

A GIAMBELLI FORMULA FOR ISOTROPIC GRASSMANNIANS

ANDERS SKOVSTED BUCH, ANDREW KRESCH, AND HARRY TAMVAKIS

ABSTRACT. Let X be a symplectic or odd orthogonal Grassmannian parametrizing isotropic subspaces in a vector space equipped with a nondegenerate (skew) symmetric form. We prove a Giambelli formula which expresses an arbitrary Schubert class in $H^*(X, \mathbb{Z})$ as a polynomial in certain special Schubert classes. We introduce and study *theta polynomials*, a family of polynomials which are positive linear combinations of products of Schur Q - and S -functions, and whose algebra agrees with the Schubert calculus on X .

0. INTRODUCTION

Let $G = G(m, N)$ denote the Grassmannian of m -dimensional subspaces of complex affine N -space. To each integer partition $\lambda = (\lambda_1, \dots, \lambda_m)$ whose Young diagram is contained in an $m \times (N - m)$ rectangle, we associate a Schubert class σ_λ in the cohomology ring of G . The *special* Schubert classes $\sigma_1, \dots, \sigma_{N-m}$ are the Chern classes of the universal quotient bundle \mathcal{Q} over $G(m, N)$; they generate the ring $H^*(G, \mathbb{Z})$. The classical *Giambelli formula* [G]

$$(1) \quad \sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{i,j}$$

is an explicit expression for σ_λ as a polynomial in the special classes; as is customary, we agree here and in later formulas that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r < 0$.

The relation between the Schubert calculus on the Grassmannian $G(m, N)$ and the algebra of Schur's S -functions s_λ (originally defined by Cauchy [C] and Jacobi [J]) is well known. Given $x = (x_1, x_2, \dots)$ a countably infinite set of commuting independent variables, we define the elementary symmetric functions $e_r(x)$ by the formal relation

$$\prod_{i=1}^{\infty} (1 + x_i t) = \sum_{r=0}^{\infty} e_r(x) t^r$$

and set, for any partition λ , $s_{\lambda'}(x) = \det(e_{\lambda_i + j - i}(x))_{i,j}$. Here λ' is the partition whose Young diagram is the transpose of the diagram of λ . The ring $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$ of symmetric functions in x has a free \mathbb{Z} -basis consisting of the Schur functions s_λ , for all partitions λ . These Schur S -functions enjoy many good combinatorial properties, such as nonnegativity of their coefficients, and multiply exactly like the Schubert classes on $G(m, N)$, when m and N are sufficiently large.

There is a closely analogous story to the above for the Lagrangian Grassmannian $LG(n, 2n)$ which parametrizes maximal isotropic subspaces of \mathbb{C}^{2n} , with respect to a symplectic form. The Schubert classes in $H^*(LG, \mathbb{Z})$ are indexed by *strict* partitions, i.e., partitions with distinct (non-zero) parts, whose diagrams fit in a square of side

Date: April 29, 2009.

2000 Mathematics Subject Classification. Primary 14N15; Secondary 05E15, 14M15.

The authors were supported in part by NSF Grant DMS-0603822 (Buch), the Swiss National Science Foundation (Kresch), and NSF Grant DMS-0639033 (Tamvakis).

n . The special Schubert classes $\sigma_r = c_r(\mathcal{Q})$ again generate the cohomology ring, and there is a Giambelli-type formula due to Pragacz [Pra]. This latter may be described in two steps: For partitions $\lambda = (a, b)$ with only two parts, we have

$$(2) \quad \sigma_{a,b} = \sigma_a \sigma_b - 2\sigma_{a+1} \sigma_{b-1} + 2\sigma_{a+2} \sigma_{b-2} - \cdots$$

while for λ with 3 or more parts,

$$(3) \quad \sigma_\lambda = \text{Pfaffian}(\sigma_{\lambda_i, \lambda_j})_{i < j}.$$

The identities (2) and (3) in fact also go back to the work of Schur [S], who considered a family of symmetric functions $\{Q_\lambda\}$ known as Schur Q -functions. We define $q_r(x)$ by the equation

$$\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} = \sum_{r=0}^{\infty} q_r(x) t^r$$

and then use the same relations (2) and (3) with $q_r(x)$ in place of σ_r to define $Q_{a,b}(x)$ and then $Q_\lambda(x)$, for each strict partition λ . If we let $\Gamma = \mathbb{Z}[q_1, q_2, \dots]$ denote the ring of Schur Q -functions, then the $\{Q_\lambda\}$ for λ strict form a \mathbb{Z} -basis for Γ , whose algebra agrees with Schubert calculus on $\text{LG}(n, 2n)$, as $n \rightarrow \infty$. Moreover, there is a well developed combinatorial theory for the Q -functions, analogous to that for the S -functions.

Choose $k \geq 0$ and consider now the Grassmannian $\text{IG}(n - k, 2n)$ of isotropic $(n - k)$ -dimensional subspaces of \mathbb{C}^{2n} , equipped with a symplectic form. We call a partition λ k -strict if no part greater than k is repeated, i.e., $\lambda_j > k \Rightarrow \lambda_j > \lambda_{j+1}$. The Schubert classes on IG are indexed by k -strict partitions whose diagrams fit in an $(n - k) \times (n + k)$ rectangle. Given such a λ and a complete flag of subspaces $F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n}$ such that $F_{n+i} = F_{n-i}^\perp$ for $0 \leq i \leq n$, we have a Schubert variety

$$X_\lambda(F_\bullet) := \{\Sigma \in \text{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \quad \forall 1 \leq j \leq \ell(\lambda)\},$$

where $\ell(\lambda)$ denotes the number of (non-zero) parts of λ and

$$(4) \quad p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}.$$

This variety has codimension $|\lambda| = \sum \lambda_i$ and defines, using Poincaré duality, a Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)]$ in $H^{2|\lambda|}(\text{IG}, \mathbb{Z})$. As above, we consider the special Schubert classes $\sigma_r = [X_r(F_\bullet)] = c_r(\mathcal{Q})$ for $1 \leq r \leq n + k$.

In [BKT1], we proved a Pieri rule for the products $\sigma_r \sigma_\lambda$ in $H^*(\text{IG})$. Equipped with this rule and the help of a computer, we observed that (i) when $\lambda_j \leq k$ for all j , then σ_λ is given by the determinantal formula (1); (ii) when $\lambda_j > k$ for all non-zero λ_j , then λ is strict and σ_λ is given by the Pfaffian formulas (2), (3). It is tempting to ask for an analogous Giambelli formula for σ_λ when λ is a general k -strict partition. Note that the formula is determined only up to an ideal of relations; whatever the answer, it must naturally interpolate between the Jacobi-Trudi determinant (1) and the Schur Pfaffian (3). This question was also raised by Pragacz and Ratajski [PR], who were using a different set of special Schubert classes.

The answer we give depends crucially on our choice of k -strict partitions to index the Schubert classes, and uses Young's *raising operators* [Y, p. 199]. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ with finite support and $i < j$, we define $R_{ij}(\alpha) = (\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_j - 1, \dots)$; a raising operator R is any monomial in

these R_{ij} 's. Set $m_\alpha = \prod_i \sigma_{\alpha_i}$ and $Rm_\alpha = m_{R\alpha}$ for any raising operator R ¹. Using these operators, the Giambelli formulas (1) and (2)–(3) can be expressed as

$$(5) \quad \sigma_\lambda = \prod_{i < j} (1 - R_{ij}) m_\lambda \quad \text{and} \quad \sigma_\lambda = \prod_{i < j} \frac{1 - R_{ij}}{1 + R_{ij}} m_\lambda,$$

respectively (compare with the identities (6) below).

Definition 1. For a general k -strict partition λ , we define the operator

$$R^\lambda = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j > 2k + j - i} (1 + R_{ij})^{-1}$$

where the first product is over all pairs $i < j$ and second product is over pairs $i < j$ such that $\lambda_i + \lambda_j > 2k + j - i$.

Theorem 1. For any k -strict partition λ , we have $\sigma_\lambda = R^\lambda m_\lambda$ in the cohomology ring of $\text{IG}(n - k, 2n)$.

For example, in the ring $H^*(\text{IG}(4, 10))$ (where $k = 1$) we have

$$\begin{aligned} \sigma_{321} &= \frac{1 - R_{12}}{1 + R_{12}} (1 - R_{13})(1 - R_{23}) m_{321} = (1 - 2R_{12} + 2R_{12}^2 - 2R_{12}^3)(1 - R_{13} - R_{23}) m_{321} \\ &= m_{321} - 2m_{411} + m_{42} + 2m_{51} - m_{33} = \sigma_3 \sigma_2 \sigma_1 - 2\sigma_4 \sigma_1^2 + \sigma_4 \sigma_2 + 2\sigma_5 \sigma_1 - \sigma_3^2. \end{aligned}$$

We introduce a family of polynomials $\{\Theta_\lambda\}$ indexed by k -strict partitions whose algebra is the same as the Schubert calculus in the stable cohomology ring of IG . Fix an integer $k \geq 0$ and consider a finite set of variables $y = (y_1, \dots, y_k)$. For any $r \geq 0$, define ϑ_r by the equation

$$\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} \prod_{j=1}^k (1 + y_j t) = \sum_{r=0}^{\infty} \vartheta_r(x; y) t^r,$$

so that $\vartheta_r(x; y) = \sum_i q_{r-i}(x) e_i(y)$. We call $\Gamma^{(k)} := \mathbb{Z}[\vartheta_1, \vartheta_2, \dots]$ the ring of *theta polynomials*. For any finite integer sequence α , let $\vartheta_\alpha = \prod_i \vartheta_{\alpha_i}$, and for any k -strict partition λ , define the theta polynomial

$$\Theta_\lambda := R^\lambda \vartheta_\lambda.$$

As a first application of Theorem 1, we obtain the next two results.

Theorem 2. The Θ_λ , for λ k -strict, form a \mathbb{Z} -basis of $\Gamma^{(k)}$. The algebra of theta polynomials agrees with the Schubert calculus on isotropic Grassmannians $\text{IG}(n - k, 2n)$, when n is sufficiently large.

Theorem 3. Let λ be a k -strict partition.

a) If $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, then

$$\Theta_\lambda(x; y) = \sum_{\mu \subset \lambda} S_\mu(x) s_{\lambda'/\mu'}(y), \quad \text{where} \quad S_\mu(x) = \det(q_{\mu_i + j - i}(x)).$$

b) If $\lambda_i + \lambda_j > 2k + j - i$ for all $i < j \leq \ell(\lambda)$, then

$$\Theta_\lambda(x; y) = \sum_{\mu \subset \lambda: \mu \text{ strict}} Q_\mu(x) s_{\mathcal{S}(\lambda/\mu)'}(y),$$

where $\mathcal{S}(\lambda/\mu)$ denotes a shifted skew diagram.

¹As is customary, we slightly abuse the notation and consider that the raising operator R acts on the index α , and not on the monomial m_α itself.

It turns out that the coefficients of the polynomial $\Theta_\lambda(x; y)$ are always nonnegative integers, which have a combinatorial interpretation.

Theorem 4. *For any k -strict partition λ , the polynomial Θ_λ is a linear combination of products of Schur Q - and S -functions:*

$$\Theta_\lambda(x; y) = \sum_{\mu, \nu} e_{\mu\nu}^\lambda Q_\mu(x) s_{\nu'}(y).$$

Moreover, the coefficients $e_{\mu\nu}^\lambda$ are nonnegative integers, equal to the number of certain Kraskiewicz tableaux [Kr] of shape μ .

Our proof of Theorem 1 proceeds by showing directly that the raising operator expression $R^\lambda m_\lambda$ satisfies the Pieri rule for isotropic Grassmannians from [BKT1]. This is sufficient because the Pieri rule can be used recursively to show that a general Schubert class may be written as a polynomial in the special Schubert classes. The argument is challenging since the raising operator R^λ changes as the partition λ does, in contrast with the fixed operator expressions in (5). We remark that the equations corresponding to (5) for the Schur S - and Q -functions may be deduced from the formal identities

$$(6) \quad \det(x_i^{\ell-j}) = \prod_{i < j} (x_i - x_j) \quad \text{and} \quad \text{Pfaffian} \left(\frac{x_i - x_j}{x_i + x_j} \right) = \prod_{i < j} \frac{x_i - x_j}{x_i + x_j}$$

due to Vandermonde and Schur, respectively (see e.g. [M, I.3 and III.8]).

Theorem 4 is proved by showing that the theta polynomial Θ_λ agrees with the type C Schubert polynomial of Billey and Haiman [BH] indexed by the corresponding Grassmannian element w_λ of the hyperoctahedral group. Our Giambelli formula may therefore be used to further understand these and related polynomials. For instance, it follows that the type C Stanley symmetric function $F_{w_\lambda}(x)$ of [BH, FK, L] is equal to $R^\lambda q_\lambda(x)$.

We have described the theory here in the symplectic case, but there are entirely analogous results for the odd orthogonal groups. In fact, for technical reasons, our proof of Theorem 1 is obtained in the setting of orthogonal type B. In a sequel to this paper, we will discuss the Giambelli formula for even orthogonal Grassmannians, which is more involved. We also obtain analogues of these Giambelli formulas for the quantum cohomology rings of symplectic and odd orthogonal Grassmannians; this application will appear in [BKT2].

This article is organized as follows. The proof of Theorem 1 occupies §1–§4. Section 5 introduces and studies theta polynomials, and contains our proofs of Theorems 2 and 3. Finally, in §6 we show that theta polynomials are Schubert polynomials for the hyperoctahedral group, and prove Theorem 4.

1. PRELIMINARY RESULTS

1.1. The Schubert varieties in $\text{IG} = \text{IG}(n-k, 2n)$ are indexed by k -strict partitions λ which are contained in an $(n-k) \times (n+k)$ rectangle; we denote the set of all such partitions by $\mathcal{P}(k, n)$. Consider the exact sequence of vector bundles over IG

$$0 \rightarrow \mathcal{S} \rightarrow E \rightarrow \mathcal{Q} \rightarrow 0,$$

where E denotes the trivial bundle of rank $2n$ and \mathcal{S} is the tautological subbundle of rank $n-k$. The special Schubert class σ_p is equal to the Chern class $c_p(\mathcal{Q})$.

The symplectic form on E gives a pairing $\mathcal{S} \otimes \mathcal{Q} \rightarrow \mathcal{O}_{\text{IG}}$, which in turn produces an injection $\mathcal{S} \hookrightarrow \mathcal{Q}^*$. For $r > k$ we therefore have

$$c_{2r}(\mathcal{Q} \oplus \mathcal{Q}^*) = c_{2r}(E/\mathcal{S} \oplus \mathcal{Q}^*) = c_{2r}(\mathcal{Q}^*/\mathcal{S}) = 0,$$

which implies that the relations

$$(7) \quad \sigma_r^2 + 2 \sum_{i=1}^{n+k-r} (-1)^i \sigma_{r+i} \sigma_{r-i} = 0 \quad \text{for } r > k$$

hold in $H^*(\text{IG}, \mathbb{Z})$.

1.2. Let $\Delta^\circ = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j\}$ and define a partial order on Δ° by agreeing that $(i', j') \leq (i, j)$ if $i' \leq i$ and $j' \leq j$. We call a finite subset D of Δ° a *valid set of pairs* if it is an order ideal, i.e., $(i, j) \in D$ implies $(i', j') \in D$ for all $(i', j') \in \Delta^\circ$ with $(i', j') \leq (i, j)$. Given a k -strict partition λ , we obtain a valid set of pairs $\mathcal{C} = \mathcal{C}(\lambda)$ by

$$\mathcal{C}(\lambda) = \{(i, j) \in \Delta^\circ \mid \lambda_i + \lambda_j > 2k + j - i \text{ and } j \leq \ell(\lambda)\}.$$

It is easy to see that when $k > 0$, a set $D \subset \Delta^\circ$ is a valid set of pairs if and only if there exists a k -strict partition λ for which $\mathcal{C}(\lambda) = D$.

A *composition* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$ is a vector of integers from the set $\mathbb{N} = \{0, 1, 2, \dots\}$; we let $|\alpha| = \sum \alpha_i$. For λ any sequence of (possibly negative) integers, we say that λ has length ℓ if $\lambda_i = 0$ for all $i > \ell$ and $\ell \geq 0$ is the smallest number with this property. All integer sequences in this paper have finite length, and we will identify any integer sequence of length ℓ with the vector consisting of its first ℓ terms. In analogy with Young diagrams of partitions, we will say that a pair $[i, j]$ is a *box* of the integer sequence λ if $i \geq 1$ and $1 \leq j \leq \lambda_i$.

Any valid set of pairs D defines the raising operator

$$R^D = \prod_{i < j} (1 - R_{ij}) \prod_{i < j : (i, j) \in D} (1 + R_{ij})^{-1}.$$

Given a composition α and an integer $\ell > 0$, we denote by $m(D, \alpha, \ell)$ the number of non-zero coordinates α_i such that $(i, \ell) \in D$. We say that α is (D, ℓ) -*compatible* if $\alpha_i \in \{0, 1\}$ whenever $(i, \ell) \notin D$.

Definition 1.1. For any valid set of pairs D and any integer sequence λ of length ℓ we define a cohomology class $T_\lambda = T(D, \lambda)$ recursively as follows. Set $T_0 = 1$, $T_p = \sigma_p$, and $T_p = 0$ for $p < 0$. For an arbitrary integer sequence $\mu = (\mu_1, \dots, \mu_{\ell-1})$ and $r \in \mathbb{Z}$, set

$$(8) \quad T_{\mu, r} = \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D, \alpha, \ell)} T_{\mu + \alpha} T_{r - |\alpha|},$$

where the sum is over all (D, ℓ) -compatible vectors $\alpha \in \mathbb{N}^{\ell-1}$.

The sum (8) is well defined because only finitely many of its summands are non-zero; we have $T_{\mu, r} = 0$ if $r < 0$. Notice that definition (8) of $T(D, \lambda)$ is equivalent to expanding the raising operator formula

$$R^D m_\lambda = \prod_{i < j < \ell} (1 - R_{ij}) \prod_{i < j < \ell : (i, j) \in D} (1 + R_{ij})^{-1} \prod_{i=1}^{\ell-1} (1 - R_{i\ell}) \prod_{i : (i, \ell) \in D} (1 + R_{i\ell})^{-1} m_{\mu, r}$$

after the last (i.e., the ℓ -th) entry of $\lambda = (\mu, r)$. Therefore $T_\lambda = R^D m_\lambda$.

1.3. If $D = \emptyset$ then for any integers r and s we have

$$T_{r,s} = T_r T_s - T_{r+1} T_{s-1}$$

and so $T_{r,r+1} = 0$, while more generally $T_{r,s} = -T_{s-1,r+1}$.

We claim that if $D \neq \emptyset$ and $r, s \in \mathbb{Z}$ are such that $r + s > 2k$, then $T_{s,r} = -T_{r,s}$; in particular $T_{r,r} = 0$ whenever $r > k$. Indeed, from the definition we obtain

$$T_{r,s} = \sigma_r \sigma_s - 2 \sigma_{r+1} \sigma_{s-1} + 2 \sigma_{r+2} \sigma_{s-2} - \cdots$$

and hence $T_{s,r} = -T_{r,s}$ whenever $r + s$ is odd. If $r + s = 2m > 2k$ is even, we see that

$$(9) \quad T_{r,s} + T_{s,r} = (-1)^{\frac{r-s}{2}} 2 (\sigma_m^2 - 2 \sigma_{m+1} \sigma_{m-1} + 2 \sigma_{m+2} \sigma_{m-2} - \cdots) = 0$$

using the relations (7) in the cohomology ring of IG.

The previous observations are generalized in the next two lemmas.

