ is not empty in $\overline{\mathcal{M}}_{0,2}(X,d)$. The later condition is equivalent to $\Gamma_d(X(w)) \cap X(u)^{\mathrm{op}} \neq \emptyset$, which holds if and only if $w_0 u W_P \leq w \cdot z_d W_P$ in the Bruhat order of W/W_P.

Define the Gromov-Witten variety $\mathrm{GW}_d(w) = \mathrm{ev}_2^{-1}(X(w)) \subset \overline{\mathcal{M}}_{0,2}(X,d)$ and consider the surjective map $\mathrm{ev}_1 : \mathrm{GW}_d(w) \to \Gamma_d(X(w))$. It was proved in [3] that the general fibers of this map are unirational. This implies that the pushforward $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \in H_T^*(X;\mathbb{Z})$ is equal to $[X(w \cdot z_d)]$ whenever $\dim \mathrm{GW}_d(w) = \dim X(w \cdot z_d)$, and otherwise $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] = 0$. It follows that any (equivariant) two-point Gromov-Witten invariant of X is given by

$$
\begin{aligned}
I_d([X(u)^{\mathrm{op}}], [X(w)]) &= \int_{\overline{\mathcal{M}}_{0,2}(X,d)} \mathrm{ev}_1^*[X(u)^{\mathrm{op}}] \cdot \mathrm{ev}_2^*[X(w)] \\
&= \begin{cases} 1 & \text{if } \dim \mathrm{GW}_d(w) = \dim X(w \cdot z_d) \text{ and } X(w \cdot z_d) = X(w_0 u)\,; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{1}
$$

We apply our methods to give a new proof of the (equivariant) quantum Chevalley formula for any product of a Schubert divisor with an arbitrary Schubert class in the equivariant quantum ring $\mathrm{QH}_T(X)$. In fact, all Gromov-Witten invariants required in such a product can be obtained from (1) combined with the divisor axiom in Gromov-Witten theory [12]. The quantum Chevalley formula was first stated in a lecture given by Dale Peterson at M.I.T., and a proof was later supplied by Fulton and Woodward [8]. The equivariant generalization is due to the second author [16] and states that all terms of a product involving a Schubert divisor in $\mathrm{QH}_T(X)$ are also visible in the equivariant cohomology $H_T^*(X)$ or in $\mathrm{QH}(X)$.

We remark that if P is not a Borel subgroup of G, then the ring $\mathrm{QH}_T(X)$ is not generated by divisor classes. However, it was demonstrated in [16] that all (3 point, genus zero) equivariant Gromov-Witten invariants of X can be computed with an explicit algorithm based on the equivariant quantum Chevalley formula. This has been applied by Lam and Shimozono to prove that the equivariant Gromov-Witten invariants of X coincide with certain structure constants of the equivariant homology of the affine Grassmannian [14].

Our paper is organized as follows. In section 2 we recall some basic facts about Schubert classes on X. Section 3 proves a bound on the length of a reflection in a Weyl group. In Section 4 we define the Demazure product and give combinatorial proofs of its main properties. Section 5 contains the proof of Theorems 1 and 2 as well as a bound on the length of z_d. In Section 6 we give a formula for equivariant two-point Gromov-Witten invariants. Section 7 finally proves the equivariant quantum Chevalley formula.

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2. Schubert varieties

In this section we fix our notation for Schubert varieties and state some basic facts. Proofs can be found in e.g. [9]. Let $X = G/P$ be a homogeneous space defined by a connected, simply connected, semisimple complex Lie group G and a parabolic subgroup P. Fix also a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. Let R be the associated root system, with positive roots R^+ and simple roots $\Delta \subset R^+$. Let $W = N_G(T)/T$ be the Weyl group of G and $W_P = N_P(T)/T$ the Weyl group of P. The parabolic subgroup P corresponds to the set of simple roots $\Delta_P = \{\beta \in \Delta \mid s_\beta \in W_P\}$. The group W_P is generated by the simple reflections s_β for $\beta \in \Delta_P$. Set $R_P = R \cap \mathbb{Z}\Delta_P$ and $R_P^+ = R^+ \cap \mathbb{Z}\Delta_P$, where $\mathbb{Z}\Delta_P = \mathrm{Span}_{\mathbb{Z}}(\Delta_P)$ is the group spanned by Δ_P.

For each element $w \in W$ we let $I(w) = R^+ \cap w^{-1}(-R^+) = \{\alpha \in R^+ \mid w(\alpha) < 0\}$ denote the *inversion set* of w. The second expression uses the partial order $\leq$ on $\mathbb{R}\Delta = \mathrm{Span}_{\mathbb{R}}(\Delta)$ defined by $\alpha \geq \beta$ if and only if $\alpha - \beta$ is a linear combination with non-negative coefficients of the simple roots Δ. The length of w is defined by $\ell(w) = |I(w)|$. Equivalently, $\ell(w)$ is the minimal number of simple reflections that w can be a product of. Define the length of the coset $wW_P \in W/W_P$ to be $\ell(wW_P) = |I(w) \smallsetminus R_P^+|$. The element w can be written uniquely as $w = uv$ such that $I(u) \cap R_P^+ = \emptyset$ and $v \in W_P$. We then have $uW_P = wW_P$ and $\ell(u) = \ell(wW_P)$. The element u is called the *minimal representative* for the coset wW_P. Similarly, if w_P denotes the longest element of W_P, then uw_P is the *maximal representative* for wW_P. Let $W^P \subset W$ be the set of all minimal representatives for cosets in W/W_P.

Let w_0 be the longest element in W and let $B^{\mathrm{op}} = w_0Bw_0 \subset G$ be the Borel subgroup opposite to B. For $w \in W$ we define the B-stable Schubert variety $X(w) = \overline{Bw.P} \subset X$ and the B^{op}-stable Schubert variety $Y(w) = \overline{B^{\mathrm{op}}w.P} \subset X$. These varieties depend only on the coset wW_P and we have $\dim X(w) = \operatorname{codim} Y(w) = \ell(wW_P)$. We also have $X(w) \cap Y(w) = \{w.P\}$. The collection of points $w.P$ for $w \in W^P$ is the set of all T-fixed points in X.

The *Bruhat order* on W/W_P is defined by $uW_P \leq wW_P$ if and only if $X(u) \subset X(w)$. This order is compatible with the Bruhat order on W in the sense that $uW_P \leq wW_P$ whenever $u \leq w$ in W. This follows because $X(u)$ is the image of $\overline{Bu.B}$ under the projection $G/B \to X$.

Let $(-,-)$ denote the W-invariant inner product on $\mathbb{R}\Delta$. Each root $\alpha \in R$ has a coroot $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. The coroots form the dual root system $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$, with basis of simple coroots $\Delta^\vee = \{\beta^\vee \mid \beta \in \Delta\}$. For $\beta \in \Delta$ we let $\omega_\beta \in \mathbb{R}\Delta$ denote the corresponding fundamental weight, defined by $(\omega_\beta, \alpha^\vee) = \delta_{\alpha,\beta}$ for $\alpha \in \Delta$.

