QUANTUM GIAMBELLI FORMULAS FOR ISOTROPIC GRASSMANNIANS

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Abstract. Let X be a symplectic or odd orthogonal Grassmannian which parametrizes isotropic subspaces in a vector space equipped with a nondegenerate (skew) symmetric form. We prove quantum Giambelli formulas which express an arbitrary Schubert class in the small quantum cohomology ring of X as a polynomial in certain special Schubert classes, extending the cohomological Giambelli formulas of [BKT2].

0. Introduction

Let E be an even (respectively, odd) dimensional complex vector space equipped with a nondegenerate skew-symmetric (respectively, symmetric) bilinear form. Let X denote the Grassmannian which parametrizes the isotropic subspaces of E. The cohomology ring $\mathrm{H}^*(X, \mathbb{Z})$ is generated by certain special Schubert classes, which for us are (up to a factor of two) the Chern classes of the universal quotient vector bundle over X. These special classes also generate the small quantum cohomology ring $\mathrm{QH}(X)$, a q-deformation of $\mathrm{H}^*(X, \mathbb{Z})$ whose structure constants are given by the three point, genus zero Gromov-Witten invariants of X. In [BKT2], we proved a Giambelli formula in $\mathrm{H}^*(X, \mathbb{Z})$, that is, a formula expressing a general Schubert class as an explicit polynomial in the special classes. Our goal in the present work is to extend this result to a formula that holds in $\mathrm{QH}(X)$.

The quantum Giambelli formula for the usual type A Grassmannian was obtained by Bertram [Be], and is in fact identical to the classical Giambelli formula. In the case of maximal isotropic Grassmannians, the corresponding questions were answered in [KT1, KT2]. The main conclusions here are similar to those of loc. cit., provided that one uses the raising operator Giambelli formulas of [BKT2] as the classical starting point. For an odd orthogonal Grassmannian, we prove that the quantum Giambelli formula is the same as the classical one. The result is more interesting when X is the Grassmannian $\mathrm{IG}(n-k, 2n)$ parametrizing $(n-k)$-dimensional isotropic subspaces of a symplectic vector space E of dimension $2n$. Our theorem in this case states that the quantum Giambelli formula for $\mathrm{IG}(n-k, 2n)$ coincides with the classical Giambelli formula for $\mathrm{IG}(n+1-k, 2n+2)$, provided that the special Schubert class σ_{n+k+1} is replaced with $q/2$. In a sequel to this paper, we will discuss the classical and quantum Giambelli formulas for even orthogonal Grassmannians.

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1. Preliminary Results

1.1. Choose $k \geq 0$ and consider the Grassmannian $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ of isotropic $(n-k)$-dimensional subspaces of $\mathbb{C}^{2n}$, equipped with a symplectic form. A partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_\ell)$ is *k-strict* if all of its parts greater than k are distinct integers. Following [BKT1], the Schubert classes on IG are parametrized by the k-strict partitions whose diagrams fit in an $(n-k) \times (n+k)$ rectangle; we denote the set of all such partitions by $\mathcal{P}(k,n)$. Given any partition $\lambda \in \mathcal{P}(k,n)$ and a complete flag of subspaces

$$F_\bullet \,:\, 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = \mathbb{C}^{2n}$$

such that $F_{n+i} = F_{n-i}^{\perp}$ for $0 \leq i \leq n$, we have a Schubert variety

$$X_\lambda(F_\bullet) := \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \;\; \forall\, 1 \leq j \leq \ell(\lambda)\}\,,$$

where $\ell(\lambda)$ denotes the number of (non-zero) parts of λ and

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j : \lambda_i + \lambda_j > 2k + j - i\}.$$

This variety has codimension $|\lambda| = \sum \lambda_i$ and defines, via Poincaré duality, a Schubert class $\sigma_\lambda = [X_\lambda(F_\bullet)]$ in $\mathrm{H}^{2|\lambda|}(\mathrm{IG}, \mathbb{Z})$. The Schubert classes σ_λ for $\lambda \in \mathcal{P}(k,n)$ form a free $\mathbb{Z}$-basis for the cohomology ring of IG. The *special Schubert classes* are defined by $\sigma_r = [X_r(F_\bullet)] = c_r(\mathcal{Q})$ for $1 \leq r \leq n+k$, where $\mathcal{Q}$ denotes the universal quotient bundle over IG.