Lemma 1.1. *Let $\lambda = (\lambda_1, \dots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \dots, \mu_\ell)$ be integer vectors. Assume that $(j, j+1) \notin D$ and that for each $h < j$, $(h, j) \notin D$ iff $(h, j+1) \notin D$. Then for any integers r and s we have*

$$T_{\lambda,r,s,\mu} = -T_{\lambda,s-1,r+1,\mu}.$$

In particular, $T_{\lambda,r,r+1,\mu} = 0$.

Proof. If $\mu = (\tau, t)$ has positive length, we set $\rho = (\lambda, r, s, \tau)$ and the identity follows by induction, because

$$T_{\rho,t} = \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho+\alpha} T_{t-|\alpha|}.$$

Therefore, we may assume that μ is empty. Set $\ell = j + 1$. Then we have

$$\begin{aligned} T_{\lambda,r,s} &= \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\lambda+\alpha,r} T_{s-|\alpha|} - \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\lambda+\alpha,r+1} T_{s-|\alpha|-1} \\ &= \sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|} 2^{m(D,\alpha,\ell)+m(D,\beta,\ell-1)} T_{\lambda+\alpha+\beta} T_{r-|\beta|} T_{s-|\alpha|} \\ &\quad - \sum_{\alpha,\beta} (-1)^{|\alpha|+|\beta|} 2^{m(D,\alpha,\ell)+m(D,\beta,\ell-1)} T_{\lambda+\alpha+\beta} T_{r+1-|\beta|} T_{s-1-|\alpha|} \end{aligned}$$

where the sums are over all (D, ℓ) -compatible sequences $\alpha \in \mathbb{N}^{j-1}$ and $(D, \ell - 1)$ -compatible sequences $\beta \in \mathbb{N}^{j-1}$. The assumptions on D imply that these two sets of sequences coincide, and this proves the lemma. \square

Lemma 1.2. *Let $\lambda = (\lambda_1, \dots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \dots, \mu_\ell)$ be integer vectors, assume $(j, j+1) \in D$, and that for each $h > j + 1$, $(j, h) \in D$ iff $(j + 1, h) \in D$. If $r, s \in \mathbb{Z}$ are such that $r + s > 2k$, then we have*

$$T_{\lambda,r,s,\mu} = -T_{\lambda,s,r,\mu}.$$

In particular, $T_{\lambda,r,r,\mu} = 0$ for any $r > k$.

Proof. If $\mu = (\tau, t)$ has positive length, we set $\rho = (\lambda, r, s, \tau)$ and $\rho' = (\lambda, s, r, \tau)$, and the identity follows by induction, because

$$T_{\rho,t} = \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho+\alpha} T_{t-|\alpha|} = - \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D,\alpha,\ell)} T_{\rho'+\alpha} T_{t-|\alpha|} = -T_{\rho',t}.$$

Thus we may assume that μ is empty. Set $\ell = j + 1$, and note that $(h, h') \in D$ for all $h < h' \leq \ell$. If $m > 0$ is the least integer such that $2m \geq \ell$, we claim that $T_\rho = T_{\lambda, r, s}$ satisfies the relation

$$(10) \quad T_\rho = \sum_{i=2}^{2m} (-1)^i T_{\rho_1, \rho_i} T_{\rho_2, \dots, \widehat{\rho_i}, \dots, \rho_{2m}}.$$

Equation (10) follows from the formal identity of raising operators

$$\prod_{1 \leq h < h' \leq 2m} \frac{1 - R_{hh'}}{1 + R_{hh'}} = \sum_{i=2}^{2m} (-1)^i \frac{1 - R_{1i}}{1 + R_{1i}} \prod_{\substack{2 \leq h < h' \leq 2m \\ h \neq i \neq h'}} \frac{1 - R_{hh'}}{1 + R_{hh'}},$$

which is equivalent to the classical formula

$$\prod_{1 \leq h < h' \leq 2m} \frac{x_h - x_{h'}}{x_h + x_{h'}} = \text{Pfaffian} \left(\frac{x_h - x_{h'}}{x_h + x_{h'}} \right)_{1 \leq h, h' \leq 2m}$$

due to Schur [S, Sec. IX]. The proof is completed using induction, starting from the base case of $j = 1$, which was obtained in (9). \square

During the above discussion the set D has remained fixed, but in subsequent arguments we will need to modify it. For this, we use a simple observation.

Lemma 1.3. *If $(i, j) \notin D$ and $D \cup (i, j)$ is a valid set of pairs, then*

$$T(D, \lambda) = T(D \cup (i, j), \lambda) + T(D \cup (i, j), R_{ij}\lambda).$$

Proof. The assertion follows immediately from the identity

$$1 - R_{ij} = \frac{1 - R_{ij}}{1 + R_{ij}} + \frac{1 - R_{ij}}{1 + R_{ij}} R_{ij}. \quad \square$$

2. FROM $\text{IG}(n - k, 2n)$ TO $\text{OG}(n - k, 2n + 1)$

2.1. For each $k \geq 0$, the odd orthogonal Grassmannian $\text{OG} = \text{OG}(n - k, 2n + 1)$ parametrizes the $(n - k)$ -dimensional isotropic subspaces in \mathbb{C}^{2n+1} , equipped with a nondegenerate symmetric bilinear form. Our aim is to show that if λ is any k -strict partition, then $T(\mathcal{C}(\lambda), \lambda) = \sigma_\lambda$ in $H^*(\text{IG}, \mathbb{Z})$. For technical reasons, we will use an isomorphism to transfer this relation to the cohomology ring of OG , and work with the latter space.

The Schubert varieties in OG are indexed by the same set of k -strict partitions $\mathcal{P}(k, n)$ as for $\text{IG}(n - k, 2n)$. Given a complete flag F_\bullet of subspaces of \mathbb{C}^{2n+1} such that $F_{n+i} = F_{n+1-i}^\perp$ for $1 \leq i \leq n + 1$ and $\lambda \in \mathcal{P}(k, n)$, we define the codimension $|\lambda|$ Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \text{OG} \mid \dim(\Sigma \cap F_{\bar{p}_j(\lambda)}) \geq j \quad \forall 1 \leq j \leq \ell(\lambda)\},$$

where

$$(11) \quad \bar{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}.$$

Let $\tau_\lambda \in H^{2|\lambda|}(\text{OG}, \mathbb{Z})$ be the cohomology class dual to the cycle given by $X_\lambda(F_\bullet)$.

For any $\lambda \in \mathcal{P}(k, n)$, let $\ell_k(\lambda)$ be the number of parts λ_i which are strictly greater than k . Let \mathcal{Q}_{IG} and \mathcal{Q}_{OG} be the universal quotient vector bundles over $\text{IG}(n - k, 2n)$ and $\text{OG}(n - k, 2n + 1)$, respectively. It is known (see e.g. [BS, §3.1]) that the map which sends $\sigma_p = c_p(\mathcal{Q}_{\text{IG}})$ to $c_p(\mathcal{Q}_{\text{OG}})$ for all p extends to a ring

isomorphism $\phi : H^*(\text{IG}, \mathbb{Q}) \rightarrow H^*(\text{OG}, \mathbb{Q})$. Moreover, we have $\phi(\sigma_\lambda) = 2^{\ell_k(\lambda)} \tau_\lambda$ for all $\lambda \in \mathcal{P}(k, n)$.

We let $c_p = c_p(\mathcal{Q}_{\text{OG}})$. The Chern classes c_p are related to the special Schubert classes τ_p on OG by the equations

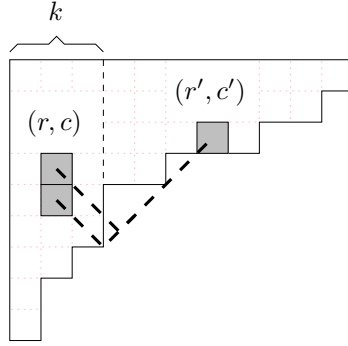
$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

Using the isomorphism ϕ , we can therefore describe the Giambelli formula for $\text{OG}(n - k, 2n + 1)$ as follows. For any integer sequence α , set $m_\alpha = \prod_i c_{\alpha_i}$; then for every $\lambda \in \mathcal{P}(k, n)$, we have

$$(12) \quad \tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda m_\lambda$$

in $H^*(\text{OG}, \mathbb{Z})$.

2.2. For λ any k -strict partition, we say that the box $[r, c]$ in row r and column c of λ is k -related to the box $[r', c']$ if $|c - k - 1| + r = |c' - k - 1| + r'$. We say that box $[r, c]$ of λ is k' -related to $[r', c']$ if $|c - k - \frac{1}{2}| + r = |c' - k - \frac{1}{2}| + r'$ (we think of k' as equal to $k - \frac{1}{2}$). For example, in the following partition the grey box $[r, c]$ is k -related to $[r', c']$, while $[r + 1, c]$ is k' -related to $[r', c']$. The definitions of k - and k' -related boxes make sense even when the boxes in question are not contained in the Young diagram of λ .



Given two Young diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row). For any two k -strict partitions λ and μ , we write $\lambda \rightarrow \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ , then the bottom box of this column is k -related to at most one box of $\mu \setminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ , then the removed boxes and the bottom box of μ in this column must each be k -related to exactly one box of $\mu \setminus \lambda$, and these boxes of $\mu \setminus \lambda$ must all lie in the same row.

Equivalently, $\lambda \rightarrow \mu$ means that $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for each j , $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and conditions (1) and (2) are true. Let \mathbb{A} be the set of boxes of $\mu \setminus \lambda$ in columns $k+1$ through $k+n$ which are not mentioned in (1) or (2), and define $\mathfrak{N}(\lambda, \mu)$

to be the number of connected components of \mathbb{A} . Here two boxes are connected if they share at least a vertex. In [BKT1, Theorem 2.1] we proved that the Pieri rule

$$(13) \quad c_p \cdot \tau_\lambda = \sum_{\lambda \rightarrow \mu, |\mu| = |\lambda| + p} 2^{\mathfrak{n}(\lambda, \mu)} \tau_\mu$$

holds, for any $p \in [1, n + k]$.

2.3. A comparison of (4) with (11) suggests modifying the definition of valid sets of pairs from §1 to include elements along the diagonal $\{(i, i) \mid i > 0\}$. This convention will make the formalism of our proof of Theorem 1 cleaner, and is in fact crucial in the corresponding proof of Giambelli for even orthogonal Grassmannians.

Set $\Delta = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq j\}$ with the same partial order as in §1.2, and define the notion of a valid set of pairs and the set $\mathcal{C}(\lambda)$ for a k -strict partition λ exactly as before, replacing Δ° by Δ . Thus $\mathcal{C}(\lambda)$ includes the pairs (i, i) such that $\lambda_i > k$.

Definition 2.1. For any valid set of pairs $D \subset \Delta$ and any integer sequence λ we define the cohomology class $T(D, \lambda) \in H^*(\text{OG})$ by

$$T(D, \lambda) = 2^{-\#\{(i, i) \in D\}} \phi(T(D \cap \Delta^\circ, \lambda)),$$

where $T(D \cap \Delta^\circ, \lambda) \in H^*(\text{IG})$ is defined as in (8).

To prove (12) and hence also establish Theorem 1, it suffices to show that if λ is a k -strict partition, the Pieri rule

$$(14) \quad c_p \cdot T(\mathcal{C}(\lambda), \lambda) = \sum_{\lambda \rightarrow \mu, |\mu| = |\lambda| + p} 2^{\mathfrak{n}(\lambda, \mu)} T(\mathcal{C}(\mu), \mu)$$

holds in $H^*(\text{OG}, \mathbb{Z})$, for all p . To see this, write $\mu \succ \lambda$ if μ strictly dominates λ , i.e., $\mu \neq \lambda$ and $\mu_1 + \dots + \mu_i \geq \lambda_1 + \dots + \lambda_i$ for each $i \geq 1$. We deduce from (13) and (14) that

$$2^{\ell_k(\lambda)} \tau_\lambda + \sum_{\mu \succ \lambda} a_\mu \tau_\mu = c_{\lambda_1} \cdots c_{\lambda_\ell} = 2^{\ell_k(\lambda)} T(\mathcal{C}(\lambda), \lambda) + \sum_{\mu \succ \lambda} a_\mu T(\mathcal{C}(\mu), \mu),$$

for some constants $a_\mu \in \mathbb{Z}$. The proof now follows by induction.

Observe that Lemmas 1.1, 1.2, and 1.3 carry over verbatim to our current setting where $D \subset \Delta$. These lemmas are the main properties of the cohomology classes $T(D, \lambda)$ that we use, and as such constitute the technical core of our proof of Theorem 1. But the non-trivial scheme that puts them to work together is an algorithm with a substitution rule; this is explained in the next section.

3. THE SUBSTITUTION RULE

3.1. Throughout the next two sections we fix $p > 0$, the k -strict partition λ of length ℓ , and choose n sufficiently large so that we can ignore it in the sequel. Set $\mathcal{C} = \mathcal{C}(\lambda)$ and for any $d \geq 1$ define the raising operator R_d^λ by

$$R_d^\lambda = \prod_{1 \leq i < j \leq d} (1 - R_{ij}) \prod_{i < j : (i, j) \in \mathcal{C}} (1 + R_{ij})^{-1}.$$

We compute that

$$c_p \cdot T(\mathcal{C}, \lambda) = c_p \cdot 2^{-\ell_k(\lambda)} R_\ell^\lambda m_\lambda = 2^{-\ell_k(\lambda)} R_{\ell+1}^\lambda \cdot \prod_{i=1}^{\ell} (1 - R_{i, \ell+1})^{-1} m_{\lambda, p}$$

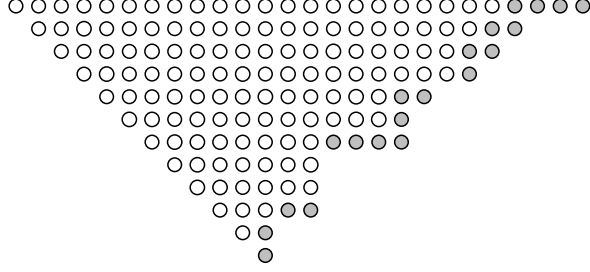


FIGURE 1. A valid set of pairs \mathcal{C} (white dots) and a subset of $\partial\mathcal{C}$ (grey dots).

$$= 2^{-\ell k(\lambda)} R_{\ell+1}^\lambda \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1} + R_{i,\ell+1}^2 + \cdots) m_{\lambda,p}$$

and therefore

$$(15) \quad c_p \cdot T(\mathcal{C}, \lambda) = \sum_{\nu \in \mathcal{N}} T(\mathcal{C}, \nu),$$

where $\mathcal{N} = \mathcal{N}(\lambda, p)$ is the set of all compositions $\nu \geq \lambda$ such that $|\nu| = |\lambda| + p$ and $\nu_j = 0$ for $j > \ell + 1$. Our strategy for proving Theorem 1 is to show that the right hand side of equation (15) is equal to the right hand side of the Pieri rule (14).

3.2. The *outside rim* $\partial\mathcal{C}$ of \mathcal{C} is the set of pairs $(i, j) \in \Delta \setminus \mathcal{C}$ such that $i = 1$ or $(i - 1, j - 1) \in \mathcal{C}$. Let $m \geq 1$ be minimal such that $\lambda_m \leq k$; we call m the *middle* row of λ . Notice that m is the least positive integer such that $(m, m) \notin \mathcal{C}$.

We consider 4-tuples (D, μ, S, h) , where D is a valid set of pairs such that $\mathcal{C} \subset D \subset \mathcal{C} \cup \partial\mathcal{C}$ and D contains no pairs (i, j) with $j > \ell + 1$, μ is an integer sequence of length at most $\ell + 1$, S is a subset of $D \cap \partial\mathcal{C}$, and $h \in [0, \ell + 1]$. We call any such tuple a *valid 4-tuple* of level h . There is an evaluation map ev from the set of all valid 4-tuples to $\mathbb{H}^*(\text{OG}, \mathbb{Z})$, defined by $\text{ev}(D, \mu, S, h) = T(D, \mu)$.

We will represent the set Δ as the positions on or above the main diagonal of a matrix, and the various sets of pairs D as sets of entries in this matrix. In Figure 1 the white dots represent a set of pairs \mathcal{C} and the grey dots are a subset of the outside rim of \mathcal{C} ; we have $m = 12$. The union of the white and grey dots form the set D in a typical valid 4-tuple (D, μ, S, h) .

Definition 3.1. Suppose that the level h of the valid 4-tuple (D, μ, S, h) satisfies $1 \leq h \leq m$. If $h \geq 2$, we let $g = 1 + \max\{j \mid (h - 1, j) \in \mathcal{C}(\lambda)\}$, while if $h = 1$, set $g = \ell + 1$. We define a to be the least positive integer such that $(a, g) \notin \mathcal{C}$, and b the least integer such that $b \geq h$ and $(h, b) \notin \mathcal{C}$.

Figure 2 illustrates a part of the set \mathcal{C} near the level $h \leq m$. In this figure, x denotes the pair $(h, g) \in \partial\mathcal{C}$, while a and b are the row and column numbers from Definition 3.1. In the sequel it will be convenient to set $\lambda_0 = \mu_0 = \infty$.

Definition 3.2. Consider a valid 4-tuple (D, μ, S, h) .

a) Let R denote the set of boxes of $\mu \setminus \lambda$ in columns $k + 1$ and higher which are k -related to any box B of λ in the first k columns, such that the box directly below B is not in μ .

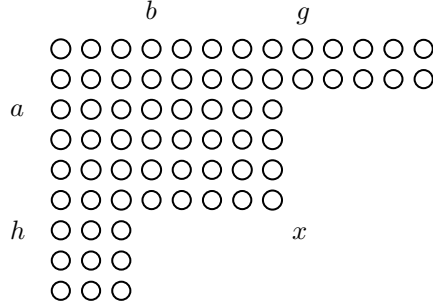


FIGURE 2. The set $\mathcal{C}(\lambda)$ near the pair (h, g)

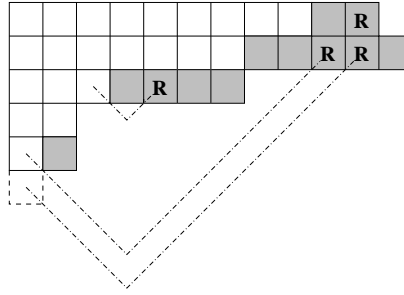


FIGURE 3. The shapes λ and μ , with $\mu \setminus \lambda$ shaded

b) Assume that $(h, h) \in D$ and $\mu_h \geq \lambda_{h-1}$. Let e be the smallest integer greater than k such that $[h, e] \notin \lambda$ and for all j with $e \leq j \leq \lambda_{h-1}$ we have $[h, j] \notin R$. If $[h, \lambda_{h-1}] \in R$ then set $f = g$. Otherwise, let f be the smaller of $\ell + 1$ and the maximum among $h + e - 2k - 1$ and the row numbers of the boxes in the first k columns of μ that are k' -related to box $[h, e]$.

Example 3.1. a) If μ is a k -strict partition and $\lambda \rightarrow \mu$, then \mathbb{A} is the set of boxes of $\mu \setminus \lambda$ in columns $k + 1$ and higher which are not in R .

b) Suppose $k = 3$ and $\lambda = (9, 7, 3, 2, 1, 1)$, so that

$$\mathcal{C}(\lambda) = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}.$$

Consider a valid 4-tuple (D, μ, S, h) with $\mu = (11, 12, 7, 2, 2)$. Figure 3 illustrates λ and μ , with the boxes in $\mu \setminus \lambda$ shaded, and the boxes in R marked. Note that there is one box in $\lambda \setminus \mu$. If $h = 2$ then we obtain $(e, f, g) = (8, 5, 5)$, while if $h = 3$ (and $(3, 3) \in D$) we get $(e, f, g) = (6, 4, 5)$.

