The many facets of definite integration

Colloquium, Rutgers University

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Tulane University

18 September 2009
Chilean Independence Day

What do you do?

People gave me funny looks when I responde
I compute integrals
it took me a while to say that

The gurus of the field have decided to fix this, so now I do
EXPERIMENTAL MATHEMATICS

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My research goal

Given a function

$$f : [a, b] \to \mathbb{R}$$

express

$$I(f; a, b) := \int_a^b f(x)\, dx$$

in terms of a, b and the parameters of f.

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Dream goal

Input:

$$\texttt{function } f,\ \{a, b\},\ \texttt{parameters}$$

`Output`: the computer gives an analytic expression for

$$I(f; a, b) := \int_a^b f(x)\, dx$$

and a good theorem about the answer.

Not possible yet

There is no theory for definite integrals

Complexity level is hard to predict

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x + 1)^2 + \pi^2} = \frac{1}{2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(e^x - x)^2 + \pi^2} = \frac{1}{1 - W(1)}$$

$W(z)$ is the Lambert function

$$W(z) \ \exp \ W(z) = z$$

Collections of Integrals

Tables of Integrals and Products

I. S. Gradshteyn and I. M. Ryzhik

Seventh edition, 2007

The GEORGE BOROS project

Prove all the formulas and put them in context

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Euler appears everywhere on this table

$$\int_0^1 \frac{x^{p-1} + x^{q-p-1}}{1 + x^q}\, dx = \frac{\pi}{q} \left(\sin\left(\frac{p\pi}{q} \right) \right)^{-1}$$

This appears as 3.244.1
Read the details in
Euler through time: A new look at old themes

V. S. Varadarajan, AMS 2006

Euler's constant

$$\gamma := \lim_{n\to\infty} H_n - \log n$$

$$H_n := \sum_{k=1}^{n} \frac{1}{k}$$

$$\gamma = -\Gamma'(1) = -\int_0^1 \log\log\frac{1}{x}\,dx$$

$$\gamma = -\int_0^\infty e^{-x}\ln x\,dx$$

Generalizations: Luis Medina

$$Q(x) := \sum_{k=0}^{\infty} a_n x^n,$$

$$L_Q(x) := \sum_{k=0}^{\infty} \frac{a_n}{(n+1)^s}$$

$$\int_0^1 Q(x) \log\log\frac{1}{x}\,dx = -\gamma L_Q(1) + L_Q'(1)$$

Examples

$$a_n = \frac{1}{n+1}$$

gives

$$\int_0^1 \frac{\log(1-x)}{x} \log\log\frac{1}{x}\,dx = \gamma\zeta(2) - \zeta'(2)$$

$$a_n = \frac{(-1)^n}{n+1}$$

gives

$$\int_0^\infty \log t\ \log\tanh t\,dt = \frac{\gamma\pi^2}{8} + \frac{\pi^2\log 2}{12} - \frac{3}{4}\zeta'(2)$$

Some logarithmic integrals

$$L_a := \int_0^1 \log^a \Gamma(x)\, dx$$

$$L_1 = \log \sqrt{2\pi}$$

Euler

$$\begin{aligned} L_2 &= \frac{\gamma^2}{12} + \frac{\pi^2}{48} + \frac{\gamma L_1}{3} + \frac{4}{3} L_1^2 \\ &- (\gamma + 2L_1)\frac{\zeta'(2)}{\pi^2} + \frac{\zeta''(2)}{2\pi^2} \end{aligned}$$

O. Espinosa - V. M.

What about L_3?

$$T(a,b,c) := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{n^a m^b (n+m)^c}$$

$T(a,b,c)$ with $a,b,c \in \mathbb{N}$ can be expressed as a finite sum of

$$K_{m,n} := \int_0^1 \psi^{(-m)}(q) B_n(q) \log\Gamma(q)\, dq$$

and three other variations.

$$\psi(z,q) = e^{-\gamma z} \frac{\partial}{\partial z}\left[e^{\gamma z} \frac{\zeta(z+1,q)}{\Gamma(-z)} \right]$$

Theorem. L_3 can be expressed in terms of $T(a,b,c)$.