The classical Giambelli formula for IG is expressed using Young's *raising operators* [Y, p. 199]. We first agree that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r < 0$. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite support and $i < j$, we set $R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots)$; a raising operator R is any monomial in these R_{ij}'s. Define $m_\alpha = \prod_i \sigma_{\alpha_i}$ and $R\, m_\alpha = m_{R\alpha}$ for any raising operator R. For any k-strict partition λ, we consider the operator

$$R^\lambda = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j > 2k + j - i} (1 + R_{ij})^{-1}$$

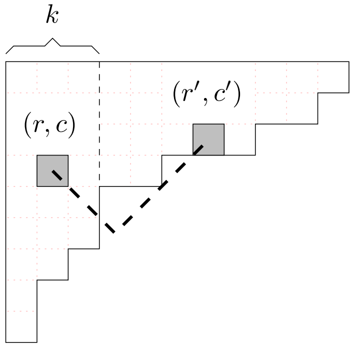
where the first product is over all pairs $i < j$ and second product is over pairs $i < j$ such that $\lambda_i + \lambda_j > 2k + j - i$. The main result of [BKT2] states that the *Giambelli formula*

$$\sigma_\lambda = R^\lambda\, m_\lambda \tag{1}$$

holds in the cohomology ring of $\mathrm{IG}(n-k, 2n)$.

1.2. As is customary, we will represent a partition by its Young diagram of boxes; this is used to define the containment relation for partitions. Given two diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ (i.e., the set-theoretic difference $\nu \smallsetminus \mu$) is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row).

We say that the box $[r,c]$ in row r and column c of a k-strict partition λ is *k-related* to the box $[r',c']$ if $|c - k - 1| + r = |c' - k - 1| + r'$. For instance, the grey boxes in the following partition are k-related.



For any two k-strict partitions λ and μ, we write $\lambda \to \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

Let $\mathbb{A}$ denote the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ which are not mentioned in (1) or (2) above, and define $N(\lambda, \mu)$ to be the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$. Here two boxes are connected if they share at least a vertex. In [BKT1, Theorem 1.1] we proved that the Pieri rule

$$\sigma_p \cdot \sigma_\lambda = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda,\mu)}\, \sigma_\mu \tag{2}$$

holds in $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$, for any $p \in [1, n+k]$.

1.3. In the following sections we will work in the stable cohomology ring $\mathbb{H}(\mathrm{IG}_k)$, which is the inverse limit in the category of graded rings of the system

$$\cdots \leftarrow \mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Z}) \leftarrow \mathrm{H}^*(\mathrm{IG}(n+1-k, 2n+2), \mathbb{Z}) \leftarrow \cdots$$

The ring $\mathbb{H}(\mathrm{IG}_k)$ has a free $\mathbb{Z}$-basis of Schubert classes σ_λ, one for each k-strict partition λ, and may be presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \sigma_2, \ldots]$ modulo the relations

$$\sigma_r^2 + 2\sum_{i=1}^{r} (-1)^i \sigma_{r+i}\sigma_{r-i} = 0 \quad \text{for } r > k. \tag{3}$$

There is a natural surjective ring homomorphism $\mathbb{H}(\mathrm{IG}_k) \to \mathrm{H}(\mathrm{IG}(n-k, 2n), \mathbb{Z})$ that maps σ_λ to σ_λ, when $\lambda \in \mathcal{P}(k, n)$, and to zero, otherwise. The Giambelli formula (1) and Pieri rule (2) are both valid in $\mathbb{H}(\mathrm{IG}_k)$. We begin with some elementary consequences of these theorems.

For any k-strict partition λ of length ℓ, we define the sets of pairs

$$\mathcal{A}(\lambda) = \{(i, j) \mid \lambda_i + \lambda_j \leq 2k + j - i \text{ and } 1 \leq i < j \leq \ell\}$$

$$\mathcal{C}(\lambda) = \{(i,j) \mid \lambda_i + \lambda_j > 2k + j - i \text{ and } 1 \leq i < j \leq \ell\}$$

and two integer vectors $a = (a_1, \ldots, a_\ell)$ and $c = (c_1, \ldots, c_\ell)$ by setting

$$a_i = \#\{j \mid (i,j) \in \mathcal{A}(\lambda)\}, \quad c_i = \#\{j \mid (i,j) \in \mathcal{C}(\lambda)\}$$

for each i.