Lemma 2.1. *Let $\alpha \in R$ and let $S \subset R$ be any set of roots such that $s_\alpha(S) = S$. Then we have*

$$\sum_{\gamma\in S}(\alpha,\gamma^\vee) = \sum_{\gamma\in S}(\gamma,\alpha^\vee) = 0\,.$$

Proof. Since s_α is an involution of S defined by $s_\alpha(\gamma) = \gamma - (\gamma,\alpha^\vee)\alpha$, we obtain

$$\sum_{\gamma\in S}(\gamma,\alpha^\vee) = \sum_{\gamma\in S}\frac{\gamma - s_\alpha(\gamma)}{\alpha} = 0\,.$$

Since we have $s_{\alpha^\vee}(S^\vee) = S^\vee$, the first sum of the lemma is also equal to zero. □

We need the following observation, which can also be found in [8, p. 648].

Lemma 2.2. *Let $\alpha \in R^+ \smallsetminus R^+_P$. Then α is uniquely determined by the coset $s_\alpha W_P \in W/W_P$.*

Proof. Set $\lambda = \sum_{\beta\in\Delta\smallsetminus\Delta_P}\omega_\beta$. Then W_P acts trivially on λ, while $\lambda - s_\alpha.\lambda = (\lambda,\alpha^\vee)\,\alpha$ is a non-zero multiple of α. The lemma follows from this because distinct positive roots are never parallel. □

It is convenient to identify $H^2(X;\mathbb{Z})$ with the span $\mathbb{Z}\{\omega_\beta \mid \beta \in \Delta\smallsetminus\Delta_P\}$ and $H_2(X;\mathbb{Z})$ with the quotient $\mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$. More precisely, for each $\beta \in \Delta\smallsetminus\Delta_P$ we identify the class $[X(s_\beta)] \in H_2(X;\mathbb{Z})$ with $\beta^\vee + \mathbb{Z}\Delta_P^\vee \in \mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$ and we identify $[Y(s_\beta)] \in H^2(X;\mathbb{Z})$ with ω_β. The Poincare pairing $H^2(X;\mathbb{Z})\otimes H_2(X;\mathbb{Z}) \to \mathbb{Z}$ is then given by the W-invariant inner product $(-,-)$ on $\mathbb{R}\Delta$.

For each positive root $\alpha \in R^+\smallsetminus R^+_P$ there is a unique irreducible T-stable curve $C_\alpha \subset X$ that contains $1.P$ and $s_\alpha.P$. We can restate [8, Lemma 3.4] as the identity

$$[C_\alpha] = \alpha^\vee + \mathbb{Z}\Delta_P^\vee \in H_2(X;\mathbb{Z})\,. \tag{2}$$

According to [8, Lemma 3.5] we have

$$c_1(T_X) \;=\; \sum_{\gamma\in R^+\smallsetminus R^+_P}\gamma \;\in H^2(X;\mathbb{Z})\,. \tag{3}$$

Indeed, if we let $c_1 = \sum_{\gamma\in R^+\smallsetminus R^+_P}\gamma$ denote the right hand side of (3), then the cited lemma implies that $(c_1,\beta^\vee) = \int_{X(s_\beta)} c_1(T_X)$ for $\beta \in \Delta\smallsetminus\Delta_P$, and Lemma 2.1 shows that $(c_1,\beta^\vee) = 0$ for each $\beta \in \Delta_P$, hence $c_1 \in \mathbb{Z}\{\omega_\beta \mid \beta \in \Delta\smallsetminus\Delta_P\} = H^2(X;\mathbb{Z})$.

If $\lambda \in \mathbb{Z}\{\omega_\beta \mid \beta \in \Delta\smallsetminus\Delta_P\}$ is an integral weight, then the assumption that G is simply connected implies that λ is represented by a character $\lambda : T \to \mathbb{C}^*$. It therefore defines the line bundle $L_\lambda := G \times^P \mathbb{C}_\lambda = (G\times\mathbb{C})/P$ over X, where P acts on $G\times\mathbb{C}$ by $p.(g,z) = (gp^{-1},\lambda(p)z)$. By [8, Lemma 3.2] we then have

$$c_1(L_\lambda) = \lambda \in H^2(X;\mathbb{Z})\,. \tag{4}$$

3. Small and large roots

Set $\rho = \frac{1}{2}\sum_{\gamma\in R^+}\gamma = \frac{1}{2}c_1(T_{G/B})$. For each simple root $\beta \in \Delta$ it follows from Lemma 2.1 applied to $R^+\smallsetminus\{\beta\}$ that $(\rho,\beta^\vee) = (\frac{1}{2}\beta,\beta^\vee) = 1$, so we also have $\rho = \sum_{\beta\in\Delta}\omega_\beta$. The *height* $\mathrm{ht}(\alpha)$ of a root α is the sum of the coefficients obtained when α is written as a linear combination of simple roots. Notice that $\mathrm{ht}(\alpha^\vee) = (\rho,\alpha^\vee)$. Similarly we have $\mathrm{ht}(\alpha) = \frac{1}{2}\sum_{\gamma\in R^+}(\alpha,\gamma^\vee)$.

Definition 3.1. A positive root $\alpha \in R^+$ is *large* if it is long and can be written as the sum of two short positive roots. Otherwise α is *small.* The root α is *colarge* if $\alpha^\vee$ is large, and α is *cosmall* if $\alpha^\vee$ is small.

Notice that all short positive roots are small and all long positive roots are cosmall. If R is simply laced, then all positive roots are both small and cosmall.

Theorem 3.2. *For any positive root $\alpha \in R^+$ we have $\ell(s_\alpha) \leq 2\,\mathrm{ht}(\alpha) - 1$. Moreover, the following are equivalent.*

(a) *The root α is small.*

(b) *We have $\ell(s_\alpha) = 2\,\mathrm{ht}(\alpha) - 1$.*

(c) *For all $\gamma \in I(s_\alpha) \smallsetminus \{\alpha\}$ we have $(\alpha, \gamma^\vee) = 1$.*

Proof. For any root $\gamma \in I(s_\alpha)$ we have $s_\alpha(\gamma) = \gamma - (\gamma, \alpha^\vee)\alpha < 0$, hence $(\gamma, \alpha^\vee) \geq 1$ and $(\alpha, \gamma^\vee) \geq 1$. Since s_α stabilizes the set $R^+ \smallsetminus I(s_\alpha)$, Lemma 2.1 implies that

$$(5)\quad 2\,\mathrm{ht}(\alpha) = \sum_{\gamma \in R^+} (\alpha, \gamma^\vee) = \sum_{\gamma \in I(s_\alpha)} (\alpha, \gamma^\vee) = 2 + \sum_{\gamma \in I(s_\alpha) \smallsetminus \{\alpha\}} (\alpha, \gamma^\vee) \geq \ell(s_\alpha) + 1\,.$$

This proves the inequality as well as the equivalence of (b) and (c). We will show that α is large if and only if (c) is false.

If α is large, then α is long and we can write $\alpha = \beta + \gamma$ where β and γ are short positive roots. Since $(\beta, \alpha^\vee) + (\gamma, \alpha^\vee) = (\alpha, \alpha^\vee) = 2$ we may assume that $(\gamma, \alpha^\vee) \geq 1$. Since $1 \leq (\gamma, \alpha^\vee) < (\alpha, \gamma^\vee)$, Table 1 in [10, p. 45] shows that we must have $(\gamma, \alpha^\vee) = 1$ and $(\alpha, \gamma^\vee) \geq 2$. Finally we have $s_\alpha(\gamma) = \gamma - \alpha = -\beta < 0$, hence (c) is false. Conversely, if (c) is false, then choose $\gamma \in I(s_\alpha) \smallsetminus \{\alpha\}$ such that $(\alpha, \gamma^\vee) \geq 2$ and set $\beta = -s_\alpha(\gamma) = \alpha - \gamma$. Then α is long, γ and β are short, and $\alpha = \beta + \gamma$. This shows that α is large, as required. ☐

Corollary 3.3. *For $\alpha \in R^+$ we have $\ell(s_\alpha) = 2\min(\mathrm{ht}(\alpha), \mathrm{ht}(\alpha^\vee)) - 1$.*

Proof. This follows because $s_\alpha = s_{\alpha^\vee}$ and either α or $\alpha^\vee$ is small. ☐

Theorem 3.2 has the following 'parabolic' version. Recall the expression (3) for the Chern class $c_1(T_X)$.