Back to basics and more Euler sums

$$C(n,p) := \int_0^{\pi/2} x^p \cos^n x \, dx$$

Assume n is even, then

$$C(n,p) = \sum_{j=0}^{\lfloor \frac{p+1}{2} \rfloor} I_j(n,p)$$

where

$$\begin{aligned} I_j(n,p) &= 2^{-2n} \binom{2n}{n} \binom{p}{2j} \frac{2^{2j}\zeta(1-2j)}{2\zeta(2j)} \left(\frac{\pi}{2}\right)^{p+1} \\ &\times \sum_{1 \le k_1 \le k_2 \le \cdots \le k_j \le n} \frac{1}{k_1^2 k_2^2 \cdots k_j^2}. \end{aligned}$$

Example from Gradshteyn and Ryzhik

$$\int_0^\infty x\, e^{-x}\, \frac{1-e^{-x}}{1+e^{-3x}}\, dx = \frac{2\pi^2}{27}$$

An interesting new addition to the 6^{th} edition: 3.248.5

$$\varphi(x) = 1 + \frac{4x^2}{3(1+x^2)}$$

$$\int_0^\infty \frac{dx}{(1+x^2)^{3/2}\left[\varphi(x)+\sqrt{\varphi(x)}\right]^{1/2}} = \frac{\pi}{2\sqrt{6}}$$

And now the bad news

The evaluation is INCORRECT

And now the good news

The integral is out of the seventh edition

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Common perception

Tables and pencils are not required because we have `Mathematica` and `Maple`

Entry 3.511.9 of `Gradshteyn and Ryzhik` is

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = 1 - \pi a \cot(\pi a)$$

`Mathematica` gives

$$\int_{-\infty}^{\infty} \frac{\sinh^2 ax}{\sinh^2 x}\, dx = \frac{3}{4} - \pi a \cot(\pi a) + a \ln 2$$

Quality control

Objectives:

To evaluate definite integrals
Create algorithms
Develop context
Create pedagogical material

Collaborators: Faculty

Tewodros Amdeberhan	Tulane Univeristy
Khristo Boyadzhiev	Ohio Northern University
Marc Chamberland	Grinnell College
Olivier Espinosa	Universidad Santa Maria, Physics
Larry Glasser	Clarkson University, New York
Ivan Gonzalez	Universidad Catolica, Chile
John Hubbard	Cornell University
John Little	Holy Cross College
Dante Manna	Wesleyan University, Virginia
Luis Medina	Rutgers University
Edward Mosteig	Loyola Marimount
Ronald Posey	Baton Rouge Community College
Sinai Robins	National University, Singapore
Jeffrey Shallit	Computer Science, University of Waterloo
Richard Stanley	MIT
Xinyu Sun	Xavier University, New Orleans
Douglas Varela	California freelancer

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Dagan Karp
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What are these good for ?

$$\int_0^1 \frac{(1-x)^4\,x^4}{1+x^2}\,dx = \frac{22}{7} - \pi$$

Theorem: $\pi \neq \frac{22}{7}$

$$\frac{1}{3164}\int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{1+x^2}\,dx = \frac{355}{113} - \pi$$

Three examples: rational functions

- A quartic integral

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

- Landen transformations

$$\int_0^\infty \frac{cx^4+dx^2+e}{x^6+ax^4+bx^2+1}\,dx$$

- Wallis formula

$$\int_0^\infty \frac{dx}{(x^2+1)^{m+1}}$$

Part 1

The original sequence

$$N_{0,4}(a;m) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}}$$

Explicit formula ?

$$N_{0,4}(a;m) = \frac{\pi}{2\,[2(a+1)]^{m+\frac{1}{2}}} P_m(a)$$

$$P_m(a) = \sum_{l=0}^{m} d_{l,m} a^l$$

$$d_{l,m} = 2^{-2m} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}$$

Some great polynomials

Theorem
There are polynomials $\alpha_l(m)$, $\beta_l(m)$ such that

$$\begin{aligned} A_{l,m} &= l!m!2^{m+l}d_{l,m} \\ &= \alpha_l(m)\prod_{k=1}^{m}(4k-1) - \beta_l(m)\prod_{k=1}^{m}(4k+1) \end{aligned}$$

Theorem (John Little, 2005)
The polynomials $\alpha_l(m)$, $\beta_l(m)$ have all their roots on the vertical line $\mathrm{Re}\, m = -\frac{1}{2}$.