Proposition 1. *We have $\lambda_i - c_i \geq \lambda_j - c_j$ for each $i < j \leq \ell$.*

Proof. Observe that the desired inequality is equivalent to

$$\lambda_i - \lambda_j \geq \#\{r \leq \ell \mid (i,r) \in \mathcal{C}(\lambda)\} - \#\{r \leq \ell \mid (j,r) \in \mathcal{C}(\lambda)\}. \tag{4}$$

Let $j = i + r$ and let s (respectively t) be maximal such that $(i,s) \in \mathcal{C}(\lambda)$ (respectively, $(j,t) \in \mathcal{C}(\lambda)$). Assume first that t exists, hence s exists and $s \geq t$. The inequality (4) then becomes $\lambda_i - \lambda_{i+r} \geq s - t + r$. We have

$$\lambda_i + \lambda_s \geq 2k + 1 + s - i \quad \text{and} \quad \lambda_{i+r} + \lambda_{t+1} \leq 2k + t + 1 - i - r,$$

hence

$$\lambda_i - \lambda_{i+r} \geq s - t + r + (\lambda_{t+1} - \lambda_s).$$

If $t < s$, then $\lambda_{t+1} \geq \lambda_s$ and we are done. If $t = s$, we need to show that $\lambda_i - \lambda_{i+r} \geq r$. This is true because $(j, j+1) \in \mathcal{C}(\lambda)$ and λ is k-strict, hence $\lambda_i > \lambda_{i+1} > \cdots > \lambda_{i+r}$.

Next we assume that t does not exist, so that either $j = \ell$ or the pair $(j, j+1)$ lies in $\mathcal{A}(\lambda)$ and

$$\lambda_j + \lambda_{j+1} \leq 2k + 1. \tag{5}$$

If s does not exist, there is nothing to prove. We must show that $\lambda_i - \lambda_j \geq s - i$, knowing that $(i,s) \in \mathcal{C}(\lambda)$, that is,

$$\lambda_i + \lambda_s \geq 2k + 1 + s - i. \tag{6}$$

Assume first that $\lambda_s \geq \lambda_j$. If $\lambda_s > k$ then we have

$$\lambda_i > \lambda_{i+1} > \cdots > \lambda_s$$

and hence $\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq s - i$. Otherwise $\lambda_s \leq k$ and (6) gives

$$\lambda_i - \lambda_j \geq \lambda_i - \lambda_s \geq \lambda_i - k \geq s - i + 1 + (k - \lambda_s) \geq s - i.$$

Finally, suppose that $\lambda_s < \lambda_j$, so in particular $j + 1 \leq s$. Then (5) and (6) give

$$\begin{aligned} \lambda_i - \lambda_j \geq \lambda_i + (\lambda_{j+1} - 2k - 1) &\geq (2k + 1 + s - i - \lambda_s) + \lambda_{j+1} - 2k - 1 \\ &= (\lambda_{j+1} - \lambda_s) + (s - i) \geq s - i. \end{aligned}$$ □

Proposition 1 implies that for any λ, the composition $\lambda - c$ is a partition, while $\lambda + a$ is a strict partition.

Proposition 2. *For any k-strict partition λ, the Giambelli polynomial $R^\lambda\, m_\lambda$ for σ_λ involves only generators σ_p with $p \leq \lambda_1 + a_1 + \lambda_2 + a_2$.*

Proof. We have

$$R^\lambda\, m_\lambda = \prod_{1 \leq i < j \leq \ell} \frac{1 - R_{ij}}{1 + R_{ij}} \prod_{(i,j) \in \mathcal{A}(\lambda)} (1 + R_{ij})\, m_\lambda = \sum_{\nu \in N} \prod_{1 \leq i < j \leq \ell} \frac{1 - R_{ij}}{1 + R_{ij}}\, m_\nu$$

where N is the multiset of integer vectors defined by

$$N = \left\{ \prod_{(i,j)\in S} R_{ij}\,\lambda \;\;\middle|\;\; S \subset \mathcal{A}(\lambda) \right\}.$$

If $m > 0$ is the least integer such that $2m \geq \ell$, then we have

$$\prod_{1\leq i<j\leq m} \frac{1-R_{ij}}{1+R_{ij}} = \text{Pfaffian}\left(\frac{1-R_{ij}}{1+R_{ij}}\right)_{1\leq i,j\leq 2m}. \tag{7}$$

Equation (7) follows from Schur's classical identity [S, Sec. IX]

$$\prod_{1\leq i<j\leq 2m} \frac{x_i-x_j}{x_i+x_j} = \text{Pfaffian}\left(\frac{x_i-x_j}{x_i+x_j}\right)_{1\leq i,j\leq 2m}.$$