Theorem 3.4. *For any root $\alpha \in R^+ \smallsetminus R^+_P$ we have $\ell(s_\alpha W_P) \leq (c_1(T_X), \alpha^\vee) - 1$. Furthermore, $\ell(s_\alpha W_P) = (c_1(T_X), \alpha^\vee) - 1$ if and only if $(R^+ \smallsetminus R^+_P) \cap s_\alpha(R^+_P) = \emptyset$ and $(\gamma, \alpha^\vee) = 1$ for all $\gamma \in I(s_\alpha) \smallsetminus (R^+_P \cup \{\alpha\})$.*

Proof. Define $A = I(s_\alpha) \smallsetminus (R^+_P \cup \{\alpha\})$, $B = (R^+ \smallsetminus R^+_P) \cap s_\alpha(R^+_P)$, and $C = (R^+ \smallsetminus R^+_P) \cap s_\alpha(R^+ \smallsetminus R^+_P)$. Since $R^+ \smallsetminus R^+_P$ is the disjoint union of $\{\alpha\}$, A, B, and C, we obtain from (3) that

$$(c_1(T_X), \alpha^\vee) = 2 + \sum_{\gamma \in A} (\gamma, \alpha^\vee) + \sum_{\gamma \in B} (\gamma, \alpha^\vee) + \sum_{\gamma \in C} (\gamma, \alpha^\vee)\,.$$

The result follows because $|A| = \ell(s_\alpha W_P) - 1$, $(\gamma, \alpha^\vee) \geq 1$ for all $\gamma \in A \cup B$, and $\sum_{\gamma \in C} (\gamma, \alpha^\vee) = 0$ by Lemma 2.1. ☐

Definition 3.5. The root $\alpha \in R^+$ is *P-cosmall* if $\ell(s_\alpha W_P) = (c_1(T_X), \alpha^\vee) - 1$.

4. The Demazure product

Our description of curve neighborhoods is formulated in terms of the Demazure product, which provides a monoid structure on the Weyl group W. This product was introduced by Demazure in [6] as a combinatorial description of compositions of Demazure operators. It has been used frequently in the study of K-theory of homogeneous spaces, for example in Kostant and Kumar's paper [13]. While the Demazure product and its properties are well known, we do not know about a reference that gives a short unified exposition, so we have taken the opportunity to provide one here.

For $u \in W$ and $\beta \in \Delta$, define

$$u \cdot s_\beta = \begin{cases} us_\beta & \text{if } us_\beta > u; \\ u & \text{if } us_\beta < u. \end{cases} \tag{6}$$

Let $u, v \in W$ and let $v = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$ be any reduced expression for v. Define the *Demazure product* of u and v by

$$u \cdot v = u \cdot s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_\ell} \,,$$

where the simple reflections are multiplied to u in left to right order.

We claim that this product is independent of the chosen reduced expression for v. In fact, any reduced expression for v can be obtained from any other by using finitely many *braid relations*, i.e. by steps that replace a subexpression of the form $t_0 t_1 \cdots t_{m-1}$ with $t_1 t_2 \cdots t_m$, where $t_{2i} = s_\alpha$ and $t_{2i+1} = s_\beta$ for given simple roots $\alpha, \beta \in \Delta$ and all $i \in \mathbb{N}$. It is therefore enough to show that

$$u \cdot t_0 \cdot t_1 \cdot \ldots \cdot t_{m-1} = u \cdot t_1 \cdot t_2 \cdot \ldots \cdot t_m \,. \tag{7}$$

Let $W_{\alpha,\beta} \subset W$ denote the parabolic subgroup generated by s_α and s_β. Then $t_0 t_1 \cdots t_{m-1} = t_1 t_2 \cdots t_m$ is the longest element of $W_{\alpha,\beta}$, and both sides of (7) are equal to the unique maximal representative for the coset $uW_{\alpha,\beta}$ in $W/W_{\alpha,\beta}$.

Given $u, v \in W$, we will say that the product uv is *reduced* if $\ell(uv) = \ell(u) + \ell(v)$. This implies that $w \cdot uv = (w \cdot u) \cdot v$ for all $w \in W$. Notice also that for $\beta \in \Delta$ and $v \in W$ we have

$$s_\beta \cdot v = \begin{cases} s_\beta v & \text{if } s_\beta v > v; \\ v & \text{if } s_\beta v < v. \end{cases} \tag{8}$$

In fact, if we set $v' = s_\beta v$, then the identity is clear if $\ell(v') > \ell(v)$, and otherwise $v = s_\beta v'$ is a reduced product, hence $s_\beta \cdot v = (s_\beta \cdot s_\beta) \cdot v' = s_\beta \cdot v' = v$.

Proposition 4.1. *Let $u, v, v', w \in W$.*

(a) *The Demazure product is associative, i.e. $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.*

(b) *We have $(u \cdot v)^{-1} = v^{-1} \cdot u^{-1}$.*

(c) *If $v \leq v'$ then $u \cdot v \cdot w \leq u \cdot v' \cdot w$.*

(d) *We have $u \leq u \cdot v$, $v \leq u \cdot v$, $uv \leq u \cdot v$, and $\ell(u \cdot v) \leq \ell(u) + \ell(v)$.*

(e) *The element $u' = (u \cdot v)v^{-1}$ satisfies $u' \leq u$ and $u'v = u' \cdot v = u \cdot v$.*

(f) *$I(v) \subset I(u \cdot v)$.*

Proof. To prove (a) it is enough to show that $(u \cdot s_\beta) \cdot w = u \cdot (s_\beta \cdot w)$ for each $\beta \in \Delta$. This is clear if $s_\beta \cdot w = s_\beta w$, and it is also clear if $u \cdot s_\beta = u$ and $s_\beta \cdot w = w$. Assume that $u \cdot s_\beta = us_\beta$ and $s_\beta \cdot w = w$, and set $w' = s_\beta w$. Since $w = s_\beta w'$ is a reduced product, we obtain $(u\cdot s_\beta)\cdot w = ((u\cdot s_\beta)\cdot s_\beta)\cdot w' = (u\cdot s_\beta)\cdot w' = u\cdot w = u\cdot(s_\beta\cdot w)$, as required. Part (b) follows from the associativity together with (6) and (8). To prove (c) it is enough to show that $v\cdot s_\beta \leq v'\cdot s_\beta$ for each $\beta\in\Delta$. This is true because $W_\beta = \{1, s_\beta\}$ is a parabolic subgroup of W, $v\cdot s_\beta$ is the maximal representative for vW_β in W/W_β, $v'\cdot s_\beta$ is the maximal representative for $v'W_\beta$, and $vW_\beta \leq v'W_\beta$. For (d), the inequality $u \leq u\cdot v$ follows from (6), and $v \leq u\cdot v$ follows from (8). If we write $v = v's_\beta$ as a reduced product with $\beta\in\Delta$, then it follows from (c) and induction on $\ell(v)$ that $u\cdot v = (u\cdot v')\cdot s_\beta \geq uv'\cdot s_\beta \geq uv's_\beta = uv$. The inequality $\ell(u\cdot v) \leq \ell(u)+\ell(v)$ is clear from the definition. For (e), let $u = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_\ell}$ be a reduced expression for u, and set $y_j = s_{\alpha_j}\cdot s_{\alpha_{j+1}}\cdot\ldots\cdot s_{\alpha_\ell}\cdot v$ for each j. Let $\{i_1 < i_2 < \cdots < i_p\}$ be the set of indices j for which $y_j \neq y_{j+1}$. Then $u\cdot v = s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_p}}v$ is a reduced product, so $u' = s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_p}}$ satisfies the requirements. Finally, part (f) follows from (8). $\square$