Problem what happens as $l \to \infty$?

The original expression for $d_l(m)$ was ugly

$$
\begin{aligned}
d_l(m) &= \sum_{j=0}^{l} \sum_{s=0}^{m-j} \sum_{k=s+l}^{m} \frac{(-1)^{k-l-s}}{2^{3k}} \times \\
&\times \binom{2k}{k} \binom{2m+1}{2(s+j)} \binom{m-s-j}{m-k} \\
&\times \binom{s+j}{j} \binom{k-s-j}{l-j}
\end{aligned}
$$

What can one do with this?

A Taylor series

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1}\times \left(1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}P_{k-1}(a)}{k2^{k+1}(a+1)^k}c^k\right)$$

Problem: the expansion of

$$\sqrt{b+\sqrt{a+\sqrt{1+c}}}$$

has coefficients that involve the **homogenization** of $P_k(a)$:

$$P_k^*(a,b) := b^k P_k(a/b)$$

Find and explain

Recurrences

$$P_m(a) = \sum_{l=0}^{m} d_l(m) a^l$$

Elementary methods:

$$\begin{aligned} P_m(a) &= \frac{(2m-3)(4m-3)a}{4m(m-1)(a-1)} P_{m-2}(a) - \\ &- \frac{(4m-3)a(a+1)}{2m(m-1)(a-1)} \frac{d}{da} P_{m-2}(a) + \\ &+ \frac{4m(a^2-1)+1-2a^2}{2m(a-1)} P_{m-1}(a). \end{aligned}$$

Recurrences: symbolic methods

Paule and Kauers, 2007.

$$\begin{aligned} 2(m+1)d_l(m+1) &= 2(m+l)d_{l-1}(m) \\ &+ (2l+4m+3)d_l(m) \end{aligned}$$

Uses ugly expression for $d_l(m)$

Unimodality and logconcavity

Theorem[Boros, V. M., 2000]: $d_l(m)$ are unimodal.

Theorem[Kauers, Paule, 2007]: $d_l(m)$ are logconcave.

$$\mathfrak{L}(a_n) := a_n^2 - a_{n-1}a_{n+1}$$

logconcave: $a_n \geq 0$ implies $\mathfrak{L}(a_n) \geq 0$
∞-**logconcave**: $a_n \geq 0$ implies $\mathfrak{L}^r(a_n) \geq 0$ for all $r \in \mathbb{N}$

Conjecture: $d_l(m)$ are ∞-logconcave.
The case of binomial coefficients might follow from the preprint
Iterated sequences and the geometry of zeros by `Petter Brändén`
Available as `axXiV:0909.1927`

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The coefficients $2^{2m} d_{l,m}$ are even

$$2^{2m} d_{l,m} = \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l}$$

better scaling

$$\begin{aligned} A_{l,m} &:= l!m!2^{m+l} d_{l,m} \\ &= \frac{l!\,m!}{2^{m-l}} \sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k} \binom{m+k}{m} \binom{k}{l} \in \mathbb{N} \end{aligned}$$

What is $\nu_2(A_{l,m})$?

The valuations $\nu_2(A_{l,m})$

$$\begin{aligned} A_{l,m} &:= l!m!2^{m+l}d_{l,m} \\ &= \frac{l!\,m!}{2^{m-l}}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l} \in \mathbb{N} \end{aligned}$$

Theorem
(Amdeberhan, Manna, V.M.)
Journal of Combinatorics A, 2008

$$\nu_2(A_{l,m}) = \nu_2(\,(m+1-l)_{2l}) + l$$

The sequence $\{\nu_2(A_{l,m}) :\ m \geq 1\}$ is $2^{1+\nu_2(l)}$-simple.