Note that each single entry in the Pfaffian (7) expands according to the formula

$$\frac{1-R_{12}}{1+R_{12}}\, m_{c,d} = \sigma_c\,\sigma_d - 2\,\sigma_{c+1}\,\sigma_{d-1} + 2\,\sigma_{c+2}\,\sigma_{d-2} - \cdots + (-1)^d\, 2\,\sigma_{c+d}.$$

By Proposition 1, we know that $\lambda + a = (\lambda_1 + a_1, \lambda_2 + a_2, \ldots)$ is a strict partition, hence $\lambda_i + a_i + \lambda_j + a_j \leq \lambda_1 + a_1 + \lambda_2 + a_2$ for any distinct i and j. Since we furthermore have $\nu_i \leq \lambda_i + a_i$, for any $\nu \in N$, the result follows. □

Corollary 1. *For any $\lambda \in \mathcal{P}(k,n)$ the stable Giambelli polynomial for σ_λ involves only special classes σ_p with $p \leq 2n + 2k - 1$.*

Lemma 1. *Let λ and ν be k-strict partitions such that $\nu_1 > \max(\lambda_1, \ell(\lambda) + 2k)$ and $p \geq 0$. Then the coefficient of σ_ν in the Pieri product $\sigma_p \cdot \sigma_\lambda$ is equal to the coefficient of $\sigma_{(\nu_1+1,\nu_2,\nu_3,\ldots)}$ in the product $\sigma_{p+1} \cdot \sigma_\lambda$.*

Proof. Let $c = \max(\lambda_1, \ell(\lambda)+2k)+1$. Observe that box $[1, c]$ belongs to a connected component of the subset $\mathbb{A}$ of $\nu \smallsetminus \lambda$ defined in §1.2 which extends all the way to the rightmost box of ν. The same statement is true for $(\nu_1 + 1, \nu_2, \nu_3, \ldots) \smallsetminus \lambda$, except that the component goes one box further to the right. The number of components of $\mathbb{A}$ which do not meet column $k+1$ in both cases is the same, hence the two Pieri coefficients are equal. □

Given any partition λ, we let $\lambda^* = (\lambda_2, \lambda_3, \ldots)$.

Proposition 3. *For any $\lambda \in \mathcal{P}(k,n)$, there exists a recursion formula of the form*

$$\sigma_\lambda = \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu\subset\lambda^*} a_{p,\mu}\,\sigma_p\,\sigma_\mu \tag{8}$$

with $a_{p,\mu} \in \mathbb{Z}$, valid in the stable cohomology ring $\mathbb{H}(\mathrm{IG}_k)$

Proof. The argument is done in two steps, the first one being a reduction step. We claim that it is enough to prove that there exists a nonnegative integer m such that $\sigma_{(\lambda_1+m,\lambda^*)}$ is a linear combination of $\sigma_p\,\sigma_\mu$ for $\lambda_1 + m \leq p \leq 2n + 2k - 1 + m$ and $\mu \subset \lambda^*$. Suppose that we know this, then let us try to obtain an expression for σ_λ.

If $\lambda_1 \geq \ell(\lambda) + 2k - 1$, and if we have an expression

$$\sigma_{(\lambda_1+m,\lambda^*)} = \sum_{p=\lambda_1+m}^{2n+2k-1+m} \sum_{\mu\subset\lambda^*} a_{p,\mu}\,\sigma_p\,\sigma_\mu \tag{9}$$

then we must have

$$\sigma_\lambda = \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu} a_{p+m,\mu}\, \sigma_p\, \sigma_\mu. \tag{10}$$

Indeed, upon applying the Pieri rule (2), the coefficient of σ_ν for ν with $\nu_1 > \lambda_1$ in each term in the sum (10) is equal to the coefficient of $\sigma_{(\nu_1+m,\nu_2,\ldots)}$ in the corresponding term in (9) by Lemma 1, and by (9) these sum to zero. It remains to consider $\nu_1 = \lambda_1$, i.e., $\nu = \lambda$, and the coefficient in this case is 1 since we must have $a_{\lambda_1+m,\lambda^*} = 1$.