Proposition 4.2. *Let $u, v\in W$. The following are equivalent.*

(a) *The product uv is reduced.*

(b) $\ell(u\cdot v) = \ell(u)+\ell(v)$.

(c) $u\cdot v = uv$.

(d) $I(v)\subset I(uv)$.

(e) $I(u)\cap I(v^{-1}) = \emptyset$.

Proof. All of the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a) follow easily from the definitions and Proposition 4.1. $\square$

The Demazure product also defines a product $W\times W/W_P \to W/W_P$ given by

$$u\cdot(wW_P) = (u\cdot w)W_P\,.$$

To see that this is well defined, write $w = w'w''$ with $w'\in W^P$ and $w''\in W_P$, and set $v = u\cdot w'$ and $p = v^{-1}(v\cdot w'')$. Then $u\cdot w = (u\cdot w')\cdot w'' = v\cdot w'' = vp$, and since $p\leq w''$ by Proposition 4.1(e) we must have $p\in W_P$. It follows that $(u\cdot w)W_P = (u\cdot w')W_P$, as required.

Notice also that for $\beta\in\Delta$ we have

$$s_\beta\cdot(wW_P) = \begin{cases} s_\beta wW_P & \text{if } s_\beta wW_P > wW_P; \\ wW_P & \text{if } s_\beta wW_P \leq wW_P. \end{cases} \tag{9}$$

In fact, if $s_\beta wW_P > wW_P$, then we must have $s_\beta w > w$ by compatibility of the Bruhat orders, and otherwise the inequality $wW_P \leq (s_\beta\cdot w)W_P$ implies that $s_\beta wW_P = wW_P$. The identity follows from this.

Proposition 4.3. *For $u, v\in W$ we have $\ell(u\cdot vW_P) \leq \ell(u)+\ell(vW_P)$. Moreover, if $\ell(u\cdot vW_P) = \ell(u)+\ell(vW_P)$ then we must have $u\cdot vW_P = uvW_P$.*

Proof. This follows from equation (9). $\square$

5. Curve Neighborhoods

5.1. **The element z_d.** Given an effective degree $d\in H_2(X;\mathbb{Z}) = \mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$, let $\overline{\mathcal{M}}_{0,n}(X,d)$ be the Kontsevich space of n-pointed stable maps of degree d to X, with total evaluation map $\mathrm{ev} = (\mathrm{ev}_1,\ldots,\mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X,d)\to X^n$. We have

$$\dim\overline{\mathcal{M}}_{0,n}(X,d) = \dim(X) + (c_1(T_X), d) + n - 3$$

where $c_1(T_X)$ is given by (3). Given any subvariety $Z \subset X$, define the *degree d neighborhood* of Z to be $\Gamma_d(Z) = \mathrm{ev}_1(\mathrm{ev}_2^{-1}(Z))$. If Z is a B-stable Schubert variety in X, then so is $\Gamma_d(Z)$ by [3, Cor. 3.3(a)]. We wish to determine the corresponding Weyl group element. Let $z_d \in W^P$ be the unique element such that $\Gamma_d(X(1)) = X(z_d)$. Notice that if $d' \leq d$ then $z_{d'} \leq z_d$. We also have $w_P \cdot z_d W_P = z_d W_P$ where w_P is the longest element of W_P. In fact, if we choose $w \leq w_P$ such that $w z_d W_P = w_P \cdot z_d W_P$ and let $C \subset X$ be a rational curve of degree d from $1.P$ to $z_d.P$, then $w.C$ is a rational curve of the same degree from $1.P = w.P$ to $wz_d.P = w_P \cdot z_d.P$, so $w_P \cdot z_d.P \in X(z_d)$. If $\alpha \in R^+ \smallsetminus R^+_P$, then since the curve C_α connects $1.P$ to $s_\alpha.P$ and has degree $[C_\alpha] = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$, we obtain $s_\alpha.P \in X(z_{\alpha^\vee})$, so $s_\alpha W_P \leq z_{\alpha^\vee} W_P$. Here we misuse notation and write $z_{\alpha^\vee} = z_{\alpha^\vee + \mathbb{Z}\Delta_P^\vee}$. We will see below that $z_{\alpha^\vee} W_P = s_\alpha W_P$ whenever $\alpha \in R^+$ is P-cosmall. We can now prove the results stated in the introduction.

Theorem 1. *For any $w \in W$ we have $\Gamma_d(X(w)) = X(w \cdot z_d)$.*

Proof. Since both varieties are B-stable Schubert varieties in X, it is enough to show that they have the same T-fixed points. Choose a rational curve $C \subset X$ of degree d that contains the points $1.P$ and $z_d.P$. Set $v = (w \cdot z_d) z_d^{-1}$. Then $v.P \in X(w)$ and $v.C$ is a curve of degree d from $v.P$ to $vz_d.P = w \cdot z_d.P$. This proves that $X(w \cdot z_d) \subset \Gamma_d(X(w))$.

Now let $u.P \in \Gamma_d(X(w))$ be any T-fixed point. Since the locus of curves from $X(w)$ to $u.P$ is a closed T-stable subvariety of $\overline{\mathcal{M}}_{0,2}(X, d)$, it follows that this locus contains a T-stable curve C, and this curve must contain a T-fixed point $v.P \in X(w)$ with $v \in W^P$. Then $v \leq w$, and since $v^{-1}C$ is a curve of degree d from $1.P$ to $v^{-1}u.P$, we also have $v^{-1}uW_P \leq z_d W_P$. We finally obtain $uW_P = v(v^{-1}u)W_P \leq w \cdot z_d W_P$, so $u.P \in X(w \cdot z_d)$ as required. ☐

Corollary 5.1. *For effective $d, d' \in H_2(X;\mathbb{Z})$ we have $z_d \cdot z_{d'} W_P \leq z_{d+d'} W_P$.*

Proof. $X(z_d \cdot z_{d'}) = \Gamma_{d'}(X(z_d)) = \Gamma_{d'}(\Gamma_d(X(1))) \subset \Gamma_{d+d'}(X(1)) = X(z_{d+d'})$. ☐

The following theorem can be used to compute z_d for any effective degree $d \in H_2(X;\mathbb{Z})$. Notice that $w.C_\alpha$ is the unique irreducible T-stable curve from $w.P$ to $ws_\alpha.P$, and we have $[w.C_\alpha] = [C_\alpha] = \alpha^\vee + \mathbb{Z}\Delta_P^\vee \in H_2(X;\mathbb{Z})$.