The precise value of f will play a crucial role in our proof that the Pieri terms in (14) appear in (15) with the correct multiplicities. The value $h + e - 2k - 1$ can be explained by the observation that box $[h, e]$ would be k' -related to $[h + e - 2k - 1, 0]$ if we allowed boxes in column zero. The maximum appearing in the definition of f ensures that for any 4-tuple (D, μ, S, h) with $(h, h) \in D$ and $\mu_h \geq \lambda_{h-1}$, we have $f \geq h$. Indeed, since $(h, h) \in D \subset \mathcal{C} \cup \partial\mathcal{C}$ we have $h \leq m$, and therefore $\lambda_{h-1} > k$. If $h + e - 2k - 1 < h$ then $k < e < 2k + 1$, and box $[h, e]$ is k' -related to box

$[h, 2k + 1 - e]$, which belongs to μ since $\mu_h > k$. Sharper bounds on the integer f are obtained in §4, Lemma 4.3.

Definition 3.3. Let $(i, j) \in \Delta$ be arbitrary. We define two conditions $W(i, j)$ and X on a valid 4-tuple (D, μ, S, h) as follows.

$$W(i, j) : \mu_i + \mu_j > 2k + j - i.$$

$$X : \mu_h = \mu_{h-1} \text{ or } \mu_h > \lambda_{h-1} \text{ or } (\mu_h = \lambda_{h-1} \text{ and } (h, f) \notin S).$$

We agree that X is false if $h \leq 1$ or $(h, h) \notin D$.

The next definition is motivated by Lemmas 1.1, 1.2, and 1.3.

Definition 3.4. Let D be a valid set of pairs and $h > 1$. We say that the pair $(h-1, h)$ is *D-tame* if (i) $(h-1, h-1) \notin D$ and for all $h' < h$, $(h', h-1) \notin D$ if and only if $(h', h) \notin D$, or (ii) $(h, h) \in D$ and for all $h' > h$, $(h-1, h') \in D$ if and only if $(h, h') \in D$. The pair $(h-1, h)$ is *D-wild* if it is not *D-tame*. An *outer corner* of D is a pair $(i, j) \in \Delta \setminus D$ such that $D \cup (i, j)$ is also a valid set of pairs.

3.3. Initially, we define the set $\Psi = \{(\mathcal{C}, \nu, \emptyset, \ell+1) \mid \nu \in \mathcal{N}(\lambda, p)\}$; thus $\sum_{\psi \in \Psi} \text{ev}(\psi)$ agrees with the right hand side of (15). We then apply an *algorithm* which will change this set by replacing some 4-tuples with one or two new valid 4-tuples. The algorithm applies the *substitution rule* described below to each element (D, μ, S, h) in Ψ of level $h \geq 1$. If the substitution rule results in a REPLACE statement, then the set Ψ is changed accordingly; otherwise the substitution rule results in a STOP statement, in which case the 4-tuple (D, μ, S, h) is left untouched. These substitutions are iterated until no further elements in Ψ can be REPLACED, i.e., until the substitution rule results in a STOP statement when applied to any 4-tuple in Ψ with $h \geq 1$.

Substitution Rule

Suppose that $h \geq 1$. Assume first that $(h, h) \notin D$. If

(i) there is an outer corner (i, h) of D such that $W(i, h)$ holds

then REPLACE (D, μ, S, h) with

$$(D \cup (i, h), \mu, S, h) \text{ and } (D \cup (i, h), R_{ih}\mu, S \cup (i, h), h).$$

Otherwise, if

(ii) $(h-1, h)$ is *D-tame* and $\mu_h > \lambda_{h-1}$,

then STOP.

Assume now that $(h, h) \in D$. If

(iii) there is an outer corner (h, j) of D with $j \leq \ell + 1$ such that $W(h, j)$ holds,

then REPLACE (D, μ, S, h) with

$$\begin{cases} (D \cup (h, j), \mu, S, h) \text{ and } (D \cup (h, j), R_{hj}\mu, S \cup (h, j), h) & \text{if } \mu_{j-1} \neq \mu_j - 1, \\ (D \cup (h, j), R_{hj}\mu, S \cup (h, j), h) & \text{if } \mu_{j-1} = \mu_j - 1. \end{cases}$$

Otherwise, if

(iv) $(h, g) \notin D$, and $W(h, g)$ or X holds,

then choose i minimal such that $(i, g) \notin D$ and REPLACE (D, μ, S, h) with

$$(D \cup (i, g), \mu, S, h) \quad \text{and} \quad (D \cup (i, g), R_{ig}\mu, S \cup (i, g), h).$$

Otherwise, if

$$\mathbf{(v)} \quad (h, g) \in D \text{ and X holds,}$$

then STOP.

If none of the above conditions hold, REPLACE (D, μ, S, h) with $(D, \mu, S, h-1)$.

We will prove in Corollary 4.1 below that all 4-tuples added to Ψ by the algorithm are valid 4-tuples. It is therefore clear that the algorithm will terminate after a finite number of steps. Whenever a 4-tuple $\psi = (D, \mu, S, h)$ is replaced by one or two 4-tuples ψ_i , we refer to ψ as the *parent* term and the ψ_i are its *children*. Notice that we can recover the initial term $\psi_0 = (\mathcal{C}, \nu, \emptyset, \ell + 1)$ that gave rise to ψ by the equation $\nu = \prod_{(i,j) \in S} L_{ij}\mu$. Here L_{ij} denotes the lowering operator which is the inverse of R_{ij} . Furthermore, the sequence of terms leading from ψ_0 to ψ is uniquely determined by ψ .

If a term $\psi \in \Psi$ is REPLACED by two terms ψ_1 and ψ_2 , we deduce from Lemma 1.3 that $\text{ev}(\psi) = \text{ev}(\psi_1) + \text{ev}(\psi_2)$. Moreover, if condition **(iii)** holds for $\psi = (D, \mu, S, h)$ and this term is REPLACED by the single term $\psi' = (D \cup (h, j), R_{hj}\mu, S \cup (h, j), h)$, then Lemmas 1.1 and 1.3 imply that $\text{ev}(\psi) = \text{ev}(\psi')$. Indeed, (h, j) is an outer corner of D , and according to Corollary 4.2 below, the pair $(j-1, j)$ is $(D \cup (h, j))$ -tame. We also have $\mu_{j-1} = \mu_j - 1$, and hence $\text{ev}(D \cup (h, j), \mu, S, h) = 0$, by Lemma 1.1.

When the algorithm terminates, let Ψ_0 denote the set of 4-tuples $(D, \mu, S, 0)$ in Ψ and let $\Psi_1 = \Psi \setminus \Psi_0$. It follows that

$$\sum_{\nu \in N} T(\mathcal{C}, \nu) = \sum_{\psi \in \Psi_0} \text{ev}(\psi) + \sum_{\psi \in \Psi_1} \text{ev}(\psi).$$

In the next section, we will prove the following two claims.

Claim 1. For each term $\psi = (D, \mu, S, 0)$ in Ψ_0 with $\mu_{\ell+1} \geq 0$, μ is a k -strict partition with $\lambda \rightarrow \mu$ and $\text{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$. Furthermore, for each such partition μ , there are exactly $2^{\text{gr}(\lambda, \mu)}$ such 4-tuples ψ , in accordance with the Pieri rule.

Claim 2. There exists an involution $\iota : \Psi_1 \rightarrow \Psi_1$ of the form $\iota(D, \mu, S, h) = (D, \mu', S', h)$ such that $\text{ev}(\psi) + \text{ev}(\iota(\psi)) = 0$, for every $\psi \in \Psi_1$.

We remark that the terms $\psi \in \Psi_0$ with $\mu_{\ell+1} < 0$ evaluate to zero trivially, by Definition 1.1. The two claims therefore suffice to prove the Pieri rule (14).

Definition 3.5. Let **(x)** be one of the conditions **(i)**–**(v)** of the Substitution Rule. We say that a 4-tuple $\psi = (D, \mu, S, h)$ *meets* condition **(x)** if ψ occurs in the algorithm, reaches condition **(x)** in the Substitution Rule, and condition **(x)** is satisfied. We say that a term ψ *survives the algorithm* if at least one of its successors lies in Ψ_0 .

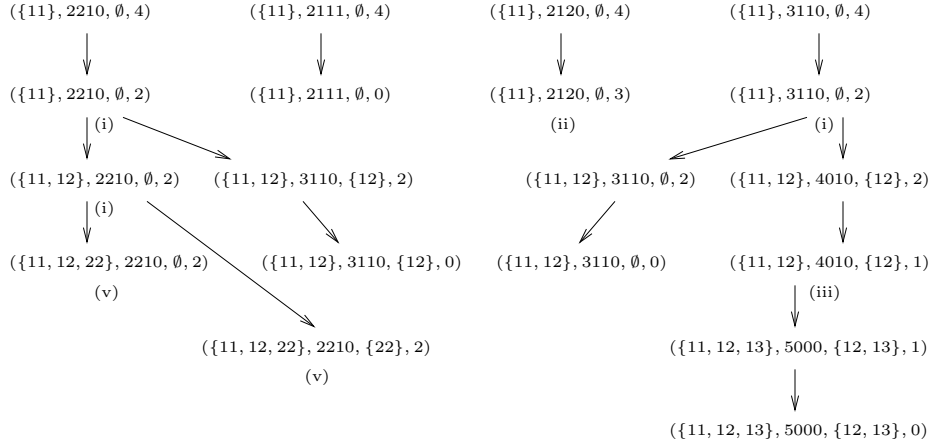
For each initial 4-tuple $\psi_0 = (\mathcal{C}, \nu, \emptyset, \ell+1)$ of the sum (15), the algorithm produces a tree of terms with root node given by ψ_0 . If the Substitution Rule REPLACES a term ψ by one or two other 4-tuples ψ_i , we have a branch in the tree from ψ to the ψ_i . The leaves of the tree are exactly the 4-tuples with $h = 0$ or where the Substitution Rule STOPS. The fate of all the terms of the sum (15) is encoded by

the collection of all the trees with root nodes $(\mathcal{C}, \nu, \emptyset, \ell + 1)$ for $\nu \in \mathcal{N}(\lambda, p)$. This collection will be called the *substitution forest*; the sum of the cohomology classes represented by the roots of the substitution forest is equal to the sum of classes given by the leaves.

Example 3.2. We discuss an example of the substitution forest in detail. Consider the Grassmannian $\text{OG}(n - 1, 2n + 1)$ for $n \geq 5$, and the Pieri product

$$c_1 \cdot \tau_{2,1,1} = \tau_{2,1,1,1} + 2\tau_{3,1,1} + \tau_5.$$

For simplicity, we will omit the commas in our notation for compositions and pairs. Thus $\lambda = 211$, $k = p = 1$, and we have $\mathcal{C}(\lambda) = \{11\}$ and $\mathcal{N}(\lambda, p) = \{2111, 2120, 2210, 3110\}$. The substitution forest is pictured below, except we have omitted those nodes (D, μ, S, h) which have $(D, \mu, S, h+1)$ as parent and $(D, \mu, S, h-1)$ as child.



Observe that the root $(\{11\}, 2120, \emptyset, 4)$ is the only initial 4-tuple that does not survive the algorithm. We have $\Psi_0 = \{ (\{11\}, 2111, \emptyset, 0), (\{11, 12\}, 3110, \{12\}, 0), (\{11, 12\}, 3110, \emptyset, 0), (\{11, 12, 13\}, 5000, \{12, 13\}, 0) \}$, which corresponds exactly to the terms in the Pieri product $c_1 \cdot \tau_{211}$. Furthermore, each 4-tuple in the set $\Psi_1 = \{ (\{11, 12, 22\}, 2210, \emptyset, 2), (\{11, 12, 22\}, 2210, \{22\}, 2), (\{11\}, 2120, \emptyset, 3) \}$ evaluates to zero in the cohomology ring of OG.

4. PROOF OF THEOREM 1

4.1. Recall the fixed choices of p , λ , ℓ , \mathcal{C} , and m from §3.1. In this section, we furthermore let $\psi = (D, \mu, S, h)$ denote a 4-tuple which occurs at some step in the algorithm, i.e., a node of the substitution forest. The symbols D , μ , S , h will refer to components of the 4-tuple ψ . We will occasionally work with more than one valid 4-tuple. If (D', μ', S', h') is an additional 4-tuple, then the sets and values that Definition 3.2 associates to this 4-tuple will be called R' , e' , f' , and g' .

The algorithm is in two phases. A 4-tuple ψ is in Phase 1 of the algorithm if $(h, h) \notin D$, and in Phase 2 if $(h, h) \in D$. The level h is always used to index an entry of the integer sequence μ in ψ ; it begins at $h = \ell + 1$ and decreases as the term proceeds through the algorithm. In Phase 1 we have $h \geq m$, while in Phase 2, $h \leq m$. Throughout the algorithm we have $i \leq m \leq j$ for each $(i, j) \in S$, that is,

μ is obtained from the initial composition ν by removing boxes from rows weakly below the middle row of λ and adding them to rows weakly above the middle row. Observe that the terms with $\mu_{\ell+1} < 0$ all evaluate to zero.

In addition, h corresponds to a row or column number of the set D ; in Phase 1, h denotes a column number of D , while in Phase 2, h is a row number of D . The set D is initially equal to \mathcal{C} and increases after each REPLACE statement by adding pairs which lie in the outside rim $\partial\mathcal{C}$ (see Corollary 4.1). In Phase 1, as h decreases from $\ell + 1$ down to m , the set D increases by adding pairs along vertical columns of $\partial\mathcal{C}$, going from right to left and top (row 1) to bottom (row m). In Phase 2, the set D increases mainly by adding pairs along horizontal rows of $\partial\mathcal{C}$, proceeding from left to right and bottom to top. It follows from Lemma 4.6 below that this variation of the set D suffices to produce any 4-tuple $(D, \mu, S, 0) \in \Psi_0$ which will contribute to the Pieri sum (14). Lemma 4.6 asserts that any term ψ that meets (iv) of the Substitution Rule does not survive the algorithm, and the set D for its successors will increase by adding pairs going down column g of $\partial\mathcal{C}$.

Our proof of Theorem 1 occupies the remainder of this section. We obtain some preliminary Lemmas and Corollaries in §4.2–§4.5. The proof of Claim 1 is then given in §4.6, while Claim 2 is justified in §4.7.

4.2. We begin by studying what happens to a 4-tuple $\psi = (D, \mu, S, h)$ in Phase 1 of the algorithm.

Lemma 4.1. a) *Suppose that $h \geq 1$, $(h, h) \notin D$, and $\mu_h > \lambda_{h-1}$. If D has an outer corner in column h , then ψ meets (i). If D does not have an outer corner in column h and $h > m$, then ψ meets (ii).*

b) *Suppose that $h \leq m < j$, or $h < m$ and (h, j) is an outer corner of D . Then $\mu_j \leq \lambda_{j-1}$.*

Proof. For part (a), suppose first that (i, h) is an outer corner of D . If $i < h$, then

$$\mu_h + \mu_i > \lambda_{h-1} + \lambda_i > 2k + (h-1) - i,$$

as $(i, h-1) \in \mathcal{C}$. If $i = h$, then $h = m$, so $\mu_h > \lambda_{m-1} > k$. In both cases $W(i, h)$ holds, therefore ψ meets (i). If D does not have an outer corner in column h and $h > m$, then $(h-1, h)$ is D -tame, hence ψ meets (ii).

If $h \leq m < j$ then part (a) implies that $\mu_j \leq \lambda_{j-1}$. Suppose that $h < m = j$ and (h, m) is an outer corner of D . Assume that $\mu_m > \lambda_{m-1}$ and let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ with $h' = m$. Then $(m, m) \notin D'$, $\mu'_m \geq \mu_m > \lambda_{m-1}$, and D' has an outer corner in column m . We deduce from part (a) that ψ' meets (i). But then ψ' is not the most recent predecessor with $h' = m$. This contradiction completes the proof of (b). \square

We will see later that the 4-tuples ψ with $(m, m) \in D$ and $\mu_m > \lambda_{m-1}$ will not survive the algorithm.

4.3. A typical part of the set \mathcal{C} at level h which will be important in Phase 2 of the algorithm is illustrated in Figure 2.

Lemma 4.2. *A 4-tuple ψ of level h with $1 \leq h \leq m$ satisfies $\lambda_{h-1} - \lambda_h \geq g - b + 1$.*

Proof. The desired inequality is clear if $b = g$, as λ is k -strict and $h \leq m$. If $b < g$, then since $(h-1, g-1) \in \mathcal{C}$ and $(h, b) \notin \mathcal{C}$, we have

$$\lambda_{h-1} + \lambda_{g-1} \geq 2k + g - h + 1 \quad \text{and} \quad 2k + b - h \geq \lambda_h + \lambda_b,$$

respectively. Adding these two inequalities gives

$$\lambda_{h-1} - \lambda_h \geq g - b + 1 + (\lambda_b - \lambda_{g-1}) \geq g - b + 1. \quad \square$$

Lemma 4.3. *If the 4-tuple ψ satisfies $(h, h) \in D$ and $\mu_h \geq \lambda_{h-1}$, then $b \leq f \leq g$.*

Proof. We first show that $f \leq g$; if this is false then we must have $h > 1$ and $[h, \lambda_{h-1}] \notin R$, and hence that $e \leq \lambda_{h-1}$. Since $(h-1, g) \notin \mathcal{C}$ we have

$$(16) \quad \lambda_{h-1} + \lambda_g \leq 2k + g - (h-1).$$

Since this implies that $h + e - 2k - 1 \leq g$, the box $[h, e]$ must be k' -related to some box $[j, c]$ with $c \leq \min(k, \mu_j)$ and $j > g$. Since $c \leq \mu_j \leq \lambda_{j-1} \leq \lambda_g$ by Lemma 4.1, we obtain

$$\lambda_{h-1} + \lambda_g \geq e + c = 2k + j - (h-1),$$

contradicting (16).

Furthermore, we claim that we have $f \geq b$. Note that $(h, h) \in D$ implies that $\mu_h > k$, and therefore that $f \geq h$, as discussed in §3.2. Hence we may assume that $b > h$, and also that $f \leq \ell$ and $[h, \lambda_{h-1}] \notin R$. It follows that $\lambda_h > k$, since otherwise $(h, h) \in D \setminus \mathcal{C}$ and therefore $h = m = b$.

Set $d = 2k + b - h - \lambda_h - \mu_b$. Using the inequality $e \geq \lambda_h + 1$, we obtain

$$f \geq h + e - 2k - 1 \geq b - (\mu_b + d).$$

We may therefore assume that $\mu_b + d > 0$. Since $(h, b-1) \in \mathcal{C}$ we have $\lambda_h + \lambda_{b-1} \geq 2k + b - h$ and hence $\mu_b + d \leq \lambda_{b-1}$. We claim that we also have $\mu_b + d \leq k$. Indeed, if $b > m$, then $\lambda_{b-1} \leq \lambda_m \leq k$, while if $b = m$, then $\lambda_h \geq k + b - h$ since λ is k -strict, and therefore $\mu_b + d \leq k$.

Define $c = 2k + 1 + b - h - e = \mu_b + d + \lambda_h + 1 - e$. Then $c \leq \mu_b + d \leq k$. If $c \leq 0$ then $f \geq h + e - 2k - 1 \geq b$. Moreover, if $1 \leq c \leq \mu_b$, then since box $[h, e]$ is k' -related to box $[b, c]$ in the first k columns of μ , we also obtain that $f \geq b$. We may therefore assume that $c > \mu_b$, or equivalently, that $e < \lambda_h + 1 + d$. In particular we have $d > 0$.