If $\lambda_1 < \ell(\lambda) + 2k - 1$, then set $\lambda' = (n + k, \lambda^*)$. By the above case, we have a recursion

$$\sigma_{\lambda'} = \sum_{p=n+k}^{2n+2k-1} \sum_{\mu\subset\lambda^*} a_{p,\mu}\, \sigma_p\, \sigma_\mu$$

for some $a_{p,\mu} \in \mathbb{Z}$. Using Lemma 1, now, we deduce that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{n+k+\lambda_1-1} \sum_{\mu\subset\lambda^*} a_{p+n+k-\lambda_1,\mu}\, \sigma_p\, \sigma_\mu + \sum_{\nu} b_{\lambda\nu}\, \sigma_\nu$$

where $b_{\lambda\nu} \in \mathbb{Z}$ and the partitions ν in the second sum satisfy $\lambda_1 < \nu_1 \leq \ell(\lambda)+2k-1$ and $\nu^* \subset \lambda^*$. By decreasing induction on ν_1, we may assume that expressions for these σ_ν as linear combinations of $\sigma_p\, \sigma_\mu$ with $\nu_1 \leq p \leq 2n + 2k - 1$ and $\mu \subset \nu^*$ exist. This completes the proof of the claim.

In the second step, given $\lambda \in \mathcal{P}(k, n)$ and $m > |\lambda|$, we show that $\sigma_{(\lambda_1+m,\lambda^*)}$ is a linear combination of products $\sigma_p\, \sigma_\mu$ for $\lambda_1 + m \leq p \leq 2n + 2k - 1 + m$ and $\mu \subset \lambda^*$. This uses the following result.

Lemma 2. *Let P_r be the set of partitions μ with $|\mu| = r$, and let m be a positive integer. Then the $\mathbb{Z}$-linear map*

$$\phi \;:\; \bigoplus_{r=0}^{\lfloor\frac{m-1}{2}\rfloor} \bigoplus_{\mu\in P_r} \mathbb{Z} \to \mathbb{H}(\mathrm{IG}_k)$$

which, for given r and $\mu \in P_r$, sends the corresponding basis element to $\sigma_{m-r}\sigma_\mu$, is injective.

Proof. The image of ϕ is contained in the span of the $\sigma_{(m-r,\mu)}$ for $0 \leq r < \frac{m}{2}$ and μ in P_r. Observe that the linear map ϕ is represented by a block triangular matrix with diagonal matrices as the blocks along the diagonal. The lemma follows. ☐

There are two elementary ways to obtain a recursion formula for a given Schubert class. First, for any k-strict partition λ, the Pieri rule (2) gives

$$\sigma_\lambda = \sigma_{\lambda_1}\sigma_{\lambda^*} - \sum_{\substack{\mu_1>\lambda_1\\ \mu^*\subset\lambda^*}} d_{\lambda\mu}\, \sigma_\mu, \tag{11}$$

where the $d_{\lambda\mu} \in \mathbb{Z}$ and the sum is over partitions μ with $\mu_1 > \lambda_1$ and $\mu^* \subset \lambda^*$. We then apply the same prescription to each of the summands σ_μ in (11), and iterate this procedure. Finally, we obtain an expression

$$\sigma_\lambda = \sum_{p=\lambda_1}^{|\lambda|} \sum_{\mu\subset\lambda^*} a_{p,\mu}\, \sigma_p\, \sigma_\mu.$$

Second, consider the stable Giambelli formula

$$\sigma_\lambda = R^\lambda\, m_\lambda = \sum_\nu b_\nu\, m_\nu \tag{12}$$

in the ring $\mathbb{H}(\mathrm{IG}_k)$. By Proposition 2 we know that the integer vectors ν in (12) all satisfy $\nu_1 \leq \lambda_1 + a_1 + \lambda_2 + a_2$. Hence we have an equation

$$\sigma_\lambda = \sum_{p=\lambda_1}^{\lambda_1+a_1+\lambda_2+a_2} \sigma_p \sum_{\nu\,:\,\nu_1=p} b_\nu\, m_{\nu^*}.$$

For $\lambda \in \mathcal{P}(k,n)$, choose $m > |\lambda|$, and set $\lambda' = (\lambda_1 + m, \lambda^*)$. Consider the expressions obtained by the two methods described in the last paragraph applied to λ':

$$\sigma_{\lambda'} = \sum_{p=\lambda_1+m}^{|\lambda|+m} \sum_{\mu\subset\lambda^*} a_{p,\mu}\, \sigma_p\, \sigma_\mu$$

and

$$\sigma_{\lambda'} = \sum_{p=\lambda_1+m}^{2n+2k-1+m} \sum_{\mu\in P_{|\lambda|+m-p}} b_{p,\mu}\, \sigma_p\, \sigma_\mu.$$