Theorem 2. *Let $0 < d \in H_2(X;\mathbb{Z})$. For any root $\alpha \in R^+ \smallsetminus R^+_P$ with $\alpha^\vee + \mathbb{Z}\Delta_P^\vee \leq d$ we have $z_d W_P \geq z_{d'} \cdot s_\alpha W_P$, where $d' = d - (\alpha^\vee + \mathbb{Z}\Delta_P^\vee)$. Furthermore, it is possible to choose α such that $z_d W_P = z_{d'} \cdot s_\alpha W_P$.*

Proof. The inequality follows from Corollary 5.1 because $z_{\alpha^\vee} W_P \geq s_\alpha W_P$. Let $C \subset X$ be a T-stable curve of degree d containing $1.P$ and $z_d.P$. This curve must be a connected union of irreducible T-stable components. At least one component contains $z_d.P$, and any such component has the form $z_d.C_\alpha$ with $\alpha \in R^+ \smallsetminus R^+_P$ and connects $z_d.P$ to $z_d s_\alpha.P$. We may choose a component $z_d.C_\alpha$ so that $C' = \overline{C \smallsetminus z_d.C_\alpha}$ connects $1.P$ to $z_d s_\alpha.P$. Write $z_d s_\alpha = uv$ with $u \in W^P$ and $v \in W_P$. Since $[C'] \leq d' := d - (\alpha^\vee + \mathbb{Z}\Delta_P^\vee)$, it follows that $u \leq z_{d'}$. We deduce that $z_d W_P = uvs_\alpha v^{-1} W_P = us_{v(\alpha)} W_P \leq z_{d'} \cdot s_{v(\alpha)} W_P \leq z_d W_P$. The result follows from this because $v(\alpha) + \mathbb{Z}\Delta_P^\vee = \alpha + \mathbb{Z}\Delta_P^\vee$. ☐

Remark 5.2. For $d' \in H_2(G/B;\mathbb{Z}) = \mathbb{Z}\Delta^\vee$ we let $z'_{d'} \in W$ be the element defining the degree d' neighborhood of $1.B$ in G/B. Theorem 2 implies that for any effective degree $d \in H_2(X;\mathbb{Z})$ we have $z_d W_P = \max\{z'_{d'} W_P \mid d' + \mathbb{Z}\Delta_P^\vee = d\}$.

Proposition 5.3. *Let $0 < d \in H^2(X;\mathbb{Z})$. Then $\ell(z_d) \leq (c_1(T_X), d) - 1$. Furthermore, if $\ell(z_d) = (c_1(T_X), d) - 1$, then there exists a unique root $\alpha \in R^+ \smallsetminus R_P^+$ such that $z_d W_P = s_\alpha W_P$. This root α is P-cosmall and we have $d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$.*

Proof. By Theorem 2 we can choose $\alpha \in R^+ \smallsetminus R_P^+$ such that $z_d W_P = z_{d'} \cdot s_\alpha W_P$ where $d' = d - (\alpha^\vee + \mathbb{Z}\Delta_P^\vee)$. Theorem 3.4 implies that $\ell(s_\alpha W_P) \leq (c_1(T_X), \alpha^\vee) - 1$. If $d' > 0$, then it follows by induction on d that $\ell(z_d W_P) \leq \ell(z_{d'}) + \ell(s_\alpha W_P) \leq (c_1(T_X), d') - 1 + (c_1(T_X), \alpha^\vee) - 1 = (c_1(T_X), d) - 2$. We deduce that, if $\ell(z_d) = (c_1(T_X), d) - 1$, then $d' = 0$, $z_d W_P = s_\alpha W_P$, and α is P-cosmall. The uniqueness of α follows from Lemma 2.2. $\square$

Corollary 5.4. *If $\alpha \in R^+ \smallsetminus R_P^+$ is a P-cosmall root, then $z_{\alpha^\vee} W_P = s_\alpha W_P$.*

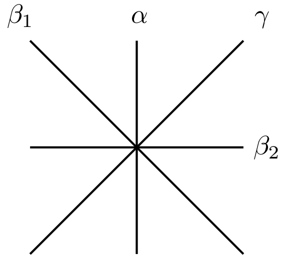
Proof. This follows because $\ell(s_\alpha W_P) \leq \ell(z_{\alpha^\vee}) \leq (c_1(T_X), \alpha^\vee) - 1 = \ell(s_\alpha W_P)$. $\square$

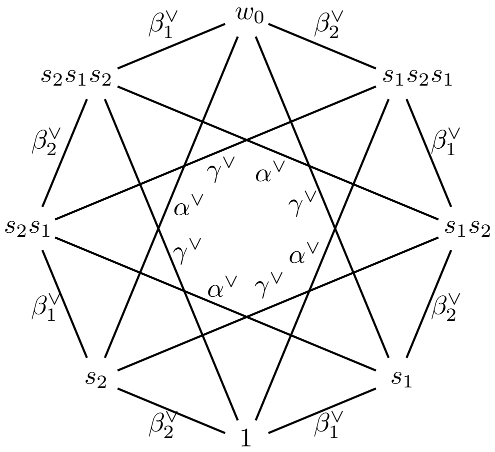
Example 5.5. Assume that $X = G/P$ is a cominuscule variety, i.e. $\Delta \smallsetminus \Delta_P = \{\gamma\}$ consists of a single cominuscule root γ. Then it follows from [3, Lemma 4.3] that $z_d W_P = (w_P s_\gamma) \cdot (w_P s_\gamma) \cdot \ldots \cdot (w_P s_\gamma) W_P$ (m-fold Demazure product).

5.2. **The moment graph.** The element $z_d \in W^P$ can also be constructed using the *moment graph* of X. The vertices of this graph are the T-fixed points X^T and the edges are the irreducible T-stable curves in X. More precisely, there is an edge between $u.P$ and $w.P$ if and only if $wW_P = us_\alpha W_P$ for some root $\alpha \in R^+ \smallsetminus R_P^+$; the corresponding T-stable curve is $u.C_\alpha \subset X$ which has degree $[C_\alpha] = \alpha^\vee + \mathbb{Z}\Delta_P \in H_2(X;\mathbb{Z})$. Define the *weight* of a path in the moment graph to be the sum of the degrees of its edges. Given $d \in H_2(X;\mathbb{Z})$, let $Z_d \subset X^T$ be the subset of points $w.P$ for which there exists a path from $1.P$ to $w.P$ of weight at most d. Then Theorem 2 implies that $z_d.P$ is the unique maximal element of Z_d in the Bruhat order on X^T, defined by $u.P \leq w.P$ if and only if $uW_P \leq wW_P$.

Example 5.6. Let $X = \mathrm{SO}(5)/B = \mathrm{OF}(5)$ be the variety of isotropic flags in the vector space $\mathbb{C}^5$ equipped with an orthogonal form. The corresponding root system has type B_2. Let $\Delta = \{\beta_1, \beta_2\}$ be the simple roots, with β_1 long and β_2 short. The remaining positive roots are $\alpha = \beta_1 + \beta_2$ and $\gamma = \beta_1 + 2\beta_2$, with coroots $\alpha^\vee = 2\beta_1^\vee + \beta_2^\vee$ and $\gamma^\vee = \beta_1^\vee + \beta_2^\vee$. Write $s_i = s_{\beta_i}$ for $i = 1, 2$. The moment graph of X is displayed below, with each edge labeled by its degree. Since the paths of weight $\gamma^\vee$ starting at 1 are $1 \to s_1 \to s_1 s_2$, $1 \to s_2 \to s_2 s_1$, and $1 \to s_\gamma = s_2 s_1 s_2$, we have $z_{\gamma^\vee} = s_\gamma$. On the other hand, the paths of weight $\alpha^\vee$ starting at 1 include

$1 \to s_1 \to w_0$, so $z_{\alpha^\vee} = w_0 \neq s_\alpha = s_1 s_2 s_1$.