For $0 \leq t < d + \min(\mu_b, 0)$, the box $B_t = [b-1, \mu_b + d - t]$ belongs to the first k columns of λ and has no box of μ directly below it. Since B_t is k -related to box $[h, \lambda_h + 1 + t]$ and $[h, \lambda_{h-1}] \notin R$, it follows that $\mu_h \geq \lambda_{h-1} > \lambda_h + d + \min(\mu_b, 0)$, and all the boxes $[h, \lambda_h + 1 + t]$ for $0 \leq t < d + \min(\mu_b, 0)$ belong to R . This implies that $e \geq \lambda_h + 1 + d + \min(\mu_b, 0)$, so we must have $\mu_b < 0$. We finally obtain $f \geq h + (\lambda_h + 1 + d + \mu_b) - 2k - 1 = b$, as required. \square

4.4. We make some important observations concerning condition **(iv)** of the Substitution Rule.

Lemma 4.4. *If the 4-tuple ψ meets **(iv)**, then $h < g$ and $(h, g-1) \in D$.*

Proof. Since ψ meets **(iv)**, we have $(h, h) \in D$ and $(h, g) \notin D$, and therefore $h < g$. Assume that $(h, g-1) \notin D$ and choose d minimal such that $(h, d) \notin D$. Then $d < g$ and $(h-1, d) \in \mathcal{C}$. In particular, (h, d) is an outer corner of D , and since ψ does not meet **(iii)** we deduce that $W(h, d)$ fails. We will show that X fails and $W(h, g)$ fails. Since \mathcal{C} and D have equally many boxes in column d it follows that $\lambda_d \leq \mu_d$, so $\mu_h + \mu_d \leq 2k + d - h < \lambda_{h-1} + \lambda_d \leq \lambda_{h-1} + \mu_d$. This shows that $\mu_h < \lambda_{h-1} \leq \mu_{h-1}$, so X fails. Notice that $g > m$; otherwise we have $h < m = g$, contradicting $(h, g-1) \notin D$. By Lemma 4.1(b) we have $\mu_g \leq \lambda_{g-1}$, and $\lambda_{g-1} \leq \lambda_d \leq \mu_d$, hence $\mu_h + \mu_g \leq \mu_h + \mu_d \leq 2k + d - h \leq 2k + g - h$ and $W(h, g)$ fails. \square

Corollary 4.1. *All 4-tuples (D, μ, S, h) that are added to Ψ during the algorithm are valid 4-tuples.*

Proof. Lemma 4.4 ensures that D remains a valid set of pairs when a pair is added in rule (iv). The same property is clear for rules (i) and (iii), so it suffices to check that all pairs added to D lie in $\partial\mathcal{C}$. Notice that the level h can only decrease throughout the process, and all applications of rule (i) happen before applications of rules (iii) and (iv). Since the addition of outer corners in (i) increases D by going down vertical columns of $\partial\mathcal{C}$, from right to left, it follows that any pair $(i, j) \neq (h, g)$ added in either (i) or (iv) satisfies that $(i, j - 1) \in \mathcal{C}$ or $i = j = m$. Since rule (iii) adds pairs (h, j) in horizontal lines, from bottom to top, this ensures that only pairs from $\partial\mathcal{C}$ can be added. \square

Lemma 4.5. *If the 4-tuple ψ is such that $(h, h) \in D$, $\mu_h \geq \lambda_{h-1}$, and $W(h, g)$ fails, then $f = g$. Moreover, if ψ meets (iv) then X holds for all successors of ψ .*

Proof. The equality $f = g$ is clear from Lemma 4.3 unless $b < g$, which implies that $\lambda_{g-1} \leq \lambda_m \leq k$. Set $c = 2k + 1 + g - h - \lambda_{h-1}$. We then have $\mu_g < c \leq \lambda_{g-1}$, the second inequality because $(h - 1, g - 1) \in \mathcal{C}$. Since box $[g - 1, c]$ is k -related to $[h, \lambda_{h-1}]$, we conclude that the latter box is in R , hence $f = g$, as required. If ψ meets (iv), then $(h, g) \notin D$. If X fails for a successor $\psi' = (D', \mu', S', h)$, then we must have $\lambda_{h-1} = \mu_h = \mu'_h$. The first statement of the lemma now shows that $f' = g$, so $(h, f') = (h, g) \notin S'$ and ψ' satisfies X anyway. \square

Lemma 4.6. *Suppose that ψ meets (iv), and let (a, g) be the outer corner of \mathcal{C} in column g . Then ψ does not survive the algorithm, and all of its successors $\psi' = (D', \mu', S', h')$ in Ψ_1 satisfy $h' > a$.*

Proof. Observe first that if $\mu_{i-1} \leq \mu_i$ for some integer i with $a < i < h$, then any successor $\psi' = (D', \mu', S', h')$ of ψ with $h' = i$ satisfies condition X. Otherwise, $\mu'_i < \mu'_{i-1}$ and $\mu'_i \leq \lambda_{i-1}$, and since $\mu_i \leq \mu'_i$ and $\lambda_{i-1} \leq \mu_{i-1}$, this implies that $\mu_i = \mu'_i = \lambda_{i-1} = \mu_{i-1}$. We deduce that $(h', f') = (i, g) \notin S'$, and so X holds for ψ' anyway. It follows that ψ will not survive the algorithm, as desired.

We may therefore assume that $\mu_a > \mu_{a+1} > \dots > \mu_{h-1}$. If $W(h - 1, g)$ holds, then since $W(i, g)$ implies $W(i - 1, g)$ whenever $\mu_{i-1} > \mu_i$, it follows that $W(i, g)$ is true for $a \leq i < h$, and moreover, these weight conditions are true for all predecessors of ψ . In particular, all pairs (i, g) for $a \leq i < h$ were added to D by (i) during Phase 1 of the algorithm, and so (h, g) is an outer corner of D by Lemma 4.4. Since ψ does not meet (iii), we deduce that $W(h, g)$ fails. Therefore X holds, hence $\mu_h \geq \lambda_{h-1}$, and the result follows by applying Lemma 4.5 to ψ .

Suppose next that $W(h - 1, g)$ fails. In this case we have $\lambda_{h-1} \leq \mu_h$; this is clear if X holds for ψ , and if $W(h, g)$ holds then it follows because $\mu_{h-1} \leq \mu_h$. It will suffice to show that X holds for every successor $\psi' = (D', \mu', S', h)$ of ψ . If X fails for ψ' , then $\mu'_h < \mu'_{h-1}$ and $\mu'_h \leq \lambda_{h-1}$, and we obtain $\lambda_{h-1} = \mu_h = \mu'_h$. It follows that $(h, g) \notin S'$, hence it is enough to prove that $f' = g$. Now $W(h - 1, g)$ fails for ψ' and $\mu'_h < \mu'_{h-1}$, therefore $W(h, g)$ also fails for ψ' . We conclude by applying Lemma 4.5 to ψ' . \square

Lemma 4.7. *If ψ satisfies condition X, then it does not survive the algorithm.*

Proof. The assumptions on ψ imply that it meets (iii), (iv), or (v). If ψ meets (iv) then it does not survive the algorithm by Lemma 4.6, while if ψ meets (v) then

it lies in Ψ_1 . Finally, if ψ meets **(iii)** then condition X also holds for its children, so the lemma follows by decreasing induction on the number of pairs in D . \square

4.5.

Lemma 4.8. *Let $\psi = (D, \mu, S, h)$ and j be a positive integer.*

- a) *If $h \geq 1$ and $(h, j) \notin \mathcal{C}$ and $(h+1, j) \in D$, then we have $\mu_j \geq \lambda_j$.*
- b) *If $h \leq 1$ or $(h-1, j) \in \mathcal{C}$, then we have $\mu_j \geq \lambda_j - 1$. Moreover, if $\mu_j = \lambda_j - 1$, then $D \setminus \mathcal{C}$ contains exactly one pair in column j , and this pair is also in S .*

Proof. Suppose that $\mu_j < \lambda_j$ and choose $i > h$ maximal such that $(i, j) \in D$. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ with $(i, j) \notin D'$. Then ψ' meets rule **(i)**, **(iii)**, or **(iv)**, which adds the pair (i, j) to D' . We have $\mu'_j \leq \lambda_j$, and if $\mu'_j = \lambda_j$ then the following holds: Any successor $\bar{\psi} = (\bar{D}, \bar{\mu}, \bar{S}, \bar{h})$ of ψ' , which is also a predecessor of ψ , satisfies that $(i, j) \in \bar{S}$, $i < j$, $\bar{\mu}_i > \mu'_i$, and $\bar{\mu}_j < \mu'_j$.

If ψ' meets **(i)**, then $h' = j$. We also have $\mu'_i \leq \lambda_{i-1}$, since otherwise condition X holds for any successor $\bar{\psi}$ with $\bar{h} = i$, and ψ could not be a successor of ψ' . Since $(i-1, j) \notin \mathcal{C}$ we have $\mu'_i + \mu'_j \leq \lambda_{i-1} + \lambda_j \leq 2k + j - i + 1$. As $W(i, j)$ holds for ψ' , it follows that $\mu'_i = \lambda_{i-1}$ and $\mu'_j = \lambda_j$. We deduce that ψ is a successor of a child $\bar{\psi}$ of ψ' with $\bar{\mu}_i > \lambda_{i-1}$, which is impossible.

Therefore ψ' meets **(iii)** with $h' = i$, or it meets **(iv)** with $h' \geq i$; in either case we have $g' = j$. Let $\bar{\psi}$ be a simultaneous successor of ψ' and predecessor of ψ , with $\bar{h} = h'$. It is enough to show that $\bar{\psi}$ satisfies condition X. For this, we may assume that $\mu'_{h'} \leq \lambda_{h'-1}$. We may also assume that $W(h', j)$ holds for ψ' , since otherwise ψ' meets **(iv)**, X holds for ψ' , and Lemma 4.5 implies that X holds for $\bar{\psi}$ as well. Since $(h'-1, j) \notin \mathcal{C}$ we obtain $\mu'_{h'} + \mu'_j \leq \lambda_{h'-1} + \lambda_j \leq 2k + j - h' + 1$. It follows that $\mu'_{h'} = \lambda_{h'-1}$ and $\mu'_j = \lambda_j$, which then implies that $\bar{\mu}_i > \mu'_i$ and $\bar{\mu}_j < \mu'_j$. We can therefore assume that $h' > i$, since otherwise $\bar{\mu}_{h'} > \lambda_{h'-1}$ and X holds for $\bar{\psi}$. We then have $\bar{\mu}_{h'} = \mu'_{h'}$, so $W(h', j)$ fails for $\bar{\psi}$. Now Lemma 4.5 shows that $\bar{f} = \bar{g} = j$, so $(h', \bar{f}) \notin \bar{S}$ and X holds for $\bar{\psi}$. This completes the proof of part (a).

If $\mu_j \leq \lambda_j - 2$, then $D \setminus \mathcal{C}$ contains at least two pairs in column j , say $(r+1, j)$ and (r, j) , and by the assumptions in (b) we can choose r with $r \geq h$. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ with $h' = r$. Part (a) applied to ψ' implies that $\mu'_j \geq \lambda_j$, a contradiction since $\mu'_j = \mu_j$. \square

Corollary 4.2. *If the 4-tuple ψ meets **(iii)** and $\mu_{j-1} = \mu_j - 1$, then the pair $(j-1, j)$ is $(D \cup (h, j))$ -tame.*

Proof. Lemma 4.1(b) implies that $\mu_j \leq \lambda_{j-1}$. Since $\mu_{j-1} < \mu_j \leq \lambda_{j-1}$ and $(h-1, j-1) \in \mathcal{C}$, we deduce from Lemma 4.8(b) that $\mu_{j-1} = \lambda_{j-1} - 1$ and D has exactly one more pair than \mathcal{C} in column $j-1$. It is enough to show that $(h, j-1) \notin \mathcal{C}$, since this implies that $(h, j-1)$ is the bottom pair of D in column $j-1$. (Notice also that $\mu_{j-1} < \lambda_{j-1}$ implies that $(j-2, j-1) \notin \mathcal{C}$, and therefore $(j-1, j-1) \notin D$, as required by Definition 3.4(i).)

Suppose $(h, j-1) \in \mathcal{C}$. Then $j = g$, since otherwise (h, j) would be an outer corner of \mathcal{C} , and this pair would have been added to D by **(i)** during Phase 1 of the algorithm. Let $(r, g-1)$ be the unique pair in column $g-1$ of $D \setminus \mathcal{C}$. Lemma 4.6 implies that this pair was added by **(i)** or **(iii)**, so $W(r, g-1)$ holds for ψ , and we obtain $\mu_r + \mu_g = \mu_r + \mu_{g-1} + 1 > 2k + g - r$. Let $\psi' = (D', \mu', S', h')$ be the most

recent predecessor of ψ with $h' = r$. Then $(h', g - 1) \in D'$. Since $\mu'_r = \mu_r$ and $\mu'_g \geq \mu_g$, $W(r, g)$ holds for ψ' . But then ψ' meets **(iv)** and is not the most recent predecessor. This contradiction shows that $(h, j - 1) \notin \mathcal{C}$, as required. \square

Lemma 4.9. *Assume that $j > m$. If $h = 0$ or $(h, j) \in D$, then $\mu_j \leq \mu_{j-1}$.*

Proof. We have $\mu_j \leq \lambda_{j-1}$ by Lemma 4.1(b), and Lemma 4.8(b) implies that $\lambda_{j-1} \leq \mu_{j-1} + 1$. Assume that $\mu_j > \mu_{j-1}$. Then $\mu_j = \lambda_{j-1} = \mu_{j-1} + 1$, and $D \setminus \mathcal{C}$ contains a unique pair $(r, j - 1)$ in column $j - 1$, with $r \geq h$. Lemma 4.6 now implies that $(r, j - 1)$ was added to D by **(i)** or **(iii)**, so ψ satisfies $W(r, j - 1)$. Since $\mu_j > \mu_{j-1}$, ψ also satisfies $W(r, j)$.

Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ for which $h' = r$ and $(r, j) \notin D'$. The assumptions of the lemma then imply that $\psi' \neq \psi$. Furthermore, since ψ' satisfies $W(r, j)$, it meets **(iii)** or **(iv)**. The choice of ψ' then implies that (r, j) is an outer corner of D' , so in fact ψ' meets **(iii)**.

Lemma 4.1(b) shows that $\mu'_j \leq \lambda_{j-1} = \mu_j$, so we must have $\mu'_j = \mu_j = \mu_{j-1} + 1 = \mu'_{j-1} + 1$. We deduce from the statement of rule **(iii)** that $(r, j) \in S$, which in turn implies that $\mu'_j > \mu_j$. This contradiction finishes the proof. \square

Lemma 4.10. *Suppose that $h < m$. Then $\mu_j \leq \lambda_{j-1}$ and $\mu_j < \mu_{j-1}$ for $h < j \leq m$.*

Proof. If the statement is false, then choose $r > h$ minimal such that $\mu_r > \lambda_{r-1}$ or $\mu_r \geq \mu_{r-1}$, and let $\psi' = (D', \mu', S', r)$ be the most recent predecessor of ψ of level r . Since $\mu'_r > \lambda_{r-1}$ or $\mu'_r \geq \mu'_{r-1}$ we must have $(r, r) \notin D'$; otherwise ψ' satisfies condition X and meets **(iii)**, **(iv)**, or **(v)**. We deduce that $r = m$. Since ψ' does not meet **(i)**, $W(m, m)$ fails for ψ' , so $\mu'_m \leq k < \lambda_{m-1} \leq \mu'_{m-1}$, a contradiction. \square

Lemma 4.11. *Suppose that $(h, h) \in D$. If $\mu_h = \lambda_{h-1}$ and $[h, \lambda_{h-1}] \in R$ then ψ does not survive the algorithm.*

Proof. If $(h, g) \notin D$, then since $f = g$ we see that ψ satisfies X, and the result follows from Lemma 4.7. Assume next that $(h, g) \in D$. The condition $[h, \lambda_{h-1}] \in R$ implies that there is a box $B = [r, c]$ of λ in the first k columns such that $\lambda_{h-1} = 2k + 2 + r - h - c$ and the box below B is not in μ . We claim that $r < g$. Indeed, if $r \geq g$ then since $(h - 1, g) \notin \mathcal{C}$ we have

$$\lambda_{h-1} \leq 2k + 1 + g - h - \lambda_g \leq 2k + 1 + r - h - \lambda_r \leq 2k + 1 + r - h - c$$

which is a contradiction. Since $\mu_{r+1} < c \leq k$, we have $r + 1 \geq m$, so Lemma 4.9 implies that $\mu_g \leq \mu_{r+1}$. We deduce that

$$\lambda_{h-1} + \mu_g \leq 2k + 2 + r - h - c + \mu_{r+1} \leq 2k + 1 + g - h - c + \mu_{r+1} \leq 2k + g - h,$$

and hence $W(h, g)$ fails. Since $\mu_h > \mu_{h+1} > \dots > \mu_m$ by Lemma 4.10, $W(r, g)$ fails for $m \leq r \leq h$. If $i \geq h$ is maximal such that $(i, g) \in D$, it follows that the pair (i, g) was added to D when a predecessor of ψ met **(iv)**. We conclude from Lemma 4.6 that ψ does not survive the algorithm. \square

4.6. In this section we will study the 4-tuples $\psi \in \Psi_0$ with $\mu_{\ell+1} \geq 0$.

Proposition 4.1. *Suppose that $\psi = (D, \mu, S, 0)$ and $\mu_{\ell+1} \geq 0$. Then μ is a k -strict partition with $|\mu| = |\lambda| + p$, satisfying $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for every $j \geq 1$, and $\lambda_j \leq \mu_j$ when $\lambda_j > k$. The set D consists of the pairs $(i, j) \in \Delta$ such that $j \leq \ell + 1$ and $W(i, j)$ holds. In particular, $\mathcal{C}(\mu)$ consists of the first $\ell(\mu)$ columns of D .*

Proof. By Lemma 4.10 we have $\mu_j \leq \lambda_{j-1}$ and $\mu_j < \mu_{j-1}$ for $1 \leq j \leq m$, and Lemmas 4.1(b) and 4.9 show that $\mu_j \leq \min(\lambda_{j-1}, \mu_{j-1})$ for $j > m$. We deduce that μ is a k -strict partition. Lemma 4.8(b) implies that $\lambda_j - 1 \leq \mu_j$ for every j . Clearly $\lambda_j \leq \mu_j$ when $\lambda_j > k$, and $|\mu| = |\lambda| + p$.