By Lemma 2, we have $a_{p,\mu} = b_{p,\mu}$. Hence, in particular, $a_{p,\mu} = 0$ whenever $p > 2n + 2k - 1 + m$. Therefore we have a recursion formula (8) for $\sigma_{\lambda'}$, as desired. □

Remark. One can be more precise about the recursion formula (8) in the case when the k-strict partition $\lambda \in \mathcal{P}(k,n)$ satisfies $\lambda_1 \geq \ell(\lambda) + 2k - 1$. If the Pieri rule reads

$$\sigma_{\lambda_1} \cdot \sigma_{\lambda^*} = \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu\subset\lambda^*} 2^{n(p,\mu)}\, \sigma_{p,\mu}$$

then we have

$$\sigma_\lambda = \sum_{p=\lambda_1}^{2n+2k-1} \sum_{\mu\subset\lambda^*} (-1)^{p-\lambda_1}\, 2^{n(p,\mu)}\, \sigma_p\, \sigma_\mu.$$

This result is proved in [T].

2. Quantum Giambelli for $\mathrm{IG}(n-k, 2n)$

The quantum cohomology ring $\mathrm{QH}^*(\mathrm{IG})$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of the formal variable q here is $n + k + 1$. We begin by recalling the quantum Pieri rule of [BKT1]. This states that for any k-strict partition $\lambda \in \mathcal{P}(k,n)$ and integer $p \in [1, n+k]$, we have

$$\sigma_p \cdot \sigma_\lambda = \sum_{\lambda\to\mu} 2^{N(\lambda,\mu)}\, \sigma_\mu + \sum_{\lambda\to\nu} 2^{N(\lambda,\nu)-1}\, \sigma_{\nu^*}\, q \tag{13}$$

in the quantum cohomology ring of $\mathrm{IG}(n-k, 2n)$. The first sum in (13) is over partitions $\mu \in \mathcal{P}(k,n)$ such that $|\mu| = |\lambda| + p$, and the second sum is over partitions $\nu \in \mathcal{P}(k, n+1)$ with $|\nu| = |\lambda| + p$ and $\nu_1 = n + k + 1$.

We work now with rational coefficients and introduce an important tool: a ring homomorphism

$$\pi : \mathbb{H}(\mathrm{IG}_k) \to \mathrm{QH}(\mathrm{IG}(n-k, 2n)).$$

The map π is determined by setting

$$\pi(\sigma_i) = \begin{cases} \sigma_i & \text{if } 1 \leq i \leq n+k, \\ q/2 & \text{if } i = n+k+1, \\ 0 & \text{if } n+k+1 < i \leq 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k. \end{cases}$$

The relations (3) then uniquely specify the values $\pi(\sigma_i)$ for i even and $i > 2n+2k$.

Theorem 1 (Quantum Giambelli for IG). *For every $\lambda \in \mathcal{P}(k,n)$, the quantum Giambelli formula for σ_λ in $\mathrm{QH}(\mathrm{IG}(n-k,2n))$ is obtained from the classical Giambelli formula $\sigma_\lambda = R^\lambda\, m_\lambda$ in $\mathrm{H}^*(\mathrm{IG}(n+1-k,2n+2),\mathbb{Z})$ by replacing the special Schubert class σ_{n+k+1} with $q/2$.*

Proof. We claim that the ring homomorphism π satisfies $\pi(\sigma_\lambda) = \sigma_\lambda$ for all $\lambda \in \mathcal{P}(k,n)$. The proof of the claim is by induction on the length of λ, with the case of length one being clear. For the inductive step, Proposition 3 implies that

$$\sigma_\lambda = \sum_{p=\lambda_1}^{n+k+1} \sum_{\mu \subset \lambda^*} a_{p,\mu}\, \sigma_p\, \sigma_\mu \tag{14}$$

holds in the cohomology ring of $\mathrm{IG}(n+1-k,2n+2)$. Furthermore, if we we apply the ring homomorphism π to both sides of (8) and use the induction hypothesis, we find that

$$\pi(\sigma_\lambda) = \sum_{p=\lambda_1}^{n+k} \sum_{\mu \subset \lambda^*} a_{p,\mu}\, \sigma_p\, \sigma_\mu + \frac{q}{2} \sum_{\mu \subset \lambda^*} a_{n+k+1,\mu}\, \sigma_\mu \tag{15}$$

holds in $\mathrm{QH}^*(\mathrm{IG}(n-k,2n))$. The right hand side of (15) can be evaluated using the quantum Pieri formula (13). We perform this computation using (14) and deduce that the expression evaluates to σ_λ, proving the claim.