5.3. **A criterion for $z_{\alpha^\vee} = s_\alpha$.** In this subsection we assume that $X = G/B$. In this case the converse of Corollary 5.4 is also true.

Corollary 5.7. *Let $\alpha \in R^+$ be any positive root. The following are equivalent.*
(a) *The root α is cosmall.*
(b) *We have $\ell(z_{\alpha^\vee}) = (c_1(T_{G/B}), \alpha^\vee) - 1$.*
(c) *We have $z_{\alpha^\vee} = s_\alpha$.*

Our proof is based on the following lemma, which we have proved by classifying the large roots of all non-simply laced Weyl groups. The proof is not illuminating and is omitted here.

Lemma 5.8. *Assume that $\alpha \in R^+$ is large, and write $\alpha = \beta + \gamma$ with β and γ short positive roots. Then $\ell(s_\alpha) < \ell(s_\beta \cdot s_\gamma)$.*

We remark that the stronger inequality $\ell(s_\alpha) < \ell(s_\beta s_\gamma)$ is also true, except when α is the highest root of type G_2.

Proof of Corollary 5.7. The implication (a) ⇒ (b) follows from Corollary 5.4, and (b) ⇒ (c) follows from Proposition 5.3. Assume that α is colarge and write $\alpha^\vee = \beta^\vee + \gamma^\vee$ where $\beta^\vee$ and $\gamma^\vee$ are short coroots. Then Lemma 5.8 and Theorem 2 imply that $\ell(s_\alpha) < \ell(s_\beta \cdot s_\gamma) \leq \ell(z_{\alpha^\vee})$, so $z_{\alpha^\vee} \neq s_\alpha$. This proves (c) ⇒ (a). □

It would be interesting to find a similar criterion in the more general case when $X = G/P$.

5.4. **Varieties of complete flags of type A.** We give an explicit formula for z_d when $X = \mathrm{SL}_{n+1}(\mathbb{C})/B$ is a complete flag variety of type A_n. We have $W = S_{n+1}$, $\Delta = \{\beta_{i,i+1} \mid 1 \leq i \leq n\}$, $R^+ = \{\beta_{ij} \mid 1 \leq i < j \leq n+1\}$, and $s_{\beta_{ij}} = t_{ij} = (i,j)$, where $\beta_{ij} = e_i - e_j \in \mathbb{R}^{n+1}$. For each $i \in [1, n+1] = \{1, 2, \dots, n+1\}$ we set $t_{ii} = 1 \in S_{n+1}$.

Lemma 5.9. *Let α_{ij} and α_{kl} be positive roots of type A.*
(a) *We have $t_{ij} \cdot t_{kl} = t_{kl} \cdot t_{ij}$ if and only if $[i,j] \cap [k,l] = \emptyset$ or $[i,j] \subset [k,l]$ or $[k,l] \subset [i,j]$.*

(b) *Assume that* $t_{ij} \cdot t_{kl} \neq t_{kl} \cdot t_{ij}$. *Then we have* $i \leq l$, $k \leq j$, $t_{ij} \cdot t_{kl} \leq t_{il} \cdot t_{kj}$, $t_{kl} \cdot t_{ij} \leq t_{il} \cdot t_{kj}$, *and* $t_{il} \cdot t_{kj} = t_{kj} \cdot t_{il}$.

Proof. The easy verification is left to the reader. $\square$

Theorem 5.10. *Let* $0 < d \in H_2(\mathrm{SL}_{n+1}(\mathbb{C})/B; \mathbb{Z})$. *Choose* $\alpha \in R^+$ *maximal such that* $\alpha^\vee \leq d$, *i.e. there is no root* $\gamma \in R^+$ *such that* $\alpha^\vee < \gamma^\vee \leq d$. *Then* $z_d = z_{d-\alpha^\vee} \cdot s_\alpha = s_\alpha \cdot z_{d-\alpha^\vee}$.

Proof. By Theorem 2 we can choose a sequence of positive roots $\alpha_1, \alpha_2, \dots, \alpha_p$ such that $d = \sum \alpha_i^\vee$ and $z_d = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \dots \cdot s_{\alpha_p}$. Then choose i such that $\alpha_i^\vee$ is maximal. Assume that s_{α_i} does not Demazure-commute with all other reflections s_{α_j}. By moving s_{α_i} through some commuting reflections, we can assume that $j = i + 1$ (or $j = i - 1$ which is treated by a symmetric case). Lemma 5.9(b) implies that there are positive roots $\gamma_i, \gamma_j \in R^+$ such that $\gamma_i + \gamma_j = \alpha_i + \alpha_j$, $\gamma_i > \alpha_i > \alpha_j > \gamma_j$, $s_{\gamma_i} \cdot s_{\gamma_j} \geq s_{\alpha_i} \cdot s_{\alpha_j}$, and $s_{\gamma_i} \cdot s_{\gamma_j} = s_{\gamma_j} \cdot s_{\gamma_i}$. Theorem 2 implies that $z_d = s_{\alpha_1} \cdots s_{\gamma_i} \cdot s_{\gamma_j} \cdots s_{\alpha_p}$, so we can replace α_i, α_j with γ_i, γ_j. By repeating this procedure finitely many times and then placing α_i at the end of the sequence, we may assume that α_p is maximal and s_{α_p} Demazure-commutes with all reflections s_{α_j} for $j < p$. We finally deduce from Theorem 2 and Corollary 5.1 that $z_d = s_{\alpha_1} \cdot \dots \cdot s_{\alpha_{p-1}} \cdot s_{\alpha_p} \leq z_{d-\alpha_p^\vee} \cdot s_{\alpha_p} \leq z_d$, as required. $\square$

6. Gromov-Witten invariants

Given $w \in W$ and $d \in H_2(X; \mathbb{Z})$, define the Gromov-Witten variety

$$\mathrm{GW}_d(w) = \mathrm{ev}_2^{-1}(X(w)) \ \subset \overline{\mathcal{M}}_{0,2}(X, d) \,.$$

We have $\dim \mathrm{GW}_d(w) = \ell(wW_P) + (c_1(T_X), d) - 1$, and Theorem 1 implies that $\mathrm{ev}_1(\mathrm{GW}_d(w)) = \Gamma_d(X(w)) = X(w \cdot z_d)$. We need the following consequence of [3, Prop. 3.3].

Proposition 6.1 ([3])**.** *The variety* $\mathrm{GW}_d(w)$ *is unirational, and the evaluation map* $\mathrm{ev}_1 : \mathrm{GW}_d(w) \to \Gamma_d(X(w))$ *is a locally trivial fibration over the open* B*-orbit in* $\Gamma_d(X(w))$. *In particular, the general fibers of* ev_1 *are unirational.*

Theorem 6.2. *Let* $w \in W^P$ *and* $0 < d \in H_2(X; \mathbb{Z})$. *Then* $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)]$ *is non-zero in* $H^*(X; \mathbb{Z})$ *if and only if* $\ell(wz_dW_P) = \ell(w) + (c_1(T_X), d) - 1$. *In this case we have* $\ell(z_d) = (c_1(T_X), d) - 1$, $wz_d \in W^P$, *and* $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] = [X(wz_d)]$.