We claim that for any pair $(i, j) \in \Delta$ with $j \leq \ell + 1$, we have $(i, j) \in D$ if and only if ψ satisfies $W(i, j)$. By descending induction on $i + j$, we may assume that $(i + 1, j) \notin D$ and $(i, j + 1) \notin D$. Suppose that $(i, j) \in D$. If $(i, j) \in \mathcal{C}$ then $\mu_i + \mu_j \geq \lambda_i + \lambda_j > 2k + j - i$. Otherwise Lemma 4.6 shows that (i, j) was added by **(i)** or **(iii)**, and $W(i, j)$ holds since μ_i and μ_j did not change since this event. Conversely, if $W(i, j)$ holds and $(i, j) \notin D$, then the most recent predecessor of ψ of level i would meet **(i)** or **(iii)**, which is impossible. It follows that $\mathcal{C}(\mu)$ agrees with the first $\ell(\mu)$ columns of D . \square

Lemma 4.12. *Suppose that $\psi = (D, \mu, S, 0)$ and $i \leq m$ satisfies $\mu_i = \lambda_{i-1}$. Let f, g , and R be computed as in Definition 3.2 for the 4-tuple (D, μ, S, i) . If $\psi' = (D', \mu', S', i)$ is the most recent predecessor of ψ of level i , then R' and R are identical in row i , $e' = e$, $f' = f$, and $g' = g$.*

Proof. The equality $g' = g$ is clear. Suppose that a box $[i, d]$ with $k < d \leq \lambda_{i-1}$ is k -related to a box $[j, c]$ of λ in the first k columns. Then we have

$$2k + 2 + j - i = c + d \leq \lambda_j + \lambda_{i-1} \leq \lambda_j + 2k + g - (i - 1) - \lambda_g,$$

therefore $j - g < \lambda_j - \lambda_g$, and hence $j < g$. Notice that any pair $(r, c) \in D \setminus D'$ satisfies $c > g$, therefore $\mu'_j = \mu_j$ for $m \leq j \leq g$. Since we also have $\mu'_i = \mu_i$, it follows that R' agrees with R in row i , and $e' = e$. To see that $f' = f$, we can assume that $[i, \lambda_{i-1}] \notin R$ and $i + e - 2k - 1 < \ell + 1$. If $f' > i + e - 2k - 1$ then box $[i, e]$ is k' -related to some box $[f', c]$ of μ' . Since $m \leq f' \leq g$ by Lemma 4.3, we then have $[f', c] \in \mu$. Finally, since no boxes of μ' in rows $f' + 1$ and higher are k' -related to $[i, e]$ and $\mu_j \leq \mu'_j$ for $j \geq m$, we conclude that $f = f'$. \square

Lemma 4.13. *Assume that $\psi = (D, \mu, S, 0)$ and j are such that $\mu_j = \lambda_j - 1$, and let (i, j) be the unique pair in column j of $D \setminus \mathcal{C}$. Then the removed box $[j, \lambda_j]$ and the above box $[j - 1, \lambda_j]$ are k -related to the boxes $[i, c]$ and $[i, c - 1]$, respectively, where $c = 2k + 2 + j - i - \lambda_j$, and these latter boxes belong to R .*

Proof. Let $\psi' = (D', \mu', S', h')$ be the most recent predecessor of ψ for which $(i, j) \notin D'$. Since $\mu'_j = \lambda_j$ and ψ' satisfies $W(i, j)$, we obtain $\mu_i \geq \mu'_i + 1 \geq c$, and since $(i, j) \notin \mathcal{C}$ we similarly have $\lambda_i \leq c - 2$. \square

Proposition 4.2. *If $\psi = (D, \mu, S, 0)$ and $\mu_{\ell+1} \geq 0$, then we have $\lambda \rightarrow \mu$.*

Proof. By Proposition 4.1, it suffices to check that conditions (1) and (2) of §2.2 are true. If $\psi' = (D', \mu', S', i)$ is the most recent predecessor of ψ of level $i \leq m$, then $[i, \lambda_{i-1}] \notin R'$, by Lemma 4.11. Using Lemma 4.12, we deduce that for any $i \leq m$, box $[i, \lambda_{i-1}] \notin R$; in particular, condition (1) holds.

Suppose that $\mu_j + 1 = \lambda_j = d$ for $j_1 \leq j \leq j_2$. According to Lemma 4.13, each removed box $[j, d]$ for $j_1 \leq j \leq j_2$ is k -related to some box $[i_j, c_j] \in \mu \setminus \lambda$, and the box $[i_j, c_j - 1]$ is also in $\mu \setminus \lambda$. Lemma 4.11 implies that each box $[j, d]$ is k -related to at most one box of $\mu \setminus \lambda$. It follows that if $j < j_2$, then $[i_j, c_j] = [i_{j+1}, c_{j+1} - 1]$, so all the boxes $[i_j, c_j]$ lie in the same row of $\mu \setminus \lambda$. The result follows from this since we also know that the box $[j_1 - 1, d]$ is k -related to $[i_{j_1}, c_{j_1} - 1]$. \square

Propositions 4.1 and 4.2 tell us that if $\psi = (D, \mu, S, 0)$ is any 4-tuple in Ψ_0 with $\mu_{\ell+1} \geq 0$, then D is uniquely determined by μ and $\text{ev}(\psi) = T(\mathcal{C}(\mu), \mu)$ is a term appearing in the Pieri rule (14). To account for the multiplicities, we give an explicit construction of the possible sets S in these 4-tuples. For the rest of this section we fix a k -strict partition μ such that $\lambda \rightarrow \mu$ and $|\mu| = |\lambda| + p$, and define $D = \{(i, j) \in \Delta \mid j \leq \ell + 1 \text{ and } \mu_i + \mu_j > 2k + j - i\}$.

Lemma 4.14. *We have $\mathcal{C} \subset D \subset \mathcal{C} \cup \partial\mathcal{C}$.*

Proof. If $(i, j) \in D$, then $\lambda_{i-1} + \lambda_{j-1} \geq \mu_i + \mu_j > 2k + j - i$. This proves that $D \subset \mathcal{C} \cup \partial\mathcal{C}$. If there exists a pair $(i, j) \in \mathcal{C} \setminus D$, then $\lambda_i + \lambda_j > 2k + j - i \geq \mu_i + \mu_j$, so we must have $\mu_i = \lambda_i$, $\mu_j = \lambda_j - 1$, and $\lambda_i + \lambda_j = 2k + 1 + j - i$. Condition (2) of §2.2 implies that some box $[h, c]$ of $\mu \setminus \lambda$ is k -related to $[j, \lambda_j]$, and $[h, c - 1]$ is also in $\mu \setminus \lambda$ since this box is k -related to $[j - 1, \lambda_j]$. The equality $\lambda_j + c = 2k + 2 + j - h$ implies that $(h, j) \in D$, and since D is a valid set of pairs, we must have $h < i$. But we also obtain $\lambda_h < c - 1 = 2k + 1 + j - h - \lambda_j = \lambda_i + i - h$, contradicting the fact that λ is k -strict. This proves that $\mathcal{C} \subset D$. \square

A *component* means an (edge or vertex) connected component of the set \mathbb{A} of §2.2. We say that a box B of \mathbb{A} is *distinguished* if the box directly to the left of B does not lie in \mathbb{A} . We say that B is *optional* if it is the rightmost distinguished box in its component. Notice that $\mathfrak{N}(\lambda, \mu)$ is equal to the number of optional distinguished boxes in $\mu \setminus \lambda$.

To each distinguished box $B \in \mathbb{A}$ we associate the pair $(i, j) \in \Delta$, where i is the row number of B , and j is maximal such that there exists a box $[j, c]$ in the first k columns of μ which is k' -related to B ; if there is no such box then let $j = \ell + 1$. Let E (respectively F) be the set of pairs associated to optional (respectively non-optional) distinguished boxes. The following lemma implies that any pair $(i, j) \in \Delta$ is associated to at most one distinguished box of \mathbb{A} . Let R be the set of boxes of $\mu \setminus \lambda$ in columns $k + 1$ and higher which are not in \mathbb{A} .

Lemma 4.15. *Let $B = [i, d] \in \mathbb{A}$ be distinguished, and assume that $B' = [i, d - 1]$ lies in R . Let (i, j) be the pair associated to B . Then B is k' -related to $[j, \mu_j]$ (we allow here the possibility $[j, \mu_j] = [\ell + 1, 0]$.)*

Proof. Let B be k' -related to the box $[j, c] \in \mu$. Since $B \notin R$, B is k -related to $[j, c + 1]$, and $[j + 1, c + 1] \notin \mu$, it follows that $[j, c + 1] \notin \lambda$. Since B' belongs to R and is k -related to $[j - 1, c + 1]$, we must have $[j - 1, c + 1] \in \lambda$ and $[j, c + 1] \notin \mu$. \square

Lemma 4.16. *Suppose $(i, j) \in F$ and let f be the value computed in Definition 3.2 for the valid 4-tuple (D, μ, S, i) given by any subset $S \subset D \cap \partial\mathcal{C}$. Then $f = j$.*

Proof. The set \mathbb{A} contains a non-optional distinguished box B in row i and $\mu_i = \lambda_{i-1}$. We have $B = [i, e]$, where e is computed in Definition 3.2 for the 4-tuple (D, μ, S, i) . To see that $f = j$, it is enough to show that if $r := i + e - 2k - 1 \leq \ell$ then B is k' -related to a box in the first k columns of μ . This is true because B is k' -related to $[r + 1, 1]$, and this box must be in μ since B is k -related to $[r, 1] \in \lambda$ and $B \notin R$. \square

Let G be the set of all pairs (i, j) for which some box in row i of $\mu \setminus \lambda$ is k -related to a box in row j of $\lambda \setminus \mu$. We note that if $(i, j) \in G$, then (i, j) is not associated to a distinguished box of \mathbb{A} . In fact, we must have $\mu_j = \lambda_j - 1$ and $[j, \lambda_j]$ is k -related to some box $[i, c] \in R$. If (i, j) is associated to a distinguished box $[i, d] \in \mathbb{A}$, then

$c < d$ and Lemma 4.15 shows that $[i, d]$ is k' -related to $[j, \mu_j]$. But then $[i, d] = [i, c]$, a contradiction.

To every subset E' of E we define the set of pairs $S(E') := E' \cup F \cup G$. This is a disjoint union, and there are exactly $2^{\mathfrak{R}(\lambda, \mu)}$ sets of this form.

Proposition 4.3. *If $(D, \mu, S, 0) \in \Psi_0$ then $S = S(E')$ for some subset $E' \subset E$.*

Proof. Suppose that $(i, j) \in G$. Then $\mu_j = \lambda_j - 1$ and $[j, \lambda_j]$ is k -related to a box in row i of $\mu \setminus \lambda$. If (r, j) is the unique pair in column j of S , then Lemma 4.13 implies that $[j, \lambda_j]$ is also k -related to a box in row r of $\mu \setminus \lambda$, and it follows from Lemma 4.11 that $r = i$. This shows that $G \subset S$.

Now let $(i, j) \in F$. Lemma 4.16 states that $f = j$, where f is the value computed in Definition 3.2 for the 4-tuple (D, μ, S, i) . Let $\psi' = (D', \mu', S', i)$ be the most recent predecessor of ψ of level i . Since ψ' does not satisfy condition X, we deduce from Lemma 4.12 that $(i, f) \in S' \subset S$. This shows that $F \subset S$.

Let $(i, j) \in S \setminus G$. We will show that (i, j) is the pair associated to a distinguished box of \mathbb{A} . If $i = j = m$, then $\lambda_i \leq k < \mu_i$ and (i, j) is associated to the distinguished box $[m, k+1] \in \mathbb{A}$. We can therefore assume that $i < j$. We have $\mu_i > \lambda_i$, and since $(i, j) \notin G$, it follows from Lemma 4.13 that $\lambda_j \leq \mu_j$.

If $\lambda_i + \mu_j \geq 2k + j - i$, then set $x = 2k + j - i - \lambda_i$. Since $(i, j) \notin \mathcal{C}$ we have $\lambda_j \leq x \leq \mu_j$. We also have $x \leq k$; if $j = m$ this follows because $\lambda_i \geq m - i + k$. Since $[j, x]$ is k' -related to $[i, \lambda_i + 1]$, it follows that $[i, \lambda_i + 1] \in \mathbb{A}$ is distinguished, and (i, j) is the associated pair.

Otherwise we have $\lambda_i + \mu_j < 2k + j - i$. In this case we set $c = 2k + j - i - \mu_j$. Since $(i, j) \in D$ we have $\lambda_i < c < \mu_i$. We also have $c > k$; if $i = m$ this follows because $\mu_j \leq \lambda_m \leq k$. We claim that $\mu_j < \lambda_{j-1}$. If $(i, j-1) \in \mathcal{C}$, then this follows because $\mu_j < 2k + j - i - \lambda_i \leq \lambda_{j-1}$, so assume that $(i, j-1) \notin \mathcal{C}$. In this case we must have $j > m$, and (i, j) was added to S in Phase 2 of the algorithm. By Lemma 4.1(b), the first predecessor (D', μ', S', i) of ψ of level i satisfies that $\mu'_j \leq \lambda_{j-1}$. Since $(i, j) \notin S'$, this implies that $\mu_j < \lambda_{j-1}$, as claimed.

Since $\lambda_j \leq \mu_j < \lambda_{j-1}$ and the boxes $[j-1, \mu_j]$ and $[j-1, \mu_j+1]$ are k -related to $[i, c+1]$ and $[i, c]$, respectively, we deduce that $[i, c] \in R$, and $[i, c+1] \in \mathbb{A}$ is a distinguished box. Since $[i, c+1]$ is k' -related to $[j, \mu_j]$, the associated pair is (i, j) .

We conclude that the set $E' := S \setminus (F \cup G)$ is a subset of E , hence $S = S(E')$ has the required form. \square

Lemma 4.17. *We have $E \cup F \cup G \subset D \cap \partial \mathcal{C}$.*

Proof. Let $(i, j) \in G$. Then $\mu_j = \lambda_j - 1$ and the boxes $[j, \lambda_j]$ and $[j-1, \lambda_j]$ are k -related to $[i, c]$ and $[i, c-1]$, where $c = 2k+2+j-i-\lambda_j$. We also have $\lambda_i+1 < c \leq \mu_i$. Therefore $\lambda_i + \lambda_j < c + \lambda_j - 1 = 2k+1+j-i$ and $\mu_i + \mu_j \geq c + \mu_j = 2k+1+j-i$, so $(i, j) \in D \setminus \mathcal{C}$.

Next let $(i, j) \in E \cup F$, i.e. (i, j) is the pair associated to a distinguished box $B = [i, c]$ of \mathbb{A} . We have $\lambda_i < c \leq \mu_i$ and B is k' -related to the box $[j, x]$ of μ where $c+x = 2k+1+j-i$. Since $[j+1, x+1] \notin \mu$ and $B \notin R$ we must have $\lambda_j \leq x \leq \mu_j$. We obtain $(i, j) \in D \setminus \mathcal{C}$ since $\lambda_i + \lambda_j < x + c$ and $\mu_i + \mu_j \geq x + c$. \square

Lemma 4.18. *For any $E' \subset E$, the set $S = S(E')$ has the following properties:*

- (i) *If $(i, j) \in S$ then $\mu_i > \lambda_i$.*
- (ii) *If $i \leq m$ and $\mu_i = \lambda_{i-1}$, then $(i, g-1) \in D$ and $(i, f) \in F$, where f, g are computed as in Definition 3.2 for the 4-tuple (D, μ, S, i) .*

- (iii) If $\mu_j = \lambda_j - 1$ then there is a unique $i < j$ with $(i, j) \in D \setminus \mathcal{C}$, and $(i, j) \in G$ for this i . If we also have $\lambda_j = \lambda_{j+1}$, then $(i, j+1) \in G$ as well.
- (iv) If $(i, j) \in S$, $i < j$, and $(i, j-1) \notin \mathcal{C}$, then $\mu_j < \lambda_{j-1}$.

Proof. Condition (i) follows directly from the definition of $S(E')$. In the situation of (ii), the inequalities

$$\mu_i + \mu_{g-1} \geq \lambda_{i-1} + (\lambda_{g-1} - 1) > 2k + (g-1) - i$$

prove that $(i, g-1) \in D$. Lemma 4.16 implies that $(i, f) \in S$. To prove that the i in (iii) is unique, it suffices to show that if $(i, j) \notin \mathcal{C}$ and $(i+1, j) \in D$, then $\mu_j \geq \lambda_j$. Indeed, if this fails, we have $\mu_j = \lambda_j - 1$ and $\mu_{i+1} + \mu_j \leq \lambda_i + \lambda_j - 1 \leq 2k + j - (i+1)$, which is a contradiction. The rest of (iii) follows since $\lambda \rightarrow \mu$. Finally, suppose that $(i, j) \in S$, $i < j$, and $(i, j-1) \notin \mathcal{C}$. If $\mu_j \geq \lambda_{j-1}$ then $\mu_j = \lambda_{j-1}$ since $\lambda \rightarrow \mu$, and hence $\lambda_i + \mu_j \leq 2k + j - 1 - i$. There is a distinguished box $B = [i, c]$ which is k' -related to a box $[j, c']$ of μ , with j maximal. Since $c + \mu_j \geq c + c' = 2k + 1 + j - i$, it follows that $c > \lambda_i + 1$. We deduce that $c = k + 1$ or the box directly to the left of B lies in R , which contradict $i < j$ and $\mu_j = \lambda_{j-1}$, respectively. \square

The proof of Claim 1 is completed by the following converse to Proposition 4.3.

Proposition 4.4. *If $S = S(E')$ for some subset $E' \subset E$, then $(D, \mu, S, 0) \in \Psi_0$.*

Proof. Set $\nu = \prod_{(i,j) \in S} L_{ij}\mu$. Observe that the definition of S ensures that $\nu \geq \lambda$ and therefore $\nu \in \mathcal{N}(\lambda, p)$. We will show that the substitution forest has a path starting with $\psi_0 = (\mathcal{C}, \nu, \emptyset, \ell + 1)$ and ending with $(D, \mu, S, 0)$. The required path is formed by first applying rule (i) repeatedly for the pairs $(i, h) \in D \setminus \mathcal{C}$ with $(i, h-1) \in \mathcal{C}$ or $i = h$, in north-to-south order, and then applying rule (iii) for the remaining pairs $(h, j) \in D \setminus \mathcal{C}$, in west-to-east order. Lemma 4.14 implies that the boxes of $D \setminus \mathcal{C}$ can be added in this order. Whenever a pair (i, j) is added to the set D' of an intermediate 4-tuple (D', μ', S', h) , we add this pair also to S' (and replace μ' with $R_{ij}\mu'$) if and only if $(i, j) \in S$. Lemma 4.17 shows that all pairs of S will be added in this way. We claim that each step in the above path coincides with one of the choices offered by the substitution rule of §3.3.

We begin by studying Phase 1 of the algorithm applied to ψ_0 . Suppose that $(i, h) \in D$ is an outer corner of \mathcal{C} . We have

$$\nu_i + \nu_h = \mu_i + \mu_h - \#\{h' \geq h \mid (i, h') \in S\} + \#\{i' \geq i \mid (i', h) \in S\}.$$

If h_1 is maximal such that $(i, h_1) \in D$, then $\mu_i + \mu_h \geq \mu_i + \mu_{h_1} > 2k + h_1 - i$, and therefore

$$\begin{aligned} \nu_i + \nu_h &> 2k + h - i + (h_1 - h) - \#\{h' \geq h \mid (i, h') \in S\} + \#\{i' \geq i \mid (i', h) \in S\} \\ &\geq 2k + h - i. \end{aligned}$$

It follows that $W(i, h)$ holds for ν . If i_1 is maximal such that $(i_1, h) \in D$ and $(i_1, h-1) \in \mathcal{C}$ or $i_1 = h$, we deduce that the algorithm will apply (i) and add all pairs (x, h) with $i \leq x \leq i_1$ to D and the corresponding pairs to S , in this order. Property (iv) of Lemma 4.18 ensures that none of the successors of ν will meet (ii). Moreover, property (iii) of loc. cit. shows that if $\mu_h = \lambda_{h-1}$, then (i, h) is the only pair of $D \setminus \mathcal{C}$ in column h .

We next pass to Phase 2, and consider a successor $\psi' = (D', \nu', S', h)$ of ψ_0 with $(h, h) \in D'$. Let r be maximal such that $(h, r) \in D$; we claim that rule (iii) may be applied ψ' and its successors by adding all pairs (h, j) with $b < j \leq r$, as specified

by the set S . Since the composition which results when (h, r) is added agrees with μ in rows h and b through r , it suffices to check that the weight conditions $W(h, j)$ hold throughout this process. We prove this by descending induction on j ; the case $j = r$ is clear because $W(h, r)$ holds for $(D, \mu, S, 0)$. Suppose that for a successor $(\overline{D}, \overline{\mu}, \overline{S}, h)$ of ψ' we have $\overline{\mu}_h + \overline{\mu}_j > 2k + j - h$ for j maximal such that $(h, j) \in \overline{D}$. Observe that $\overline{\mu}_s = \mu_s$ for $m \leq s \leq j$. We deduce that if $(h, j) \in \overline{S}$, then $\overline{\mu}_j < \lambda_{j-1}$, by Lemma 4.18(iv). It follows that for the predecessor $(\overline{D} \setminus (h, j), \tilde{\mu}, \overline{S} \setminus (h, j), h)$ of $(\overline{D}, \overline{\mu}, \overline{S}, h)$ with $\tilde{\mu} = L_{h,j}\overline{\mu}$, we have

$$\tilde{\mu}_h + \tilde{\mu}_{j-1} \geq \tilde{\mu}_h + \lambda_{j-1} - 1 \geq \overline{\mu}_h + \overline{\mu}_j - 1 > 2k + (j - 1) - h.$$

On the other hand, if $(h, j) \notin \overline{S}$, then for the predecessor $(\overline{D} \setminus (h, j), \overline{\mu}, \overline{S}, h)$ of $(\overline{D}, \overline{\mu}, \overline{S}, h)$, we also obtain that $\overline{\mu}_h + \overline{\mu}_{j-1} \geq \overline{\mu}_h + \overline{\mu}_j > 2k + (j - 1) - h$.

