The *p*-adic valuations of interesting sequences

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All good things come from Integrals[1]

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[1] The world according to Victor

Collaborators

Tewodros Amdeberhan, Tulane Univeristy
Erin Beyerstedt, graduate student, Tulane University
Marc Chamberland, Grinnell College
Karen Kohl, graduate student, Tulane University
Dante Manna, Wesleyan University, Virginia
Luis Medina, Rutgers University
Armin Straub, graduate student, Tulane University
Xinyu Sun, Xavier University, New Orleans

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General problem

Given a function

$$f : [a, b] \to \mathbb{R}$$

say something interesting about

$$I(f; a, b) := \int_a^b f(x)\, dx$$

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A quartic integral

$$\frac{1}{\pi}\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} = \frac{P_m(a)}{2^{m+3/2}(a+1)^{m+1/2}}$$

$$P_m(a) = \sum_{l=0}^{m} d_l(m)a^l$$

$$d_l(m) = 2^{-2m}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}$$

$$A_{l,m} := l!m!2^{m+l}d_l(m) = \frac{l!m!}{2^{m-l}}\sum_{k=l}^{m} 2^k \binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}$$

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I hoped for a million dollars

There are polynomials α_l and β_l:

$$d_l(m) = \alpha_l(m) \times \text{Easy}_1 - \beta_l(m) \times \text{Easy}_2$$

and **all** zeros of α_l and β_l are on the line

$$\text{Re}(m) = -\frac{1}{2}$$

$$d_1(m) = (2m+1)\prod_{k=1}^{m}(4k-1) - \prod_{k=1}^{m}(4k+1)$$

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The original expression for $d_l(m)$ was ugly

$$\begin{aligned} d_l(m) &= \sum_{j=0}^{l}\sum_{s=0}^{m-j}\sum_{k=s+l}^{m} \frac{(-1)^{k-l-s}}{2^{3k}} \times \\ &\times \binom{2k}{k}\binom{2m+1}{2(s+j)}\binom{m-s-j}{m-k} \\ &\times \binom{s+j}{j}\binom{k-s-j}{l-j} \end{aligned}$$

What can one do with this?

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A Taylor series

$$\sqrt{a+\sqrt{1+c}} \;=\; \sqrt{a+1}\times \left(1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}P_{k-1}(a)}{k2^{k+1}(a+1)^k}c^k\right)$$

Problem: the expansion of

$$\sqrt{b+\sqrt{a+\sqrt{1+c}}}$$

has coefficients that involve the **homogenization** of $P_k(a)$:

$$P_k^*(a,b) := b^k P_k(a/b)$$

Find and explain

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Recurrences

$$P_m(a) = \sum_{l=0}^{m} d_l(m) a^l$$

Elementary methods:

$$\begin{aligned} P_m(a) &= \frac{(2m-3)(4m-3)a}{4m(m-1)(a-1)} P_{m-2}(a) - \\ &- \frac{(4m-3)a(a+1)}{2m(m-1)(a-1)} \frac{d}{da} P_{m-2}(a) + \\ &+ \frac{4m(a^2-1)+1-2a^2}{2m(a-1)} P_{m-1}(a). \end{aligned}$$

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Recurrences: symbolic methods

Paule and Kauers, 2007.

$$\begin{aligned} 2(m+1)d_l(m+1) &= 2(m+l)d_{l-1}(m) \\ &+ (2l+4m+3)d_l(m) \end{aligned}$$

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Jacobi family

$$P_m^{(\alpha,\beta)}(a) := \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} \left(\frac{1+a}{2}\right)^k$$

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$\gamma = \alpha + \beta$

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The quartic polynomial is of Jacobi type

$$P_m(a) = P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)$$

Recurrences do not match

Coefficients α and β depend on the index m

Everything has to be done from scratch

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Unimodality and logconcavity

Theorem[Boros, V. M., 2000]: $d_l(m)$ are unimodal.

Theorem[Kauers, Paule, 2007]: $d_l(m)$ are logconcave.

$$\mathfrak{L}(a_n) := a_n^2 - a_{n-1}a_{n+1}$$

logconcave: $a_n \geq 0$ implies $\mathfrak{L}(a_n) \geq 0$
∞-**logconcave**: $a_n \geq 0$ implies $\mathfrak{L}^r(a_n) \geq 0$ for all $r \in \mathbb{N}$

Conjecture: $d_l(m)$ are ∞-logconcave.
The case of binomial coefficients might follow from the preprint
Iterated sequences and the geometry of zeros by `Petter Brändén`
Available as `axXiV:0909.1927`