Proof. Notice first that $\ell(wz_dW_P) \leq \dim X(w \cdot z_d) \leq \ell(w) + \ell(z_d) \leq \ell(w) + (c_1(T_X), d) - 1 = \dim \mathrm{GW}_d(w)$. If $\ell(wz_dW_P) = \ell(w) + (c_1(T_X), d) - 1$, then we must have $\dim \mathrm{GW}_d(w) = \dim X(w \cdot z_d)$, so Theorem 1 and Proposition 6.1 imply that $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] = [X(w \cdot z_d)]$. On the other hand, if $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \neq 0$, then we obtain $\ell(w \cdot z_dW_P) = \ell(w) + \ell(z_d)$ and $\ell(z_d) = (c_1(T_X), d) - 1$, and the first equality implies that $wz_dW_P = w \cdot z_dW_P$ by Proposition 4.3. $\square$

Let $H_T^*(X; \mathbb{Z})$ denote the T-equivariant cohomology ring of X. Each T-stable closed subvariety $Z \subset X$ defines an equivariant class $[Z] \in H_T^*(X; \mathbb{Z})$. Pullback along the structure morphism $X \to \{\text{point}\}$ gives $H_T^*(X; \mathbb{Z})$ the structure of an algebra over the ring $\Lambda := H_T^*(\text{point}; \mathbb{Z})$, and $H_T^*(X; \mathbb{Z})$ is a free Λ-module with basis $\{[Y(w)] : w \in W^P\}$. For any class $\Omega \in H_T^*(X; \mathbb{Z})$, we let $\int_X \Omega \in \Lambda$ denote the proper pushforward of Ω along the structure morphism $X \to \{\text{point}\}$. The Kontsevich space $\overline{\mathcal{M}}_{0,n}(X, d)$ has a natural T-action given by $(t.f)(y) = t.f(y)$ for any stable map $f : C \to X$ and $t \in T$, and the evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$ are T-equivariant. Given classes $\Omega_1, \dots, \Omega_n \in H_T^*(X;\mathbb{Z})$ and $d \in H_2(X;\mathbb{Z})$, the associated *equivariant Gromov-Witten invariants* is defined by

$$I_d(\Omega_1, \dots, \Omega_n) = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(\Omega_1) \cdot \mathrm{ev}_2^*(\Omega_2) \cdots \mathrm{ev}_n^*(\Omega_n) \ \in \Lambda\,.$$

Notice that Theorem 6.2 holds for the equivariant class $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \in H_T^*(X)$, with the same proof.

Corollary 6.3. *Let $w, u \in W^P$ and $0 < d \in H_2(X;\mathbb{Z})$. The two-point Gromov-Witten invariant $I_d([X(w)],[Y(u)])$ is non-zero if and only if there exists a root $\alpha \in R^+ \smallsetminus R_P^+$ such that $d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$, $\ell(w s_\alpha W_P) = \ell(w) + (c_1(T_X), d) - 1$, and $uW_P = w s_\alpha W_P$. In this case α unique and P-cosmall, and $I_d([X(w)],[Y(u)]) = 1$.*

Proof. The projection formula and Theorem 6.2 imply that $I_d([X(w)],[Y(u)]) = \int_X (\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \cdot [Y(u)]$ is non-zero if and only if we have $\ell(w z_d W_P) = \ell(w) + (c_1(T_X), d) - 1$ and $u = w z_d$. In this case it follows from Proposition 5.3 that $d = \alpha^\vee + \mathbb{Z}\Delta^\vee$ for a unique P-cosmall root $\alpha \in R^+ \smallsetminus R_P^+$ with $s_\alpha W_P = z_d W_P$. □

The divisor axiom [12] (see also [7, (40)]) is valid for equivariant Gromov-Witten invariants. Let $Z \subset X$ be any T-stable divisor and $0 < d \in H_2(X;\mathbb{Z})$, and consider the variety $\mathrm{ev}_n^{-1}(Z) \subset \overline{\mathcal{M}}_{0,n}(X,d)$ and the morphism $\phi : \mathrm{ev}_n^{-1}(Z) \to \overline{\mathcal{M}}_{0,n-1}(X,d)$ that discards the n-th marked point in the domain of its argument. For a general stable map $f : C \to X$ in $\overline{\mathcal{M}}_{0,n-1}(X,d)$ we can identify the fiber $\phi^{-1}(f)$ with $f^{-1}(Z) \subset C$, so it follows from Kleiman's transversality theorem [11] that $\#\phi^{-1}(f) = ([Z], d) = \int_d [Z]$ for all points f in a dense open subset of $\overline{\mathcal{M}}_{0,n-1}(X,d)$. We deduce that $\phi_*[\mathrm{ev}_n^{-1}(Z)] = ([Z],d) \cdot [\overline{\mathcal{M}}_{0,n-1}(X,d)]$, so the projection formula implies that

$$I_d(\Omega_1, \dots, \Omega_{n-1}, [Z]) = ([Z], d) \cdot I_d(\Omega_1, \dots, \Omega_{n-1}) \ \in \Lambda \tag{10}$$

for all classes $\Omega_1, \dots, \Omega_{n-1} \in H_T^*(X;\mathbb{Z})$. In particular, the equivariant Gromov-Witten invariant $I_d(\Omega_1, \dots, \Omega_{n-1}, [Z])$ depends only on the class of Z in the ordinary cohomology ring $H^*(X;\mathbb{Z})$ and not on its equivariant class in $H_T^*(X;\mathbb{Z})$.

Corollary 6.4. *Let $u, w \in W^P$, $\beta \in \Delta$, and $0 < d \in H_2(X;\mathbb{Z})$. If the Gromov-Witten invariant $I_d([Y(u),[Y(s_\beta)],[X(w)])$ is non-zero, then there exists a unique root $\alpha \in R^+ \smallsetminus R_P^+$ such that* (i) *$d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$,* (ii) *$\ell(wW_P) = \ell(uW_P) + 1 - (c_1(T_X), \alpha^\vee)$, and* (iii) *$wW_P = u s_\alpha W_P$. If $\alpha \in R^+ \smallsetminus R_P^+$ is any root satisfying* (i)*,* (ii)*, and* (iii)*, then we have $\langle [Y(u)], [Y(s_\beta)], [X(w)] \rangle_d = (\omega_\beta, \alpha^\vee) \in \mathbb{Z}$.*

Proof. If $I_d([Y(u)],[X(w)],[Y(s_\beta)]) \neq 0$ then it follows from (10) and Corollary 6.3 that there exists a root $\gamma \in R^+ \smallsetminus R_P^+$ such $d = \gamma^\vee + \mathbb{Z}\Delta_P^\vee$, $\ell(wW_P) = \ell(uW_P) + 1 - (c_1(T_X), \gamma^\vee)$, and $uW_P = w s_\gamma W_P$. Since $u^{-1} w s_\gamma \in W_P$ we deduce that $\alpha := u^{-1}w(-\gamma) \in R^+ \smallsetminus R_P^+$ satisfies (i), (ii), and (iii). □

Remark 6.5. The K-theoretic two-point invariants are easier to compute, since Proposition 6.1 together with the K-theoretic Gysin formula of [4, Thm. 3.1] imply that $(\mathrm{ev}_2)_*[\mathcal{O}_{\mathrm{GW}_d(w)}] = [\mathcal{O}_{X(w \cdot z_d)}] \in K_T(X)$ for any degree $d \geq 0$. It follows that any equivariant K-theoretic two-point Gromov-Witten invariant of X is given by

$$\begin{aligned}\chi_{\overline{\mathcal{M}}_{0,2}(X,d)}(\mathrm{ev}_1^*[\mathcal{O}_{X(w)}]\cdot\mathrm{ev}_2^*[\mathcal{O}_{Y(u)}]) \;&=\; \chi_X([\mathcal{O}_{X(w\cdot z_d)}]\cdot[\mathcal{O}_{Y(u)}])\\ =\; \chi_X(\mathcal{O}_{X(w\cdot z_d)\cap Y(u)}) \;&=\; \begin{cases}1 & \text{if } uW_P\leq w\cdot z_dW_P\,;\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We refer to [4, §4] for notation. Unfortunately, the K-theoretic invariants do not satisfy a divisor axiom, so this formula does not reveal any 3-point invariants.