It follows from Lemma 4.18(ii) that no valid 4-tuples (D', μ', S', h') in the substitution path satisfy condition X. Moreover, if $(h', g') \notin D'$ and $W(h', g')$ holds, then $(h', g') \in D$ and hence $(h' - 1, g') \in D'$. We deduce that (D', μ', S', h') does not meet **(iv)** or **(v)**, and this completes the argument. \square

Figures 4 and 5 display two partitions λ, μ which appear in a Pieri product (14) and the sets $\mathcal{C}(\lambda) \subset \mathcal{C}(\mu)$. In Figure 4, the diagram of λ is displayed in white, while the shaded regions show the part of μ which extends beyond the diagram of λ . In Figure 5, the pairs in $\mathcal{C}(\mu) \setminus \mathcal{C}(\lambda)$ are shaded. The diagonal dotted lines in Figure 4 are reflected about the vertical line k units to the right of the left vertical axis, and point to boxes which are k' -related to the 7 optional distinguished boxes. Observe that there is a single non-optional distinguished box. The boxes marked with a question mark may or may not be part of a composition $\nu \in \mathcal{N}(\lambda, p)$ such that the initial term $(\mathcal{C}, \nu, \emptyset, \ell + 1)$ gives rise to a 4-tuple $(\mathcal{C}(\mu), \mu, S, 0)$. There are 2^7 distinct such compositions ν , each giving rise to a single Pieri term $\text{ev}(\mathcal{C}(\mu), \mu, S, 0) = T(\mathcal{C}(\mu), \mu)$ when the algorithm terminates.

4.7. In this section, we define the involution $\iota : \Psi_1 \rightarrow \Psi_1$ and prove Claim 2. Suppose that a term $\psi = (D, \mu, S, h) \in \Psi_1$ is such that $(h, h) \notin D$, and thus met condition **(ii)** in the course of the algorithm. Define μ' by setting $\mu'_{h-1} = \mu_h - 1$, $\mu'_h = \mu_{h-1} + 1$, and $\mu'_r = \mu_r$ for $r \notin \{h - 1, h\}$. Let

$$\iota(D, \mu, S, h) = (D, \mu', S, h);$$

then $\text{ev}(\psi) + \text{ev}(\iota(\psi)) = 0$, using Lemma 1. If ν and ν' are the initial compositions that gave rise to (D, μ, S, h) and (D, μ', S, h) , respectively, then note that $\mu'_h > \lambda_{h-1}$ and $\nu_j = \nu'_j$ for $j \notin \{h - 1, h\}$. Therefore it is easy to verify that $\nu' \in \mathcal{N}(\lambda, p)$ and that $(\mathcal{C}, \nu', \emptyset, \ell + 1)$ goes through the same branching history to reach (D, μ', S, h) as $(\mathcal{C}, \nu, \emptyset, \ell + 1)$ went through to reach (D, μ, S, h) . We deduce that (D, μ', S, h) also meets condition **(ii)**, and hence lies in Ψ_1 ; clearly $\iota(D, \mu', S, h) = (D, \mu, S, h)$.

Suppose next that a term $\psi = (D, \mu, S, h) \in \Psi_1$ satisfies $(h, h) \in D$, and thus met condition **(v)**. Since $(h, g) \in D$, the pair $(h - 1, h)$ is D -tame for ψ and Lemma 1.2 applies. If $\mu_h = \mu_{h-1}$, we let $\iota(D, \mu, S, h) = (D, \mu, S, h)$. Otherwise, let ϖ be the involution on Δ which exchanges $(h - 1, g)$ with (h, f) , and fixes all other pairs. Define μ' by switching parts μ_{h-1} and μ_h in the composition μ , set $S' = \varpi(S)$, and let $\iota(D, \mu, S, h) = (D, \mu', S', h)$. Then we have $\text{ev}(\psi) + \text{ev}(\iota(\psi)) = 0$, for every $\psi \in \Psi_1$, by construction.

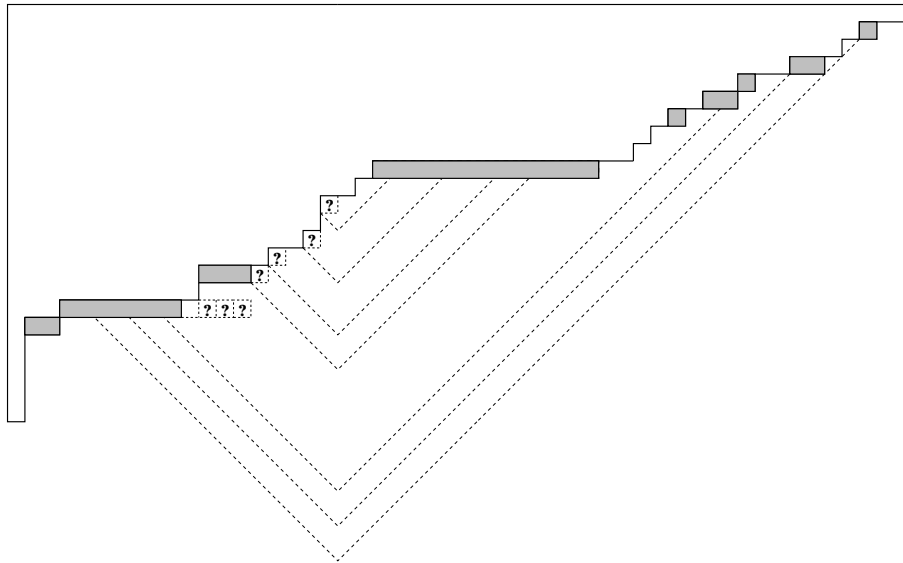


FIGURE 4. The index μ of a Pieri term, with $\mu \setminus \lambda$ shaded

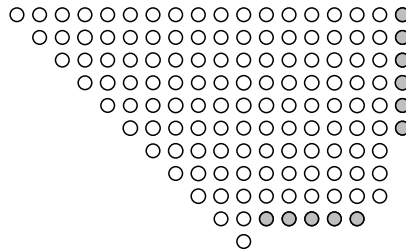


FIGURE 5. The sets of pairs $\mathcal{C}(\lambda) \subset \mathcal{C}(\mu)$

Observe that to the two terms $\psi = (D, \mu, S, h)$ and $\psi' = (D, \mu', S', h)$, there are assigned the same values of $e, f,$ and $g,$ because these three integers do not depend on $S, S',$ and $\mu_r = \mu'_r$ for all $r > h.$ It follows that $\iota(\psi') = \psi.$ We claim that ψ' also satisfies condition X. Indeed, if $\mu_{h-1} = \lambda_{h-1} < \mu_h,$ then we must have $(h-1, g) \notin S,$ and hence X holds for $\psi',$ since we have $\mu'_h = \lambda_{h-1}$ and $(h, f) \notin S'. In all other cases, the claim is clear.$

It remains to show that $\psi' \in \Psi_1.$ We assume in the following that ψ meets $(\mathbf{v}),$ and will prove that ψ' also meets $(\mathbf{v}).$

Proposition 4.5. *We have $\mu_f < \lambda_{f-1},$ or $\mu_f = \lambda_{f-1}$ and $f = b.$*

Proof. Since $(h, h) \in D,$ Lemmas 4.1 and 4.3 imply that $\mu_f \leq \lambda_{f-1}.$ Assume that $\mu_f = \lambda_{f-1}.$ If $f = h,$ then clearly $f = b,$ so suppose that $f > h.$ We claim that box $[h, \lambda_{h-1}]$ is not in $R,$ for otherwise, we may argue as in Lemma 4.11 to deduce

that $\lambda_{h-1} + \mu_g \leq 2k + g - h$. Now $f = g$, and the inequality

$$2k + g - h - \lambda_{h-1} < \lambda_{g-1} = \mu_g$$

leads to a contradiction. Following Definition 3.2, $e \leq \lambda_{h-1}$ and box $[h, e]$ is k' -related to a box $[f, r]$ with $r \leq \mu_f = \lambda_{f-1}$ (we include here the case when $r = 0$). Equivalently, $[h, e - 1]$ is k -related to $[f - 1, r + 1]$, and therefore $[h, e - 1] \notin R$. We deduce that either $e = k + 1$ or $e = \lambda_h + 1$. If $e = k + 1$ then clearly $\lambda_h \leq k$, hence $h = m = f$, which contradicts our assumption. If $e = \lambda_h + 1$ then

$$\lambda_h + \lambda_{f-1} = \lambda_h + \mu_f \geq e - 1 + r = 2k + f - h.$$

It follows that $(h, f - 1) \in \mathcal{C}$, and hence $f = b$. \square

Lemma 4.19. *If $f < g$ then $\mu_g = \lambda_{g-1}$, $(h, g) \notin S$, and ψ, ψ' both satisfy $W(h, g)$.*

Proof. Observe that $c := 2k + 1 + g - h - \lambda_{h-1} \leq \lambda_{g-1}$, since $(h - 1, g - 1) \in \mathcal{C}$. If $\mu_g < c$, then $[h, \lambda_{h-1}] \in R$, hence $f = g$. If $c \leq \mu_g < \lambda_{g-1}$, then the box $[h, e]$ is k' -related to $[g, \mu_g]$, hence $f = g$ again. Since $\mu_g \leq \lambda_{g-1}$ by Lemma 4.1, we must have $\mu_g = \lambda_{g-1}$ and $(h, g) \notin S$. As ψ satisfies X, we have $\mu_h \geq \lambda_{h-1}$, therefore

$$\mu_h + \mu_g \geq \lambda_{h-1} + \lambda_{g-1} > 2k + g - h$$

and condition $W(h, g)$ holds. The same inequalities are true for ψ' . \square

Define the retraced weight condition $\widetilde{W}(r, g)$ on a valid 4-tuple by the inequality

$$\widetilde{W}(r, g) : \mu_r + \mu_g + N_r > 2k + g - r,$$

where $N_r = \#\{i > r \mid (i, g) \in S\}$. For the 4-tuple ψ , let z be the least integer with $a \leq z \leq h - 1$ such that $\widetilde{W}(z, g)$ fails; if no such integer exists, then set $z = h$. If $z < h$, it follows that the pairs $(a, g), \dots, (z - 1, g)$ were added to D in Phase 1 of the algorithm, and the pairs $(z, g), \dots, (h, g)$ were added to D in Phase 2. We deduce by arguing as in Lemma 4.1(a) that the inequality

$$(17) \quad \mu_g + N_{z-1} \leq \lambda_{g-1}$$

holds. Define the integers N'_r and z' associated to the 4-tuple ψ' in the same fashion.

Proposition 4.6. *Assume that $z \neq z'$.*

- a) *If $f = g$, then $\{z, z'\} = \{h - 1, h\}$, and either ψ or ψ' satisfies $\widetilde{W}(h, g)$.*
- b) *If $f < g$, then $\max(z, z') = h$, and ψ, ψ' both satisfy $W(h, g)$.*

Proof. If $f = g$, then from the definition of S' we obtain that $N_r = N'_r$ for any $r < h - 1$. We deduce that $z < h - 1$ if and only if $z' < h - 1$, in which case we have $z = z'$. Since $z \neq z'$, it follows that $\{z, z'\} = \{h - 1, h\}$. If $z = h$, then $\widetilde{W}(h - 1, g)$ holds for ψ , and therefore $\widetilde{W}(h, g)$ holds for ψ' . This proves part (a).

If $f < g$, we know by Lemma 4.19 that ψ and ψ' both satisfy $W(h, g)$ and $(h, g) \notin S \cup S'$. Assume that $(h - 1, g) \in S$. Since $\mu_g = \lambda_{g-1}$, Lemma 4.1 for the term ψ shows that the pair $(h - 1, g)$ was added to D in Phase 1 of the algorithm, and hence $z = h$. A similar observation applies to the term ψ' . On the other hand, if $(h - 1, g) \notin S \cup S'$, then reasoning as in the proof of (a), we see that $\{z, z'\} = \{h - 1, h\}$. \square

Proposition 4.7. *If $(h + 1, g) \in D$, then both ψ and ψ' satisfy condition $\widetilde{W}(i, g)$ for $a \leq i \leq h + 1$. In particular, we have $f = g$ and $z = z' = h$.*

Proof. Clearly $f = g = b$, and Lemma 4.6 implies that $(h+1, g)$ was added to D in Phase 1 of the algorithm for ψ . Therefore ψ satisfies $\widetilde{W}(i, g)$ whenever $a \leq i \leq h+1$, and $z = h$. It is clear from the definitions that $\widetilde{W}(i, g)$ holds for ψ' when $a \leq i \leq h+1$ and $i \notin \{h-1, h\}$. It is also easy to see that condition $\widetilde{W}(h-1, g)$ for ψ implies that $\widetilde{W}(h, g)$ holds for ψ' . Notice that $\mu_h > \lambda_h$, since ψ meets (\mathbf{v}) and therefore satisfies X. Moreover, $\widetilde{W}(g, h+1)$ holds for ψ , and hence

$$(18) \quad \mu_h + \mu_g + N_{h+1} > \lambda_h + \mu_g + N_{h+1} \geq \mu_{h+1} + \mu_g + N_{h+1} \geq 2k + g - h.$$

We claim that $\mu_h + \mu_g + N_h > 2k + 1 + g - h$. This follows immediately from (18) if $\mu_{h+1} < \lambda_h$. If $\mu_{h+1} = \lambda_h$, then we must have $(h+1, g) \in S$, for otherwise the parent of ψ would have met (\mathbf{v}) . We conclude that $N_h = N_{h+1} + 1$, proving the claim. Since $N'_h = N_h$ and $\mu'_g = \mu_g$, we obtain $\mu'_{h-1} + \mu'_g + N'_h > 2k + g - (h-1)$, which is condition $\widetilde{W}(h-1, g)$ for ψ' . \square

We will obtain a sequence of valid 4-tuple predecessors $(\overline{D}, \overline{\mu}, \overline{S}, h)$ of ψ' by successively removing pairs (i, g) for $i \leq h$ from the set $D \cap \partial\mathcal{C}$, and applying corresponding lowering operators to μ' , as dictated by the set S' . This backtracking continues until weight considerations along column g force the sequence of predecessors to proceed by removing pairs along row h , if $b < g$, or by increasing h to $h+1$, if $b = g$. The precise point when this happens is specified by the value of z' . If $z = z'$, then the backtracking sequence for the term ψ' is essentially the same as that for ψ . The various possibilities when $z \neq z'$ are explained in Proposition 4.6; in this case either ψ or ψ' is such that each of the pairs $(a, g), \dots, (h-1, g)$ will be added to D in Phase 1.

We claim that all predecessors $\overline{\psi} = (\overline{D}, \overline{\mu}, \overline{S}, h)$ of ψ' with $(h, g-1) \in \overline{D}$ satisfy either $W(h, g)$ or X. If condition $W(h, g)$ is true for ψ' then it will also hold for all predecessors $\overline{\psi}$, so assume that $W(h, g)$ is false for ψ' . We have $f = g$ by Lemma 4.19, and since $W(h, g)$ fails we deduce that $[h, \lambda_{h-1}] \in R$. Let \overline{f} and \overline{R} be the values of f and R as computed for some predecessor $\overline{\psi}$ of ψ' . The only way that X can fail for $\overline{\psi}$ is if $\overline{\mu}_h = \lambda_{h-1}$ and $(h, \overline{f}) \in \overline{S}$. Clearly $\overline{f} < g$, therefore $[h, \lambda_{h-1}] \notin \overline{R}$, and we conclude that condition $W(h, g)$ must hold for $\overline{\psi}$.

For the remainder of the proof of Claim 2, we distinguish two cases.

Case 1. Assume that $b < g$. Then the backtracking sequence for ψ' begins by removing the pair (h, g) from D . We have seen that if $\mu_{h-1} = \lambda_{h-1} < \mu_h$, then ψ' satisfies $\mu'_h = \lambda_{h-1}$ and $(h, f) \notin S'$. On the other hand, if $\mu_h = \lambda_{h-1} < \mu_{h-1}$, we have $(h, f) \notin S$ and therefore $\mu'_{h-1} = \lambda_{h-1} < \mu'_h$ and $(h-1, g) \notin S'$. If $z' < h$, the sequence continues by successively removing the pairs $(h-1, g), \dots, (z', g)$ from D , to arrive at a 4-tuple $(\overline{D}, \overline{\mu}, \overline{S}, h)$.

At this juncture, Lemma 4.20 below ensures that $\overline{\mu}_h + \overline{\mu}_{g-1}$ is large enough so that the sequence can be traced back further by removing pairs along row h of \overline{D} until only the pair (h, b) remains in row h of $\overline{D} \cap \partial\mathcal{C}$. To see this, suppose that for a predecessor $(\overline{D}, \overline{\mu}, \overline{S}, h)$ of ψ' we have $\overline{\mu}_h + \overline{\mu}_j > 2k + j - h$ for $j \leq g$ maximal such that $(h, j) \in \overline{D}$. If $(h, j) \in \overline{S}$, then $\overline{\mu}_j < \lambda_{j-1}$. It follows that for the predecessor $(\overline{D} \setminus (h, j), \tilde{\mu}, \overline{S} \setminus (h, j), h)$ of $(\overline{D}, \overline{\mu}, \overline{S}, h)$ with $\tilde{\mu} = L_{hj}\overline{\mu}$, we have

$$\tilde{\mu}_h + \tilde{\mu}_{j-1} \geq \tilde{\mu}_h + \lambda_{j-1} - 1 \geq \overline{\mu}_h + \overline{\mu}_j - 1 > 2k + (j-1) - h.$$

On the other hand, if $(h, j) \notin \overline{S}$, note that if $\overline{\mu}_{j-1} = \lambda_{j-1} - 1 < \overline{\mu}_j$, then the term $(\overline{D}, \overline{\mu}, \overline{S}, h)$ never appears, as dictated by (iii). We therefore have $\overline{\mu}_j \leq \lambda_{j-1} \leq \overline{\mu}_{j-1}$. For the predecessor $(\overline{D} \setminus (h, j), \overline{\mu}, \overline{S}, h)$ of $(\overline{D}, \overline{\mu}, \overline{S}, h)$, we also obtain that

$$\overline{\mu}_h + \overline{\mu}_{j-1} \geq \overline{\mu}_h + \overline{\mu}_j > 2k + (j - 1) - h.$$

From the above point onwards, the backtracking process for ψ' continues exactly as it did for the 4-tuple ψ , until we reach the initial term $(\mathcal{C}, \nu', \emptyset, \ell + 1)$ with $\nu' = \prod L_{ij} \mu'$, where the product is over all pairs $(i, j) \in S'$. Indeed, the sets S and S' only differ (potentially) in the pairs $(h - 1, g)$ and (h, f) , and correspondingly the parts of μ, μ' and their predecessors can only differ in rows $h - 1, h, f$, and g . To verify that $\nu' \in \mathcal{N}(\lambda, p)$, we use this observation and Lemma 4.2, which implies that $\#\{j \mid (h, j) \in S'\} \leq \mu'_h - \lambda_h$. It is also easy to see from this that no predecessor of ψ' meets (v). Finally, Proposition 4.5 and (17) ensure that the 4-tuples in the sequence of successors of $(\mathcal{C}, \nu', \emptyset, \ell + 1)$ leading up to ψ' do not meet (ii).