One proof per talk

$$\nu_2(d_l(m)) = \nu_2((m+1-l)_{2l}) + l$$

define

$$B_{l,m} = \frac{l!\, m!\, 2^m d_{l,m}}{(m+1-l)_{2l}}$$

To show $B_{l,m}$ is odd.
The WZ-technology shows

$$B_{l-1,m} = (2m+1)B_{l,m} - (m-l)(m+l+1)B_{l+1,m}.$$

The initial values

$$B_{m,m} = 1 \text{ and } B_{m-1,m} = 2m+1.$$

The valuations $\nu_2(C_{l,m})$

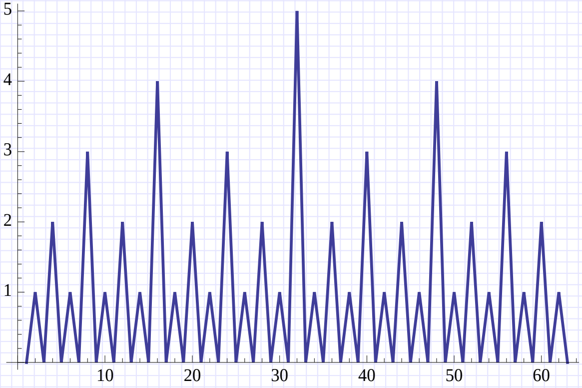
$$C_{l,m} := A_{l,l+(m-1)\times 2^{1+\nu_2(l)}}$$

What is $\nu_2(C_{l,m})$?

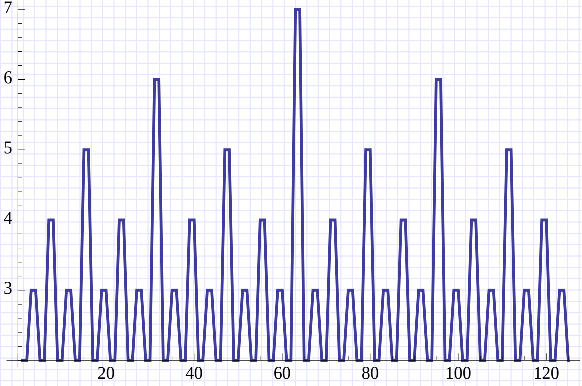
Theorem (X. Sun, V. M. 2008)

$$\nu_2(C_{3,m}) = \begin{cases} 7 + \nu_2\left(\frac{m+1}{2}\right) & \text{if } m \equiv 1 \bmod 2 \\ 9 + \nu_2\left(\frac{m}{4}\right) & \text{if } m \equiv 0 \bmod 4 \\ 9 + \nu_2\left(\frac{m+2}{4}\right) & \text{if } m \equiv 2 \bmod 4 \end{cases}$$

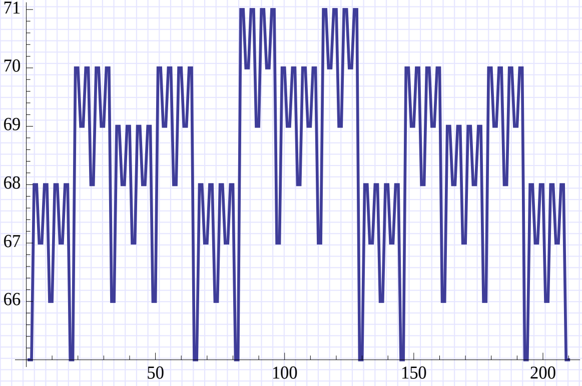
The basic picture



The 2-adic valuation of $A_{1,m}$



For higher l graph is complicated



The tree associated to l

Take one point per block

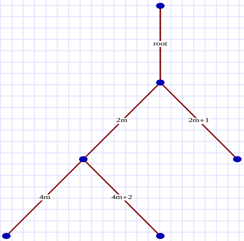
$$C_{l,m} := A\left(l, m \cdot 2^{1+\nu_2(l)}\right)$$

Start with a root vertex

Ask: is $\nu_2(C_{l,m})$ a shift of the basic $\nu_2(m)$?