According to Corollary 1, the stable Giambelli polynomial for σ_λ may be expressed as an equation

$$\sigma_\lambda = f_\lambda(\sigma_1, \ldots, \sigma_{2n+2k-1}) \tag{16}$$

in $\mathbb{H}(\mathrm{IG}_k)$, where $f_\lambda \in \mathbb{Z}[x_1, \ldots, x_{2n+2k-1}]$. We now apply the ring homomorphism π to (16) to get an identity in $\mathrm{QH}(\mathrm{IG}(n-k,2n))$. The left hand side evaluates to σ_λ by the last claim, while the right hand side maps to $f_\lambda(\sigma_1, \ldots, \sigma_{n+k}, \frac{q}{2}, 0, \ldots, 0)$. We deduce that

$$\sigma_\lambda = f_\lambda(\sigma_1, \ldots, \sigma_{n+k}, \frac{q}{2}, 0, \ldots, 0)$$

in $\mathrm{QH}(\mathrm{IG}(n-k,2n))$, which is precisely the quantum Giambelli formula. □

3. QUANTUM GIAMBELLI FOR $\mathrm{OG}(n-k,2n+1)$

3.1. For each $k \geq 0$, let $\mathrm{OG} = \mathrm{OG}(n-k,2n+1)$ denote the odd orthogonal Grassmannian which parametrizes the $(n-k)$-dimensional isotropic subspaces in $\mathbb{C}^{2n+1}$, equipped with a non-degenerate symmetric bilinear form. The Schubert varieties in OG are indexed by the same set of k-strict partitions $\mathcal{P}(k,n)$ as for $\mathrm{IG}(n-k,2n)$. Given any $\lambda \in \mathcal{P}(k,n)$ and a complete flag of subspaces

$$F_\bullet : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n+1} = \mathbb{C}^{2n+1}$$

such that $F_{n+i} = F_{n+1-i}^{\perp}$ for $1 \leq i \leq n+1$, we define the codimension $|\lambda|$ Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j(\lambda)}) \geq j \ \ \forall\, 1 \leq j \leq \ell(\lambda)\},$$

where

$$\overline{p}_j(\lambda) = n + k + 1 + j - \lambda_j - \#\{i \leq j : \lambda_i + \lambda_j > 2k + j - i\}.$$

Let $\tau_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{OG}, \mathbb{Z})$ denote the cohomology class dual to the cycle given by $X_\lambda(F_\bullet)$.

Let $\ell_k(\lambda)$ be the number of parts λ_i which are strictly greater than k, and let $\mathcal{Q}_{\mathrm{IG}}$ and $\mathcal{Q}_{\mathrm{OG}}$ denote the universal quotient vector bundles over $\mathrm{IG}(n-k, 2n)$ and $\mathrm{OG}(n-k, 2n+1)$, respectively. It is known (see e.g. [BS, §3.1]) that the map which sends $\sigma_p = c_p(\mathcal{Q}_{\mathrm{IG}})$ to $c_p(\mathcal{Q}_{\mathrm{OG}})$ for all p extends to a ring isomorphism $\varphi : \mathrm{H}^*(\mathrm{IG}, \mathbb{Q}) \to \mathrm{H}^*(\mathrm{OG}, \mathbb{Q})$ such that $\varphi(\sigma_\lambda) = 2^{\ell_k(\lambda)}\tau_\lambda$ for all $\lambda \in \mathcal{P}(k, n)$.

We let $c_p = c_p(\mathcal{Q}_{\mathrm{OG}})$. The *special Schubert classes* on OG are related to the Chern classes c_p by the equations

$$c_p = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

For any integer sequence α, set $m_\alpha = \prod_i c_{\alpha_i}$. Then for every $\lambda \in \mathcal{P}(k, n)$, the classical Giambelli formula

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \, m_\lambda \tag{17}$$

holds in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$.