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The basic picture

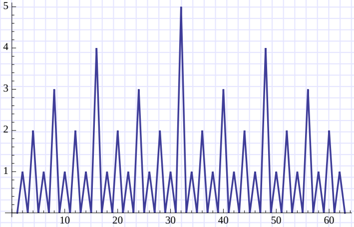


Figure: The 2-adic valuation of m

The 2-adic valuation of $A_{1,m}$

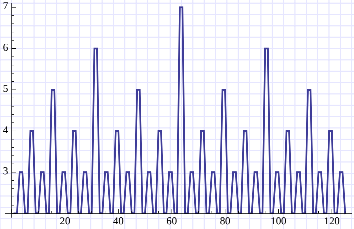


Figure: The 2-adic valuation of $A_{1,m}$

A second example

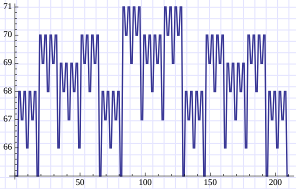


Figure: The 2-adic valuation of $A_{23,m}$

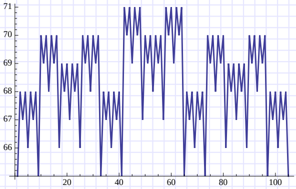


Figure: The 2-adic valuation of $A_{23,2m}$

$l = 52$

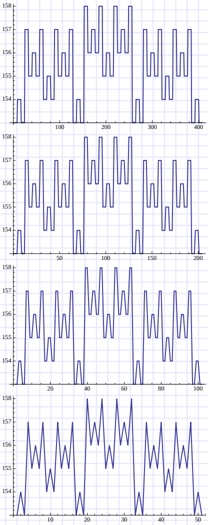


Figure: The 2-adic valuation of $A_{52,m}$

Simple sequences

Theorem: The sequence

$$\{\, \nu_2\left(A_{l,m}\right) :\ m \geq l \,\}$$

is $2^{1+\nu_2(l)}$-simple.

Take one point per block

$$C_{l,m} := A\left(l, m \cdot 2^{1+\nu_2(l)}\right)$$

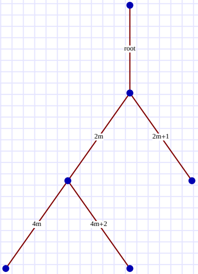
The tree associated to l

Start with a root vertex

Ask: is $\nu_2(C_{l,m})$ a shift of the basic $\nu_2(m)$?

If yes: add another vertex, connect and end

Each tree is a formula



$$f(m)=\begin{cases}9+\nu_2\left(\frac{m}{4}\right) & \text{if } m\equiv 0 \bmod 4\\ 9+\nu_2\left(\frac{m-2}{4}\right) & \text{if } m\equiv 2 \bmod 4\\ 7+\nu_2\left(\frac{m-1}{2}\right) & \text{if } m\equiv 1 \bmod 2\end{cases}$$