If yes: add another vertex, connect and end
If no: split into $\nu_2(C_{l,2m})$ and $\nu_2(C_{l,2m+1})$
Continue

Each tree is a formula



$$f(m) = \begin{cases} 9 + \nu_2\left(\frac{m}{4}\right) & \text{if } m \equiv 0 \bmod 4 \\ 9 + \nu_2\left(\frac{m-2}{4}\right) & \text{if } m \equiv 2 \bmod 4 \\ 7 + \nu_2\left(\frac{m-1}{2}\right) & \text{if } m \equiv 1 \bmod 2 \end{cases}$$

$$\nu_2\left(A_{3,2m}\right) = f(m+1) \quad \text{for } m \geq 2$$

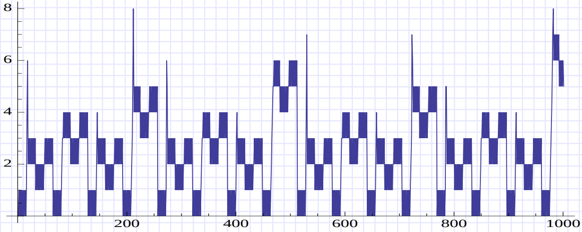
Challenge

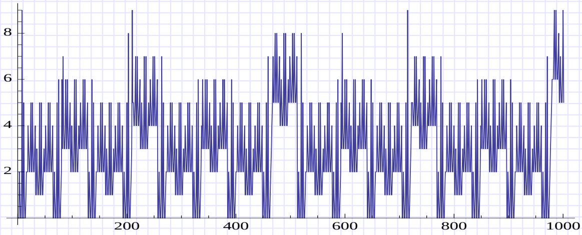
$$
\begin{aligned}
x_n &= P(n)x_{n-1} + Q(n)x_{n-2} \\
x_0 &= a \\
x_1 &= b
\end{aligned}
$$

Stirling numbers

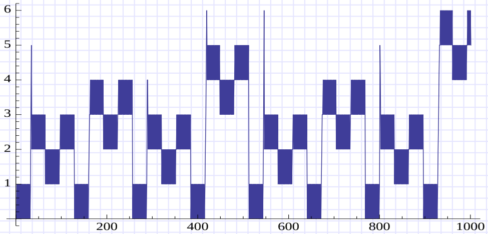
$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

The 2-adic valuation of S(n,k) with k fixed





k=195 has pretty structure



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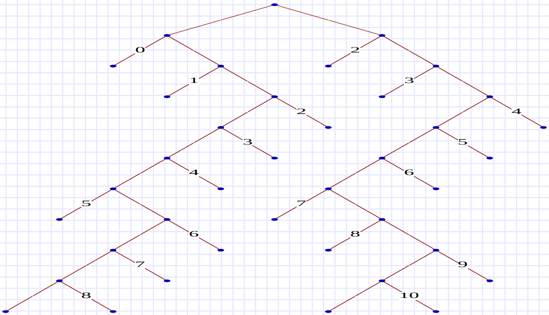
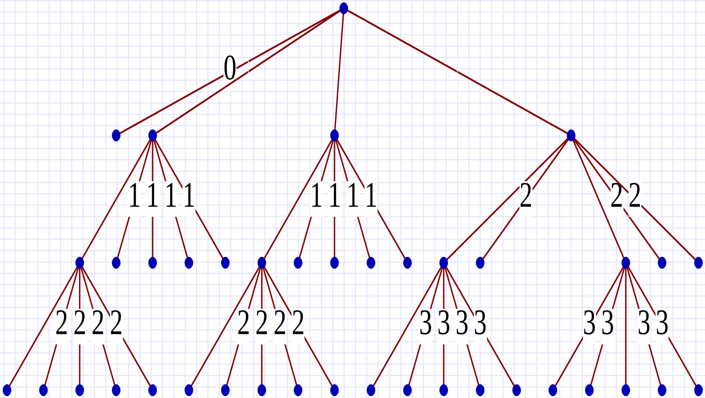


The 5-adic valuation of S(n,k) with k fixed

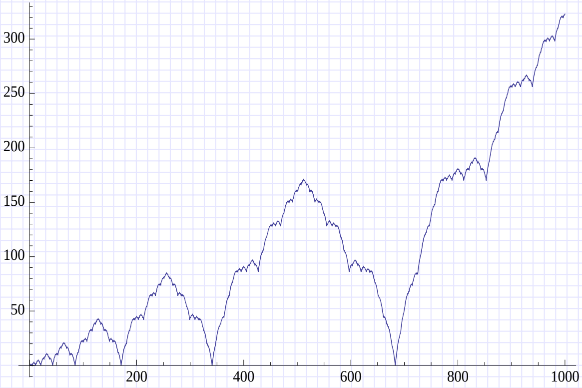


Another interesting sequence

$$T(n) := \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

counts the Alternating Sign Matrices

2-adic valuation of ASM numbers



A generalization

$$T_p(n) := \prod_{j=0}^{n-1} \frac{(pj+1)!}{(n+j)!}$$

$$\nu_p(T_p(pn)) = p\nu_p(T_p(n)) + \frac{1}{2}p(p-3)n^2$$

What does $T_p(n)$ count?