7. The equivariant quantum Chevalley formula

The *T-equivariant quantum cohomology ring* $\mathrm{QH}_T(X)$ is an algebra over the polynomial ring $\Lambda[q] := \Lambda[q_\beta : \beta\in\Delta\smallsetminus\Delta_P]$, $\Lambda = H_T^*(\text{point})$, which as a $\Lambda[q]$-module is defined by $\mathrm{QH}_T(X) = H^*(X;\mathbb{Z})\otimes_{\mathbb{Z}}\Lambda[q]$. The multiplicative structure of $\mathrm{QH}_T(X)$ is given by

$$[Y(u)]\star[Y(v)] = \sum_{w,d} I_d([Y(u)],[Y(v)],[X(w)])\,q^d\,[Y(w)]\,,$$

where the sum is over $w\in W^P$ and $0\leq d\in H_2(X;\mathbb{Z})$, and we write $q^d = \prod_\beta q_\beta^{(\omega_\beta,d)}$. It was proved in [15, 16] that if $v = s_\beta$ is a simple reflection, then the product $[Y(u)]\star[Y(s_\beta)]$ contains no *mixed* terms, i.e. if $d\neq 0$ then the coefficient of $q^d[Y(w)]$ is always an integer. This fact is also a consequence of Corollary 6.4.

To state the equivariant quantum Chevalley formula, we need some notation. Since G is simply connected, each integral weight $\lambda\in\mathbb{Z}\{\omega_\beta\mid\beta\in\Delta\}$ can be identified with a character $\lambda : T\to\mathbb{C}^*$. Let $\mathbb{C}_\lambda$ be the corresponding one-dimensional representation of T, defined by $t.z = \lambda(t)z$. This representation can be viewed as a T-equivariant vector bundle over a point, so it defines the equivariant Chern class $c_T(\lambda) := c_1^T(\mathbb{C}_\lambda)\in\Lambda$. This class should *not* be confused with the class that λ might represent in $H^2(X;\mathbb{Z}) = \mathbb{Z}\{\omega_\beta\mid\beta\in\Delta\smallsetminus\Delta_P\}$ by the notation of section 2. The ring Λ is the polynomial ring over $\mathbb{Z}$ generated by the classes $c_T(\omega_\beta)$ for $\beta\in\Delta$. The equivariant quantum Chevalley formula is the following result [5, 8, 16].

Theorem 7.1. *Let $u\in W$ and $\beta\in\Delta\smallsetminus\Delta_P$. Then we have*

$$\begin{aligned}&[Y(u)]\star[Y(s_\beta)] =\\ &\sum_\alpha(\omega_\beta,\alpha^\vee)\,[Y(us_\alpha)] \;+\; c_T(\omega_\beta-u.\omega_\beta)\,[Y(u)] \;+\; \sum_\alpha(\omega_\beta,\alpha^\vee)\,q^{\alpha^\vee}\,[Y(us_\alpha)]\;;\end{aligned}$$

the first sum is over $\alpha\in R^+\smallsetminus R_P^+$ such that $\ell(us_\alpha W_P) = \ell(uW_P)+1$, and the second sum is over $\alpha\in R^+\smallsetminus R_P^+$ such that $\ell(us_\alpha W_P) = \ell(uW_P)+1-(c_1(T_X),\alpha^\vee)$.

Proof. It follows from Corollary 6.4 that the second sum accounts for all terms with non-zero q-degrees. The remaining terms come from the equivariant product $[Y(u)]\cdot[Y(s_\beta)]\in H_T^*(X;\mathbb{Z})$, and the coefficient of $[Y(w)]$ in this product is

$$c^w_{u,s_\beta} = \int_X [Y(u)]\cdot[Y(s_\beta)]\cdot[X(w)] = \int_X [Y(u)\cap X(w)]\cdot[Y(s_\beta)] \;\in\Lambda\,.$$

This coefficient is non-zero only if $u\leq w$ and $\ell(wW_P)\leq\ell(uW_P)+1$. If $\ell(wW_P) = \ell(uW_P)+1$, then the intersection $Y(u)\cap X(w)$ is a one-dimensional closed T-stable subvariety of X whose T-fixed points consist of $u.P$ and $w.P$. It follows that $Y(u)\cap X(w) = u.C_\alpha$ and $w.P = us_\alpha.P$ for some root $\alpha\in R^+\smallsetminus R_P^+$, and we have

$c^w_{u,s_\beta} = ([Y(s_\beta)], [C_\alpha]) = (\omega_\beta, \alpha^\vee)$ as claimed. This argument can also be found in e.g. [8, Lemma 8.1] or [2, Prop. 1.4.3].

The last remaining term is $c^u_{u,s_\beta}[Y(u)]$. The projection formula implies that

$$c^u_{u,s_\beta} = \int_X [Y(s_\beta)] \cdot [u.P] = [Y(s_\beta)]_{u.P}$$

where $[Y(s_\beta)]_{u.P} \in \Lambda$ is the restriction of $[Y(s_\beta)]$ to the T-fixed point $u.P \in X$. Set $\lambda = \omega_\beta$ and notice that $L_\lambda = G \times^P \mathbb{C}_\lambda$ is a G-equivariant line bundle with action defined by $g'.[g,z] = [g'g,z]$. According to the Borel-Weil theorem [17] there exists a B^{op}-stable section $\sigma \in H^0(X, L_\lambda)$, unique up to scalar, and we have $\mathbb{C}\sigma \cong \mathbb{C}_{-\lambda}$ as a T-representation. This implies that $\sigma : X \times \mathbb{C}_{-\lambda} \to L_\lambda$ is a morphism of T-equivariant line bundles. The T-equivariant class of $Y(s_\beta)$ is therefore given by

$$[Y(s_\beta)] = [Z(\sigma)] = c_1^T(L_\lambda) - c_1^T(X \times \mathbb{C}_{-\lambda}) \in H^2_T(X;\mathbb{Z})\,.$$

Since the fiber of L_λ over $u.P$ is $L_\lambda(u.P) \cong \mathbb{C}_{-u.\lambda}$, we obtain

$$[Y(s_\beta)]_{u.P} = c_1^T(\mathbb{C}_{-u.\lambda}) - c_1^T(\mathbb{C}_{-\lambda}) = c_T(\omega_\beta - u.\omega_\beta)\,.$$

This finishes the proof. ☐

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