$$\nu_2\left(A_{3,2m}\right)=f(m+1)\qquad \text{for } m\geq 2$$

The tree of l depends only on its odd part

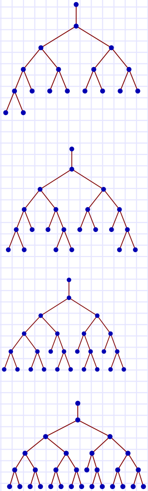


Figure: The trees for 9, 11, 13 and 15

Challenge

$$\begin{aligned} x_n &= P(n)x_{n-1} + Q(n)x_{n-2} \end{aligned}$$

$$\begin{aligned} x_0 &= a \\ x_1 &= b \end{aligned}$$

Stirling numbers

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

The 2-adic valuation of $S(n, k)$ with k fixed

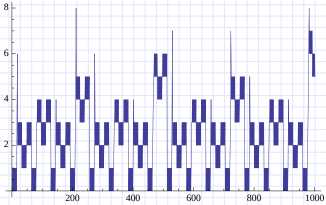


Figure: The data for $k = 100$

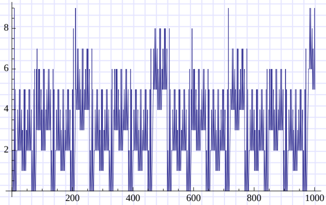


Figure: The data for $k = 101$

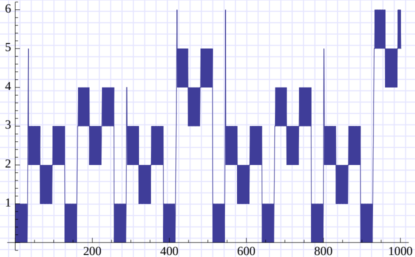


Figure: The data for $k = 195$

The 2-adic valuation of $S(n,k)$ with k fixed

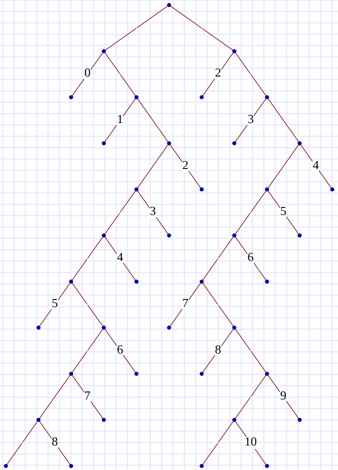


Figure: The tree for $k = 7$ and $p = 2$

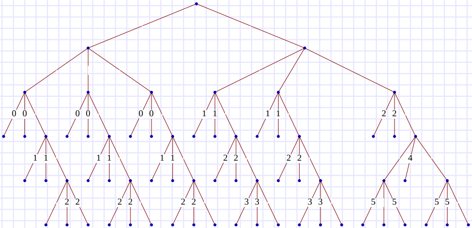


Figure: The tree for $k = 11$ and $p = 3$

Sequences counting Alternating Symmetric Matrices

$$A_3(n) := \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

$$\nu_3(A_3(3n)) = 3\nu_3(A_3(n))$$

Problem

$$A_p(n) := \prod_{j=0}^{n-1} \frac{(pj+1)!}{(n+j)!}$$

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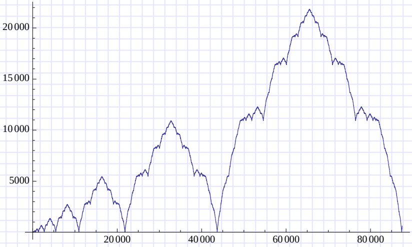

Figure: The 2-adic valuation

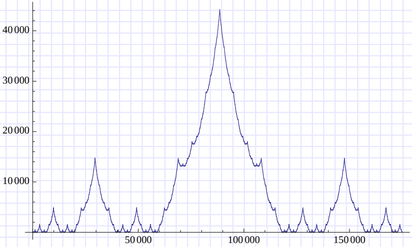

Figure: The 3-adic valuation

Theorem

Let J_n the Jacobsthal number

$$\begin{aligned} J_n &= J_{n-1} + 2J_{n-2} \\ J_0 &= J_1 = 1 \end{aligned}$$

and define $I_n := [J_n, J_{n+1}]$. The function $\nu_2 \circ A_3$ restricted to I_n is determined by its restriction to $I_{n-1} \cup I_{n-2}$.

Algorithm for the function $\nu_2 \circ A_3$

Step 1. Verify the special values $\nu_2(A_3(2^n)) = J_{n-1}$ and $\nu_2(A_3(J_n)) = 0$. The midpoint of the interval $I_n = [\, J_n,\, J_{n+1}\,]$ is 2^n.

Step 2. Given $N \in \mathbb{N}$, compute the unique index n such that $J_n \leq N < J_{n+1}$.

Step 3. For $1 \leq i \leq J_{n-1}$,

$$\nu_2(A_3(2^n + i)) = \nu_2(A_3(2^n - i)).$$

Thus, if $2^n < N < J_{n+1}$, replace N by $N^* := 2^{n+1} - N$ that satisfies $J_n < N^* < 2^n$ and $\nu_2(A_3(N)) = \nu_2(A_3(N^*))$. Therefore, the value of $\nu_2 \circ A_3$ on the interval $[J_n, J_{n+1}]$ is determined by the values on its first half $[J_n, 2^n]$.

Algorithm: continued

Step 4. For $0 < i < 2J_{n-3}$,

$$\nu_2(A_3(J_n + i)) = i + \nu_2(A_3(J_{n-2} + i)).$$

This yields the value of $\nu_2 \circ A_3$ on the first part of the interval $[\,J_n, 2^n\,]$, namely $[\,J_n,\ J_n + 2J_{n-3}\,]$, in terms of those from $I_{n-2} = [\,J_{n-2},\ J_{n-1}\,]$.

Step 5. For $0 \leq i \leq J_{n-2}$,

$$\nu_2(A_3(2^n - J_{n-2} + i)) = \nu_2(A_3(J_{n-1} + i)) + 2J_{n-3}.$$

This determines the values of $\nu_2 \circ A_3$ on the second part of the interval $[\,J_n, 2^n\,]$, namely $[\,J_n + 2J_{n-3},\ 2^n\,]$, in terms of $\nu_2 \circ A_3$ restricted to the previous interval $I_{n-1} = [\,J_{n-1},\ J_{n-2}\,]$.

Limiting function

For $n \in \mathbb{N}$, let f_n be the restriction of $\nu_2 \circ A_3$ to the interval I_n scaled to the unit square $[0,1] \times [0,1]$. Then f_n converges to the unique function $f : [0,1] \to [0,1]$ that satisfies

$$f(x) = \begin{cases} 2x + \frac{1}{4} f(4x) & \text{if } 0 \leq x < \frac{1}{4}, \\ \frac{1}{2} + \frac{1}{2} f(2x - \frac{1}{2}) & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 2(1-x) + \frac{1}{4} f(4x-3) & \text{if } \frac{3}{4} < x \leq 1. \end{cases}$$

Thanks for coming