Part 2

Landen transformations

Landen, Legendre and Gauss showed

$$G(a,b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

satisfies

$$G\left(a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}\right) = G(a,b)$$

iterate $(a,b) \mapsto (a_1, b_1)$ to reach a common limit $L(a,b)$
In the limit

$$G(a,b) = \frac{\pi}{2L(a,b)}$$

Rational Landen transformations

Maps on the coefficients of a rational function that preserve the integral

$$I_6(a, b; c, d, e) = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\, dx$$

is invariant under

$$\begin{aligned} a &\mapsto \frac{ab + 5a + 5b + 9}{(a + b + 2)^{4/3}} \\ b &\mapsto \frac{a + b + 6}{(a + b + 2)^{2/3}} \end{aligned}$$

plus equations for the other parameters

The motivating problem

$$I = \int_{-\infty}^{\infty} A(x)\, dx$$

with A a rational function.

We assume the poles of A are not real.

The change of variables $x = \tan y$ gives

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} A(\tan y)(1 + \tan^2 y)\, dy \\ &= \int_{-\pi/2}^{\pi/2} A_1(\tan y)\, dy \end{aligned}$$

In special cases

$$A_1(\tan y) = A_2(\tan 2y)$$

and we obtain

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} A_2(\tan 2y)\, dy \\ &= \frac{1}{2}\int_{-\pi}^{\pi} A_2(\tan y_1)\, dy_1 \\ &= \int_{-\pi/2}^{\pi/2} A_2(\tan y_1)\, dy_1 \\ &= \int_{-\infty}^{\infty} A_3(x)\, dx \end{aligned}$$

using symmetry in the last step.

A good change of variables is worth a thousand theorems

$$y = R_2(x) := \frac{x^2 - 1}{2x}$$

satisfies

$$\cot 2\theta = R_2(\cot\theta)$$

Theorem: if A is a rational function, then

$$\int_{-\infty}^{\infty} A(x)\,dx \quad = \quad \int_{-\infty}^{\infty} A^{+}(y)\,dy + \int_{-\infty}^{\infty} A^{-}(y)\,dy$$

where

$$A^{+}(y) = A(y + \sqrt{y^2+1}) + A(y - \sqrt{y^2+1})$$

and

$$A^{-}(y) = \left[A(y + \sqrt{y^2+1}) - A(y - \sqrt{y^2+1})\right]\ \frac{y}{\sqrt{y^2+1}}$$

Rational preservation

Theorem: if A is a rational function, then

$$\begin{aligned}\int_{-\infty}^{\infty} A(x)\,dx &= \int_{-\infty}^{\infty} A^{+}(y)\,dy + \int_{-\infty}^{\infty} A^{-}(y)\,dy \\ &= \int_{-\infty}^{\infty} A_1(y)\,dy\end{aligned}$$

where

$$A^{+}(y) = A(y+\sqrt{y^2+1}) + A(y-\sqrt{y^2+1})$$

and

$$A^{-}(y) = \left[A(y+\sqrt{y^2+1}) - A(y-\sqrt{y^2+1})\right] \frac{y}{\sqrt{y^2+1}}$$

- If A is rational, so is A_1
- If A is an even rational function, so is A_1

Landen transformations

We have: if A is a rational function, then

$$\int_{-\infty}^{\infty} A(x)\,dx = \int_{-\infty}^{\infty} A_1(y)\,dy$$

where

$$\begin{aligned} A_1(y) &= A(y+\sqrt{y^2+1}) + A(y-\sqrt{y^2+1}) + \\ &+ \frac{y}{\sqrt{y^2+1}}\left(A(y+\sqrt{y^2+1}) - A(y-\sqrt{y^2+1})\right) \end{aligned}$$

$$\begin{aligned} \mathfrak{L} \quad : \quad & Rat \longrightarrow Rat \\ & A \longrightarrow A_1 \end{aligned}$$

satisfies

$$\int_{-\infty}^{\infty} A(x)\,dx = \int_{-\infty}^{\infty} \mathfrak{L}(A)(y)\,dy$$