3.2. The quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG}(n-k, 2n+1))$ is defined similarly to that of IG, but the degree of q here is $n + k$. More notation is required to state the quantum Pieri rule for OG. For each λ and μ with $\lambda \to \mu$, we define $N'(\lambda, \mu)$ to be equal to the number (respectively, one less than the number) of connected components of $\mathbb{A}$, if $p \leq k$ (respectively, if $p > k$). Let $\mathcal{P}'(k, n+1)$ be the set of $\nu \in \mathcal{P}(k, n+1)$ for which $\ell(\nu) = n + 1 - k$, $2k \leq \nu_1 \leq n + k$, and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 1$. For any $\nu \in \mathcal{P}'(k, n+1)$, we let $\widetilde{\nu} \in \mathcal{P}(k, n)$ be the partition obtained by removing the first row of ν as well as $n + k - \nu_1$ boxes from the first column. That is,

$$\widetilde{\nu} = (\nu_2, \nu_3, \ldots, \nu_r), \text{ where } r = \nu_1 - 2k + 1.$$

According to [BKT1, Theorem 2.4], for any k-strict partition $\lambda \in \mathcal{P}(k, n)$ and integer $p \in [1, n+k]$, the following quantum Pieri rule holds in $\mathrm{QH}^*(\mathrm{OG}(n-k, 2n+1))$.

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda \to \mu} 2^{N'(\lambda,\mu)}\, \tau_\mu + \sum_{\lambda \to \nu} 2^{N'(\lambda,\nu)}\, \tau_{\widetilde{\nu}}\, q + \sum_{\lambda^* \to \rho} 2^{N'(\lambda^*,\rho)}\, \tau_{\rho^*}\, q^2 . \tag{18}$$

Here the first sum is classical, the second sum is over $\nu \in \mathcal{P}'(k, n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$, and the third sum is empty unless $\lambda_1 = n + k$, and over $\rho \in \mathcal{P}(k, n)$ such that $\rho_1 = n + k$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - k + p$.

Let $\delta_p = 1$, if $p \leq k$, and $\delta_p = 2$, otherwise. The stable cohomology ring $\mathbb{H}(\mathrm{OG}_k)$ has a free $\mathbb{Z}$-basis of Schubert classes τ_λ for k-strict partitions λ, and is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \tau_2, \ldots]$ modulo the relations

$$\tau_r^2 + 2\sum_{i=1}^{r} (-1)^i \delta_{r-i} \tau_{r+i} \tau_{r-i} = 0 \quad \text{for } r > k. \tag{19}$$

We require a ring homomorphism

$$\tilde{\pi} : \mathbb{H}(\mathrm{OG}_k) \to \mathrm{QH}(\mathrm{OG}(n-k, 2n+1))$$

analogous to the map π of §2. The morphism $\tilde{\pi}$ is determined by setting

$$\tilde{\pi}(\tau_i) = \begin{cases} \tau_i & \text{if } 1 \leq i \leq n+k, \\ 0 & \text{if } n+k < i < 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k. \end{cases}$$

The relations (19) then uniquely specify the values $\tilde{\pi}(\tau_i)$ for i even and $i \geq 2n+2k$. To verify this, we just have to check that the relations

$$\tau_r^2 + 2 \sum_{i=1}^{n+k-r} (-1)^i \delta_{r-i} \tau_{r+i} \tau_{r-i} = 0$$

are true in $\mathrm{QH}^*(\mathrm{OG}(n-k, 2n+1))$, for $(n+k)/2 \leq r \leq n+k-1$. But when $k < n-1$ the individual terms in these relations carry no q correction. Indeed, we are applying the quantum Pieri rule (18) to length 1 partitions, hence the q term vanishes (since $1 < n-k$) and the q^2 term vanishes (since $\deg(q^2) = 2n+2k$). It remains only to consider the case $k = n-1$, which uses the quantum Pieri rule for the quadric $\mathrm{OG}(1, 2n+1)$. The computation is then done as in [BKT1, Theorem 2.5] (which treats the case $r = n$), and involves computing the coefficient c of $q\,\tau_{2(r-n)+1}$ in the corresponding expression. As in loc. cit., the result is $c = 1-2+2-\cdots\pm 2 \mp 1$ when $r \leq (3n-2)/2$, and otherwise $c = 2-4+4-\cdots \pm 4 \mp 2$; hence $c = 0$ in both cases.

Theorem 2 (Quantum Giambelli for OG)**.** *For every $\lambda \in \mathcal{P}(k,n)$, we have*

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \, m_\lambda$$

in the quantum cohomology ring $\mathrm{QH}(\mathrm{OG}(n-k, 2n+1))$*. In other words, the quantum Giambelli formula for* OG *is the same as the classical Giambelli formula.*

Proof. We may use the isomorphism φ of §3.1 to translate all of the results of §1 to their images in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$ and the stable cohomology ring $\mathbb{H}(\mathrm{OG}_k)$. The proof of quantum Giambelli for OG is therefore identical to the proof of Theorem 1, using the ring homomorphism $\tilde{\pi}$ in place of π. □

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