Case 2. Assume that $b = g$. If $(h + 1, g) \notin D$, then the backtracking sequence for ψ' is similar to that in Case 1. However, (h, g) is the only pair in row h of $D \cap \partial \mathcal{C}$, and if $z' = h < m$ and $W(h, g)$ holds, then the parent of ψ' is $(D, \mu', S', h + 1)$. If $(h + 1, g) \in D$, then Proposition 4.7 shows that the pairs $(a, g), \dots, (h, g)$ were added to D in Phase 1 of the algorithm for both ψ and ψ' . In particular, the parent of ψ' is $(D, \mu', S', h + 1)$, and the 4-tuples ψ, ψ' proceed through the algorithm in the same fashion.

Lemma 4.20. *Suppose that $b < g$ and define $q = \#\{i > h \mid (i, g - 1) \in S\}$.*

a) *If $(h, g - 2) \notin \mathcal{C}$ or $g \leq h + 1$, then $\mu'_h + \mu'_{g-1} \geq 2k + g - h$, and when $(h, g) \in S'$, we have $\mu'_h + \mu'_{g-1} > 2k + g - h$.*

b) *If $(h, g - 2) \in \mathcal{C}$, then $\mu'_h + \mu'_{g-1} + q \geq 2k + g - h$, and when $(h, g) \in S'$, we have $\mu'_h + \mu'_{g-1} + q > 2k + g - h$.*

Proof. To prove part (a), note that by X we have $\mu'_h \geq \lambda_{h-1}$, and since $(g - 2, g - 1)$ is \mathcal{C} -tame or $g \leq h + 1$, it follows that $\mu'_{g-1} \geq \lambda_{g-1} - 1$. So $\mu'_h + \mu'_{g-1} \geq \lambda_{h-1} + \lambda_{g-1} - 1 \geq 2k + g - h$, since $(h - 1, g - 1) \in \mathcal{C}$. We show now that equality implies $(h, g) \notin S'$, so suppose that $\mu'_h = \lambda_{h-1}$ and $\mu'_{g-1} = \lambda_{g-1} - 1$. Since $(h - 1, g - 1) \in \mathcal{C}$ we have $2k + 1 + g - h - \lambda_{g-1} \leq \lambda_{h-1}$, now the fact that $[g - 1, \lambda_{g-1}]$ is k -related to $[h, 2k + 1 + g - h - \lambda_{g-1}]$ forces $e \geq 2k + 2 + g - h - \lambda_{g-1}$. Therefore $2k + 1 + g - h - e \leq \mu'_{g-1}$. Now $[g - 1, 2k + 1 + g - h - e]$ is k -related to $[h, e]$, hence $f = g$ or $\mu'_g \geq 2k + 1 + g - h - e$. But $[h, e]$ is k' -related to $[g, 2k + 1 + g - h - e]$, so $f = g$. Then, since ψ' satisfies X, we have $(h, g) \notin S'$.

For part (b), note that $\mu'_{g-1} = \mu_{g-1} \geq \lambda_{g-1} - 1 - q$, and by X we have $\mu'_h \geq \lambda_{h-1}$. So $\mu'_h + \mu'_{g-1} + q \geq \lambda_{h-1} + \lambda_{g-1} - 1 \geq 2k + g - h$. We show that equality implies $(h, g) \notin S'$, so suppose that $\mu'_h = \lambda_{h-1}$ and $\mu'_{g-1} = \lambda_{g-1} - 1 - q$. Since $(h - 1, g - 1) \in \mathcal{C}$ we have $2k + 1 + g - h - \lambda_{g-1} \leq \lambda_{h-1}$, now the fact that $[g - 1, \lambda_{g-1} - q]$ is k -related to $[h, 2k + 1 + g - h - \lambda_{g-1} + q]$ forces $e \geq 2k + 2 + g - h - \lambda_{g-1} + q$. So $2k + 1 + g - h - e \leq \mu'_{g-1}$. Now $[g - 1, 2k + 1 + g - h - e]$ is k -related to $[h, e]$, hence $f = g$ or $\mu'_g \geq 2k + 1 + g - h - e$. But $[h, e]$ is k' -related to $[g, 2k + 1 + g - h - e]$, so $f = g$. Then, since ψ' satisfies X, we have $(h, g) \notin S'$. \square

This completes the proof of Claim 2, and of Theorem 1.

Example 4.1. The most subtle ingredient of the Substitution Rule is the condition $W(h, g)$ in **(iv)**. This example shows that if we omit it from **(iv)**, then our cancellation scheme fails, and also illustrates the discussion in the paragraph just before Case 1 above.

Let $\lambda = (4, 3, 1)$, $p = 4$, $k = 1$, and take $n \geq 5$. We have $\mathcal{C} = \{11, 12, 13, 22, 23\}$. Consider the following sequence of 4-tuples in the algorithm, stemming from the root $\psi_0 = (\mathcal{C}, 4341, \emptyset, 4)$.

$$\begin{aligned} \psi_0 &\longrightarrow (\mathcal{C}, 4341, \emptyset, 3) \xrightarrow{(i)} (\mathcal{C} \cup \{33\}, 4341, 33, 3) \xrightarrow{(iv)} (\mathcal{C} \cup \{33, 14\}, 534, \{33, 14\}, 3) \\ &\xrightarrow{(iv)} (\mathcal{C} \cup \{33, 14, 24\}, 534, \{33, 14\}, 3) \xrightarrow{(iv)} (\mathcal{C} \cup \{33, 14, 24, 34\}, 534, \{33, 14\}, 3) = \psi. \end{aligned}$$

The substitution rule STOPS at the leaf ψ , for which we have $[3, 3] \in R$ and hence $f = g = 4$. Now $\psi' = (\mathcal{C} \cup \{33, 14, 24, 34\}, 543, \{33, 14\}, 3)$, which backtracks to the initial term $\psi'_0 = (\mathcal{C}, 4431, \emptyset, 4)$. However, the tree of the substitution forest with root ψ'_0 contains the initial path

$$\psi'_0 \longrightarrow (\mathcal{C}, 4431, \emptyset, 3) \xrightarrow{(i)} (\mathcal{C} \cup \{33\}, 4431, 33, 3)$$

Note that for the term $\bar{\psi} = (\mathcal{C} \cup \{33\}, 4431, 33, 3)$ we have $\bar{f} = 3$, hence condition X fails. If the condition $W(h, g)$ is omitted from **(iv)**, we deduce that $\bar{\psi}$ does not meet any of **(i)**–**(v)**, and is REPLACED by $(\mathcal{C} \cup \{33\}, 4431, 33, 2)$. We conclude that ψ' does not appear in the substitution forest, and the cancellation scheme fails.

5. THETA POLYNOMIALS

5.1. In this section we develop the theory of theta polynomials systematically; the exposition is influenced by that in Macdonald's text [M, III.8]. Let $x = (x_1, x_2, \dots)$ and let $\Lambda = \Lambda(x)$ be the ring of symmetric functions in x . Consider the generating functions

$$E(x; t) = \prod_{i=1}^{\infty} (1 + x_i t) = \sum_{r=0}^{\infty} e_r(x) t^r \quad \text{and} \quad H(x; t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t} = \sum_{r=0}^{\infty} h_r(x) t^r$$

for the elementary and complete symmetric functions e_r and h_r , respectively. Fix an integer $k \geq 0$, let $y = (y_1, \dots, y_k)$, and for each r define $\vartheta_r = \vartheta_r(x; y)$ by

$$\vartheta_r = \sum_{i \geq 0} q_{r-i}(x) e_i(y).$$

We let $\Gamma^{(k)}$ be the subring of $\Lambda \otimes \mathbb{Z}[y_1, \dots, y_k]^{S_k}$ generated by the ϑ_r :

$$\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \vartheta_3, \dots].$$

Set $\Theta(t) = \sum_{r \geq 0} \vartheta_r t^r$; we then have

$$\Theta(t) = \prod_i \frac{1 + tx_i}{1 - tx_i} \prod_{j=1}^k (1 + y_j t) = E(x; t) H(x; t) E(y; t)$$

and hence

$$\Theta(t) \Theta(-t) = E(y; t) E(y; -t) = \sum_{m=0}^{2k} (-1)^m e_m(y^2),$$

where y^2 denotes (y_1^2, \dots, y_k^2) . It follows that

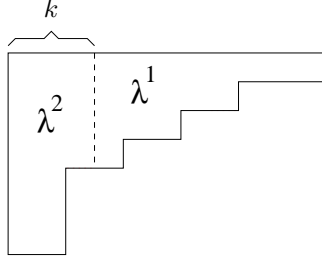
$$(19) \quad \sum_{r+s=d} (-1)^r \vartheta_r \vartheta_s = \begin{cases} 0 & \text{if } d \text{ is odd} \\ (-1)^d e_{d/2}(y^2) & \text{if } d \text{ is even.} \end{cases}$$

When $d = 2m > 2k$, equation (19) gives

$$(20) \quad \vartheta_m^2 = 2 \sum_{i=1}^m (-1)^{i+1} \vartheta_{m+i} \vartheta_{m-i}.$$

For any integer sequence ρ , let $\vartheta_\rho = \prod_i \vartheta_{\rho_i}$, and define $q_\rho = \prod_i q_{\rho_i}$ and $e_\rho = \prod_i e_{\rho_i}$ similarly. We deduce from (20) that either λ is k -strict, or ϑ_λ is a \mathbb{Z} -linear combination of the ϑ_μ such that μ is k -strict and $\mu \succ \lambda$.

Definition 5.1. Given any k -strict partition λ , we obtain two partitions λ^1 and λ^2 , with λ^1 strict, by cutting the Young diagram of λ into a disjoint union of two diagrams: λ^1 is the part of λ lying in columns $k+1$ and higher, while $\lambda^2 = \lambda \setminus \lambda^1$.



For any partition λ , we have

$$\vartheta_\lambda(x; y) = \sum_{\alpha} q_{\lambda-\alpha}(x) e_{\alpha}(y),$$

the sum over all compositions α with $0 \leq \alpha_i \leq k$ for all i . If λ is k -strict, it follows that the homogeneous summand of ϑ_λ of lowest x -degree is equal to $q_{\lambda^1}(x) e_{\lambda^2}(y)$. The sets $\{q_\lambda(x) \mid \lambda \text{ strict}\}$ and $\{e_\lambda(y) \mid \lambda_i \leq k, \forall i\}$ are linearly independent over \mathbb{Z} . We deduce that the ϑ_λ , λ k -strict, are linearly independent over \mathbb{Z} .

Equation (19) also implies that, for $m > k$, $\vartheta_{2m} \in \mathbb{Q}[\vartheta_1, \dots, \vartheta_{2m-1}]$. By induction on m it follows that

$$\vartheta_{2m} \in \mathbb{Q}[\vartheta_1, \dots, \vartheta_{2k}, \vartheta_{2k+1}, \vartheta_{2k+3}, \dots, \vartheta_{2m-1}]$$

for all $m > k$. Let $\Gamma_{\mathbb{Q}}^{(k)} = \Gamma^{(k)} \otimes \mathbb{Q}$. Then $\Gamma_{\mathbb{Q}}^{(k)}$ is generated by the ϑ_r with all $r > 2k$ odd.

We say that a partition λ is k -odd if all its parts which are greater than $2k$ are odd. For each $m \geq 0$, the number of k -odd partitions of m is equal to the number of k -strict partitions of m , because of the equality of generating functions

$$\begin{aligned} \sum_{\lambda \text{ } k\text{-odd}} t^{|\lambda|} &= \prod_{r=1}^{2k} \frac{1}{1-t^r} \prod_{r>k} \frac{1}{1-t^{2r-1}} = \prod_r \frac{1}{1-t^r} \prod_{r>k} (1-t^{2r}) \\ &= \prod_{r=1}^k \frac{1}{1-t^r} \prod_{r>k} (1+t^r) = \sum_{\lambda \text{ } k\text{-strict}} t^{|\lambda|}. \end{aligned}$$

We therefore have proved the following result.

Proposition 5.1. (i) *The ϑ_λ for λ k -strict form a \mathbb{Z} -basis of $\Gamma^{(k)}$.*

(ii) *The ϑ_λ for λ k -odd form a \mathbb{Q} -basis of $\Gamma_{\mathbb{Q}}^{(k)}$.*

5.2.

Definition 5.2. For any valid set of pairs $D \subset \Delta^\circ$ and integer sequence λ , define the polynomial $\Theta(D, \lambda)$ by the raising operator formula $\Theta(D, \lambda) = R^D \vartheta_\lambda$. Equivalently, we recursively set

$$\Theta(D, \lambda) = \sum_{\alpha} (-1)^{|\alpha|} 2^{m(D, \alpha, \ell)} \Theta(D, \mu + \alpha) \vartheta_{r - |\alpha|},$$

where $\lambda = (\mu, r)$ has length ℓ and the sum is over all (D, ℓ) -compatible vectors $\alpha \in \mathbb{N}^{\ell-1}$. For any k -strict partition λ , the *theta polynomial* $\Theta_\lambda(x; y)$ is defined by $\Theta_\lambda = \Theta(\mathcal{C}(\lambda), \lambda) = R^\lambda \vartheta_\lambda$.

It follows from the raising operator definition that each Θ_λ is of the form

$$\Theta_\lambda = \vartheta_\lambda + \sum_{\mu \succ \lambda} c_{\lambda\mu} \vartheta_\mu$$

with coefficients $c_{\lambda\mu} \in \mathbb{Z}$. We deduce that we have

$$\Theta_\lambda = \vartheta_\lambda + \sum_{\mu \succ \lambda} c'_{\lambda\mu} \vartheta_\mu$$

where now the sum on the right is restricted to k -strict partitions $\mu \succ \lambda$. Since the latter form a \mathbb{Z} -basis of $\Gamma^{(k)}$ by Proposition 5.1, it follows that the Θ_λ , as λ runs over k -strict partitions, form a \mathbb{Z} -basis of $\Gamma^{(k)}$.

Let

$$\mathbb{H}(\text{IG}_k) = \varprojlim \mathbb{H}^*(\text{IG}(n - k, 2n), \mathbb{Z})$$

be the stable cohomology ring of IG ; that is, the inverse limit in the category of *graded* rings of the system

$$\cdots \leftarrow \mathbb{H}^*(\text{IG}(n - k, 2n), \mathbb{Z}) \leftarrow \mathbb{H}^*(\text{IG}(n + 1 - k, 2n + 2), \mathbb{Z}) \leftarrow \cdots$$

From the presentation of $\mathbb{H}^*(\text{IG}(n - k, 2n), \mathbb{Z})$ given in [BKT1, Thm. 1.2], we deduce that $\mathbb{H}(\text{IG}_k)$ is isomorphic to the polynomial ring $\mathbb{Z}[\sigma_1, \sigma_2, \dots]$ modulo the relations

$$\sigma_m^2 + 2 \sum_{i=1}^m (-1)^i \sigma_{m+i} \sigma_{m-i} = 0$$

for all $m > k$. Since the generators ϑ_r of $\Gamma^{(k)}$ satisfy (20), we have a homomorphism $\phi: \mathbb{H}(\text{IG}_k) \rightarrow \Gamma^{(k)}$ sending σ_r to ϑ_r for each r . Theorem 1 implies that $\phi(\sigma_\lambda) = \Theta_\lambda$ for any k -strict partition λ . Since the Θ_λ form a basis of $\Gamma^{(k)}$, we conclude that ϕ is an isomorphism. This completes the proof of Theorem 2.

5.3. Consider the analogues of the polynomials ϑ_r when the $e_r(y)$ are replaced by complete symmetric functions $h_r(y)$. Define for each r a function $\widehat{\vartheta}_r = \widehat{\vartheta}_r(x; y)$ by

$$\widehat{\vartheta}_r = \sum_i q_{r-i}(x) h_i(y)$$

and set $\widehat{\Theta}(t) = \sum_{r \geq 0} \widehat{\vartheta}_r t^r$. We then have $\Theta(t) \widehat{\Theta}(-t) = 1$, or equivalently,

$$(21) \quad \sum_{r=0}^n (-1)^r \vartheta_r \widehat{\vartheta}_{n-r} = 0, \quad n \geq 1.$$

The equations (21) imply that for any partitions λ and μ with $\mu \subset \lambda$,

$$\det(\vartheta_{\lambda_i - \mu_j + j - i}) = \det(\widehat{\vartheta}_{\lambda'_i - \mu'_j + j - i}),$$

and in particular,

$$(22) \quad \det(\vartheta_{\lambda_i + j - i}) = \det(\widehat{\vartheta}_{\lambda'_i + j - i}).$$

Here λ' is the partition conjugate to λ , i.e., $\lambda'_i = \#\{h \mid \lambda_h \geq i\}$ for all i .

Assume that $k > 0$, and let (1^r) denote the partition $(1, \dots, 1)$ of length r . We claim that for any $r \geq 1$, we have $\Theta_{(1^r)}(x; y) = \widehat{\vartheta}_r(x; y)$. Indeed, note that $\mathcal{C}(1^r) = \emptyset$, and hence (22) gives

$$\Theta_{(1^r)} = \prod_{i < j} (1 - R_{ij}) \vartheta_{(1^r)} = \det(\vartheta_{1+j-i})_{1 \leq i, j \leq r} = \widehat{\vartheta}_r.$$

The polynomials $\widehat{\vartheta}_r = \Theta_{(1^r)}$ map to the Chern classes of the dual of the tautological subbundle $\mathcal{S} \rightarrow \text{IG}$ under the isomorphism ϕ of §5.2.

5.4. We next introduce an analogue of the Schur S -functions in the ring $\Gamma^{(k)}$.

Definition 5.3. For any two finite integer sequences λ, μ , define the function $S_{\lambda/\mu}^{(k)} \in \Gamma^{(k)}$ by setting

$$S_{\lambda/\mu}^{(k)}(x; y) = \det(\vartheta_{\lambda_i - \mu_j + j - i}(x; y))_{i, j}.$$

Assume that λ and μ are two partitions. Then, arguing as in [M, I.5], the skew function $S_{\lambda/\mu}^{(k)}(x; y)$ is zero unless $\lambda_i \geq \mu_i$ for each i , in which case it depends only on the skew diagram $\lambda - \mu$. The functions $S_{\lambda/\mu}(x) := S_{\lambda/\mu}^{(0)}(x; y)$ are well known (see [M, III.8, Example 7] and [W, Sec. 2.7]). We also let

$$s_{\lambda'/\mu'}(y) = \det(e_{\lambda_i - \mu_j + j - i}(y))_{i, j}$$

denote the (ordinary) skew Schur polynomial in the variables y . We have that $s_{\lambda'/\mu'}(y) = 0$ unless $0 \leq \lambda_i - \mu_i \leq k$ for each i . The functions $S_{\lambda/\mu}(x)$ (respectively, $s_{\lambda'/\mu'}(y)$) are known to be linear combinations of Schur Q -functions $Q_\nu(x)$ (respectively, Schur S -polynomials $s_{\nu'}(y)$) with positive integer coefficients.

Proposition 5.2. For any partitions λ, μ with $\mu \subset \lambda$, we have

$$(23) \quad S_{\lambda/\mu}^{(k)}(x; y) = \sum_{\nu} S_{\lambda/\nu}(x) s_{\nu'/\mu'}(y) = \sum_{\nu} S_{\nu/\mu}(x) s_{\lambda'/\nu'}(y)$$

summed over all partitions ν such that $\mu \subset \nu \subset \lambda$.