Quadratic example

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \int_{-\infty}^{\infty} \frac{dx}{a_1 x^2 + b_1 x + c_1}$$

where

$$a_1 = \frac{2ac}{a+c}, \quad b_1 = \frac{b(c-a)}{a+c}, \quad c_1 = \frac{(a+c)^2 - b^2}{2(a+c)}$$

Definition: the Landen transformation is the map $\mathfrak{L}_{2,2} : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\mathfrak{L}_{2,2}(a, b, c) := (a_1, b_1, c_1)$$

$$\mathfrak{L}_{d,m}$$

d is the degree of denominator and
m is the order of the transformation

Quadratic example: continuation

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \int_{-\infty}^{\infty} \frac{dx}{a_1x^2 + b_1x + c_1}$$

$$\mathfrak{L}_{2,2}(a, b, c) := \left(\frac{2ac}{a+c}, \quad \frac{b(c-a)}{a+c}, \quad \frac{(a+c)^2 - b^2}{2(a+c)} \right)$$

$$\mathfrak{L}_{2,2}^{(n)}(a, b, c) := (a_n,\ b_n,\ c_n)$$

Theorem: there exits $L \in \mathbb{R}$ such that

$$a_n \to L,\ b_n \to 0,\ c_n \to L$$

$$err_n := |a_n - L|^2 + |b_n|^2 + |c_n - L|^2$$

$$err_{n+1} \leq C err_n^2$$

$$\int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} = \frac{\pi}{L}$$

General case

Theorem: given a rational function A and $m \in \mathbb{N}$, there exists (explicit) map on the coefficients of A that produces an numerical method to compute the integral of A over $\mathbb{R}$.

The order or convergence is m.

The iterations of this map converge if and only if the original integral converges.

This is the rational Landen transformation of order m.

Open problem

Develop Landen transformations for

$$\int_0^\infty \frac{dx}{ax^2+bx+c}$$

Part 3

Wallis' formula

$$G_n := \frac{2}{\pi}\int_0^\infty \frac{dx}{(x^2+1)^n}$$

Easy recurrence: $G_{n+1} = \frac{2n-1}{2n} G_n$

$G_1 = 1$ gives $G_n = \frac{1}{2^{2n}}\binom{2n}{n}$

Hard recurrence: double angle plus binomial theorem give

$$G_{n+1} = 2^{-n}\sum_{k=0}^{n/2}\binom{n}{2k} G_{k+1}$$

Guess is equivalent to

$$\sum_k 2^{-2k}\binom{n}{2k}\binom{2k}{k} = 2^{-n}\binom{2n}{n}$$

Now ask WZ or Chu-Vandemonde.

A generalization

$$G_n(\mathbf{q}) = \frac{2}{\pi}\int_0^\infty \prod_{k=1}^{n} \frac{1}{x^2+q_k^2}$$

Ramanujan gives the case n = 4.

$$G_4(\mathbf{q}) = \frac{(q_1+q_2+q_3+q_4)^3-(q_1^3+q_2^3+q_3^3+q_4^3)}{3q_1q_2q_3q_4(q_1+q_2)(q_2+q_3)(q_1+q_3)(q_1+q_4)(q_2+q_4)(q_3+q_4)}$$

An expression in terms of Schur functions

$$\begin{aligned}\lambda(n) &= (n-1, n-2, \cdots, 2, 1)\\ a_{\lambda(n)}(\mathbf{q}) &= \prod_{1\le i<j\le n} (q_i - q_j)\\ s_\mu(\mathbf{q}) &= a_{\mu+\lambda(n)}(\mathbf{q})/a_{\lambda(n)}(\mathbf{q})\end{aligned}$$

Theorem (Amdeberhan, Espinosa, Straub, M.; 2009):

$$G_n(\mathbf{q}) = s_{\lambda(n-1)(\mathbf{q})}/s_{\lambda(n+1)(\mathbf{q})}$$

Thanks