Proof. Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots)$ be another countably infinite set of variables and define the ring $\tilde{\Lambda} = \mathbb{Z}[e_1(\tilde{x}), e_2(\tilde{x}), \dots] \otimes_{\mathbb{Z}} \mathbb{Z}[e_1(y), \dots, e_k(y)]$. Consider the ring homomorphism $\epsilon : \tilde{\Lambda} \rightarrow \Gamma^{(k)}$ defined by sending $e_i(\tilde{x})$ to $q_i(x)$ and $e_j(y)$ to $e_j(y)$. According to [M, I.(5.10)], the identity

$$s_{\lambda'/\mu'}(\tilde{x}, y) = \sum_{\nu} s_{\lambda'/\nu'}(\tilde{x}) s_{\nu'/\mu'}(y) = \sum_{\nu} s_{\nu'/\mu'}(\tilde{x}) s_{\lambda'/\nu'}(y)$$

holds in $\tilde{\Lambda}$. This is mapped to (23) under the homomorphism ϵ . \square

The definition of $S_\lambda^{(k)}$ implies that $S_\lambda^{(k)} = \vartheta_\lambda + \sum_{\mu \succ \lambda} d_{\lambda\mu} \vartheta_\mu$ for some integers $d_{\lambda\mu}$, and therefore that the set of $S_\lambda^{(k)}$ for λ k -strict forms another \mathbb{Z} -basis of $\Gamma^{(k)}$. The next result follows immediately from the definitions.

Proposition 5.3. *For any k -strict partition λ , we have*

$$\Theta_\lambda(x; y) = \prod_{(i,j) \in \mathcal{C}(\lambda)} (1 - R_{ij} + R_{ij}^2 - \cdots) S_\lambda^{(k)}(x; y).$$

5.5. In this section, we give the proof of Theorem 3. Let λ be a k -strict partition. Note that if $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, then $\mathcal{C}(\lambda) = \emptyset$, and hence $\Theta_\lambda = S_\lambda^{(k)}$. Part (a) of the theorem then follows by setting $\mu = 0$ in (23). Observe that in this case we also have

$$\Theta_\lambda(x; y) = \sum_{\mu, \nu} c_{\mu\nu}^\lambda S_\mu(x) s_{\nu'}(y) = \sum_{\mu \subset \lambda} S_{\lambda/\mu}(x) s_{\mu'}(y),$$

where $c_{\mu\nu}^\lambda$ is a Littlewood-Richardson coefficient.

Suppose now that the partition λ is such that $\lambda_i + \lambda_j > 2k + j - i$ for all $i < j \leq \ell(\lambda)$; in particular, λ is *strict*. For $\ell = \ell(\lambda)$ and $\epsilon_\ell = (0, 1, \dots, \ell - 1)$, we define the shifted skew shape

$$\mathcal{S}(\lambda/\mu) = (\lambda + \epsilon_\ell)/(\mu + \epsilon_\ell)$$

for any strict partition $\mu \subset \lambda$.

We claim that

$$(24) \quad \Theta_\lambda = \sum_{\alpha} Q_{\lambda-\alpha}(x) e_\alpha(y),$$

where the sum runs over all compositions α with $0 \leq \alpha_i \leq k$ for each i . Indeed we have

$$\Theta_\lambda = R^\lambda \vartheta_\lambda(x; y) = \sum_{\alpha} e_\alpha(y) R^\lambda q_{\lambda-\alpha}(x),$$

where the operator R^λ is acting on the partition λ . For a fixed composition α , the effect of the Schur Pfaffian operator R^λ on the term $q_{\lambda-\alpha}(x)$ is to convert it to the term $Q_{\lambda-\alpha}(x)$. Equation (24) follows.

Since λ is strict and $\lambda_{\ell-1} + \lambda_\ell > 2k + 1$, we see that $\lambda_i > \alpha_i$ for all compositions α indexing the sum (24) and every i except possibly $i = \ell$. If $\lambda_\ell < \alpha_\ell$ then $Q_{\lambda-\alpha} = 0$; therefore all non-vanishing terms in the sum are indexed by (nonnegative) compositions. If any index has a repeated part, the Q -function again vanishes. Let $\delta_\ell = (\ell - 1, \ell - 2, \dots, 1, 0)$ and $b = \delta_\ell + \epsilon_\ell = (\ell - 1)^\ell$. It follows that we may rewrite (24) as

$$\Theta_\lambda = \sum_{\mu} \sum_{w \in S_\ell} (-1)^w Q_\mu(x) e_{\lambda-w(\mu)}(y)$$

summed over strict partitions μ of length at most ℓ . For each $w \in S_n$, we have

$$\lambda - w(\mu) = \lambda + b - w(\mu + b) = (\lambda + \epsilon_\ell) + \delta_\ell - w((\mu + \epsilon_\ell) + \delta_\ell).$$

Therefore

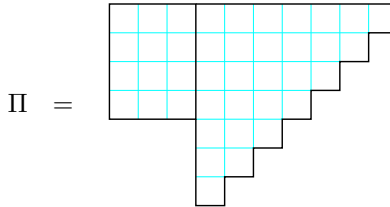
$$\Theta_\lambda = \sum_{\mu} Q_\mu(x) \sum_{w \in S_\ell} (-1)^w e_{(\lambda + \epsilon_\ell) + \delta_\ell - w((\mu + \epsilon_\ell) + \delta_\ell)}(y),$$

the sum over all strict partitions $\mu \subset \lambda$. The latter sum is equal to the one in the statement of the theorem. Note that the skew Schur function $s_{\mathcal{S}(\lambda/\mu)}(y)$ vanishes unless μ has length at least $\ell(\lambda) - 1$.

6. SCHUBERT POLYNOMIALS FOR ISOTROPIC GRASSMANNIANS

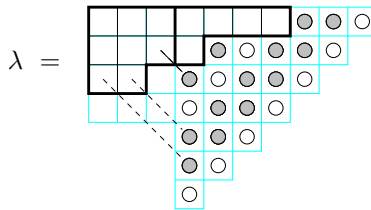
6.1. The polynomials $\Theta_\lambda(x; y)$ fall within the Billey-Haiman theory of type C Schubert polynomials $\mathfrak{C}_w(x, z)$. We will prove and discuss this in detail in this section. Let W_n be the hyperoctahedral group of signed permutations on the set $\{1, \dots, n\}$, and $W_\infty = \cup_n W_n$. The group W_∞ is generated by the simple transpositions $s_i = (i, i + 1)$ for $i > 0$, and the sign change $s_0(1) = \bar{1}$. The elements of W_n index the Schubert classes in the cohomology ring of the flag variety Sp_{2n}/B , which includes $H^*(\mathrm{IG}(n - k, 2n), \mathbb{Z})$ as a subring. In particular every k -strict partition $\lambda \in \mathcal{P}(k, n)$ corresponds to a *Grassmannian element* $w_\lambda \in W_n$, which we proceed to describe; the reader may consult [T, §4] for more details.

The elements of $\mathcal{P}(k, n)$ are exactly the k -strict partitions whose diagrams fit inside a shape Π , obtained by attaching an $m \times k$ rectangle to the left side of a staircase partition with n rows. When $n = 7$ and $k = 3$, this looks as follows.



The signed permutation $w_\lambda = (w_1, \dots, w_n)$ has a unique descent at k , that is, $w(i) < w(i + 1)$ whenever $i \neq k$. For $\lambda \in \mathcal{P}(k, n)$ we let λ^1 be the strict partition formed by the boxes of λ in columns $k + 1$ through $k + n$. The negative entries of w_λ are then given by the parts of λ^1 .

The boxes of the staircase partition which are outside λ are organized into south-west to north-east diagonals. The k diagonals which are k -related to one of the bottom boxes in the first k columns of λ are called *related*; the remaining diagonals are non-related. The first k entries of w_λ are the lengths of the related diagonals, and the last $n - k - \ell_k(\lambda)$ entries are the lengths of the non-related diagonals. For example, the partition $\lambda = (7, 4, 2) \in \mathcal{P}(3, 7)$ results in the element $w_\lambda = 2564\bar{1}37$.



6.2. A sequence $a = (a_1, \dots, a_m)$ is called *unimodal* if for some r with $0 \leq r \leq m$, we have

$$a_1 > a_2 > \dots > a_r < a_{r+1} < \dots < a_m.$$

Let $w \in W_\infty$ and λ be a Young diagram with r rows such that $|\lambda| = \ell(w)$. A *Kraškiewicz tableau* [Kr] for w of shape λ is a filling T of the boxes of λ with nonnegative integers in such a way that (a) if t_i is the sequence of entries in the i -th row of T , reading from left to right, then the row word $t_r \dots t_1$ is a reduced word for w ; and (b) for each i , t_i is a unimodal subsequence of maximum length in $t_r \dots t_{i+1} t_i$.

For each $w \in W_\infty$ one has a *type C Stanley symmetric function* $F_w(x)$, which is a positive linear combination of Schur Q -functions. There exist several combinatorial interpretations for the coefficients in this expression. We will use the following result of Lam [L]:

$$(25) \quad F_w(x) = \sum_{\lambda} e_w^\lambda Q_\lambda(x)$$

where e_w^λ equals the number of Kraśkiewicz tableaux for w of shape λ .

Example 6.1. Suppose that $k = 0$ and $w = w_\lambda$ is a maximal (Lagrangian) Grassmannian element corresponding the strict partition λ . In this case there is only one Kraśkiewicz tableau for w , which has shape λ , and is given as in the following example, for $\lambda = (6, 5, 2)$:

$$\begin{array}{cccccc} 5 & 4 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 & \\ 1 & 0 & & & & \end{array}$$

It follows that $F_{w_\lambda}(x) = Q_\lambda(x)$, for all such λ .

Following Billey and Haiman, each $w \in W_\infty$ indexes a type C Schubert polynomial $\mathfrak{C}_w(x, z)$. Here $z = (z_1, z_2, \dots)$ is another infinite set of variables and each \mathfrak{C}_w is a polynomial in the ring $A = \mathbb{Z}[q_1(x), q_2(x), \dots; z_1, z_2, \dots]$. The \mathfrak{C}_w for $w \in W_\infty$ form a \mathbb{Z} -basis of A , and their algebra agrees with the Schubert calculus on symplectic flag varieties Sp_{2n}/B , when n is sufficiently large. According to [BH, Thm. 3], for any $w \in W_n$ we have

$$(26) \quad \mathfrak{C}_w(x, z) = \sum_{uv=w} F_u(x) \mathfrak{S}_v(z),$$

the sum over all reduced factorizations $uv = w$ in W_n (i.e., such that $\ell(u) + \ell(v) = \ell(w)$) with $v \in S_n$. Here $\mathfrak{S}_v(z)$ denotes the type A Schubert polynomial of Lascoux and Schützenberger [LS].

6.3. We will show next that the theta polynomial ϑ_r agrees with the Billey-Haiman polynomial indexed by the Grassmannian permutation $w_{(r)} \in W_n$ corresponding to $\lambda = r$, for $1 \leq r \leq n + k$. It is easy to see that there is a unique reduced word for any such element; these $n + k$ words are listed below.

$$s_k, s_{k-1}s_k, \dots, s_1 \cdots s_k, s_0 s_1 \cdots s_k, s_1 s_0 s_1 \cdots s_k, \dots, s_{n-1} \cdots s_1 s_0 s_1 \cdots s_k.$$

Set $v_i = s_i s_{i+1} \cdots s_k$. For any reduced factorization $w_{(r)} = uv$ with $v \in S_n$, it is immediate that $v = v_i$ for some $i > 0$. The type A Schubert polynomial for v_i is exactly the elementary symmetric polynomial $e_{k+1-i}(z_1, \dots, z_k)$, for $1 \leq i \leq k$. Moreover, if $u_i = w_{(r)} v_i^{-1}$, then (25) implies that $F_{u_i}(x) = q_{r+i-k-1}(x)$. We conclude from (26) that

$$\mathfrak{C}_{w_{(r)}}(x, z) = \sum_{j \geq 0} q_{r-j}(x_1, x_2, \dots) e_j(z_1, \dots, z_k) = \vartheta_r(x, z),$$

as required. Since both the theta and Billey-Haiman polynomials form a basis for the ring that they span, we deduce the following result.

Corollary 6.1. *The ring $\Gamma^{(k)}$ of theta polynomials is a subring of the ring of Billey-Haiman Schubert polynomials of type C. For every k -strict partition λ , we have $\Theta_\lambda(x; y) = \mathfrak{C}_{w_\lambda}(x, y)$.*

We have shown that for every k -strict λ , there is a unique expression

$$(27) \quad \Theta_\lambda(x; y) = \sum_{uv=w_\lambda} F_u(x) \mathfrak{S}_v(y),$$

the sum over all reduced factorizations $uv = w_\lambda$ in W_∞ with $v \in S_\infty$. The right factor v in any such factorization must be a Grassmannian permutation with unique descent at k . Hence each Schubert polynomial $\mathfrak{S}_v(y)$ will be symmetric in y , and therefore equal to a Schur polynomial $s_{\nu'}(y)$ for some partition ν' ; in fact, one checks easily that $\nu \subset \lambda^2$. The coefficient of $Q_\mu(x) s_{\nu'}(y)$ in (27) is equal to the number of Kraškievich tableaux for $w_\lambda v^{-1}$ of shape μ . This completes the proof of Theorem 4.

Corollary 6.2. a) *For any k -strict partition λ , the homogeneous summand of $\Theta_\lambda(x; y)$ of highest x -degree is the type C Stanley symmetric function $F_{w_\lambda}(x)$, and satisfies $F_{w_\lambda}(x) = R^\lambda q_\lambda(x)$.*

b) *The homogeneous summand of $\Theta_\lambda(x; y)$ of lowest x -degree is $Q_{\lambda^1}(x) s_{(\lambda^2)'}(y)$.*

Proof. Part (a) is deduced by setting $y = 0$ in (27) and also in the raising operator expression $\Theta_\lambda(x; y) = R^\lambda \vartheta_\lambda(x; y)$. For part (b), notice that there is a unique permutation $v \in S_\infty$ of maximal length such that w_λ has a reduced factorization $w_\lambda = uv$. In this case u is the maximal Grassmannian Weyl group element corresponding to the strict partition λ^1 . The result now follows, using Example 6.1. \square

Example 6.2. Let $k = 1$ and $\lambda = (3, 2, 1)$, with corresponding Weyl group element $w_\lambda = 4\overline{2}13 \in W_4$. Then we have

$$(28) \quad \Theta_{321} = (Q_{42} + Q_{321}) + (Q_{41} + 2Q_{32}) s_{1'} + 2Q_{31} s_{11'} + Q_{21} s_{111'}$$

(with the variables x and y omitted). The Kraškievich tableaux which correspond to the summands on the right hand side of (28) are given in the following table.

u	v	Kraškievich tableaux for u						
$4\overline{2}13$	1234	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border-right: 1px solid black; padding: 2px 10px;">3201</td><td style="padding: 2px 10px;">321</td></tr> <tr><td style="border-right: 1px solid black; padding: 2px 10px;">01</td><td style="padding: 2px 10px;">10</td></tr> <tr><td style="border-right: 1px solid black; padding: 2px 10px;"></td><td style="padding: 2px 10px;">0</td></tr> </table>	3201	321	01	10		0
3201	321							
01	10							
	0							
$\overline{2}4\overline{1}3$	2134	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border-right: 1px solid black; padding: 2px 10px;">3102</td><td style="padding: 2px 10px;">320</td><td style="padding: 2px 10px;">302</td></tr> <tr><td style="border-right: 1px solid black; padding: 2px 10px;">0</td><td style="padding: 2px 10px;">01</td><td style="padding: 2px 10px;">01</td></tr> </table>	3102	320	302	0	01	01
3102	320	302						
0	01	01						
$\overline{2}143$	3124	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border-right: 1px solid black; padding: 2px 10px;">310</td><td style="padding: 2px 10px;">103</td></tr> <tr><td style="border-right: 1px solid black; padding: 2px 10px;">0</td><td style="padding: 2px 10px;">0</td></tr> </table>	310	103	0	0		
310	103							
0	0							
$\overline{2}134$	4123	<table style="display: inline-table; border-collapse: collapse;"> <tr><td style="border-right: 1px solid black; padding: 2px 10px;"></td><td style="padding: 2px 10px;">10</td></tr> <tr><td style="border-right: 1px solid black; padding: 2px 10px;"></td><td style="padding: 2px 10px;">0</td></tr> </table>		10		0		
	10							
	0							

6.4. We remark that to obtain polynomials that multiply like the Schubert classes on the orthogonal Grassmannians $\text{OG}(n - k, 2n + 1)$, one simply uses the $2^{-\ell_k(\lambda)} \Theta_\lambda$ for all k -strict λ . These polynomials agree with the Billey-Haiman Schubert polynomials of type B indexed by Grassmannian elements w_λ . For the even orthogonal Grassmannians $\text{OG}(n - k, 2n)$, both the Giambelli formula and the corresponding family of polynomials are more involved; we plan to develop this theory elsewhere.

REFERENCES

- [BS] N. Bergeron and F. Sottile : *A Pieri-type formula for isotropic flag manifolds*, Trans. Amer. Math. Soc. **354** (2002), 4815–4829.

- [BH] S. Billey and M. Haiman : *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), 443–482.
- [BKT1] A. S. Buch, A. Kresch, and H. Tamvakis : *Quantum Pieri rules for isotropic Grassmannians*, Preprint (2008), available at arXiv:0809.4966.
- [BKT2] A. S. Buch, A. Kresch, and H. Tamvakis : *Quantum Giambelli formulas for isotropic Grassmannians*, Preprint (2008).
- [C] A. L. Cauchy : *Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérés entre les variables qu’elles renferment*, J. École Polyt. **10** (1815), 29–112; Oeuvres, ser. 2, vol. 1, 91–169.
- [FK] S. Fomin and A. N. Kirillov : *Combinatorial B_n -analogs of Schubert polynomials*, Trans. Amer. Math. Soc. **348** (1996), 3591–3620.
- [G] G. Z. Giambelli : *Risoluzione del problema degli spazi secanti*, Mem. R. Accad. Sci. Torino (2) **52** (1902), 171–211.
- [J] C. G. J. Jacobi : *De functionibus alternantibus earumque divisione per productum e differentibus elementorum conflatum*, J. reine angew. Math. **22** (1841), 360–371. Reprinted in Gesammelte Werke **3**, 439–452, Chelsea, New York, 1969.
- [Kr] W. Kraśkiewicz : *Reduced decompositions in hyperoctahedral groups*, C. R. Acad. Sci. Paris Sér. I Math. **309** (1989), 903–907.
- [L] T. K. Lam : *B_n Stanley symmetric functions*, Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994). Discrete Math. **157** (1996), 241–270.
- [LS] A. Lascoux and M.-P. Schützenberger : *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450.
- [M] I. Macdonald : *Symmetric Functions and Hall Polynomials*, Second edition, Clarendon Press, Oxford, 1995.
- [Pra] P. Pragacz : *Algebra-geometric applications of Schur S - and Q -polynomials*, Séminaire d’Algèbre Dubreil-Malliavin 1989-1990, Springer Lecture Notes in Math. 1478 (1991), 130–191.
- [PR] P. Pragacz and J. Ratajski : *A Pieri-type theorem for Lagrangian and odd orthogonal Grassmannians*, J. reine angew. Math. **476** (1996), 143–189.
- [S] I. Schur : *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.
- [T] H. Tamvakis : *Quantum cohomology of isotropic Grassmannians*, Geometric Methods in Algebra and Number Theory, 311–338, Progress in Math. 235, Birkhäuser, 2005.
- [W] D. R. Worley : *A theory of shifted Young tableaux*, Ph.D. thesis, MIT, 1984.
- [Y] A. Young : *On quantitative substitutional analysis VI*, Proc. Lond. Math. Soc. (2) **34** (1932), 196–230.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854, USA

E-mail address: asbuch@math.rutgers.edu

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT ZÜRICH, WINTERTHURERSTRASSE 190, CH-8057 ZÜRICH, SWITZERLAND

E-mail address: andrew.kresch@math.uzh.ch

UNIVERSITY OF MARYLAND, DEPARTMENT OF MATHEMATICS, 1301 MATHEMATICS BUILDING, COLLEGE PARK, MD 20742, USA

E-mail address: harryt@math.umd.edu