Using enumeration schemes to $q$-enumerate avoidance sets

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A permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ contains pattern $\sigma \in S_k$ if there is some substring $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ which is order-isomorphic to $\sigma$. If $\pi$ does not contain $\sigma$, then $\pi$ avoids $\sigma$.

**Example:** $412563$ contains $132$, but avoids $321$. 

**Notation**

For set of patterns $B$, let $S_n(B)$ be the set of permutations of length $n$ which avoid every pattern in $B$. 

Note: Set braces for $B$ will often be omitted to reduce clutter.
A permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ contains pattern $\sigma \in S_k$ if there is some substring $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ which is order-isomorphic to $\sigma$. If $\pi$ does not contain $\sigma$, then $\pi$ avoids $\sigma$.

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Goal

Notation

For permutation $\pi$, let $\text{inv}(\pi)$ be the number of inversions of $\pi$, i.e.

$$\text{inv}(\pi) = |\{(i, j) : i < j, \pi_i > \pi_j\}|$$
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Notation

For permutation \( \pi \), let \( \text{inv}(\pi) \) be the number of inversions of \( \pi \), i.e.

\[
\text{inv}(\pi) = |\{(i, j) : i < j, \pi_i > \pi_j\}|
\]

Goal

Compute \( F(n, B, q) = \sum_{\pi \in S_n(B)} q^{\text{inv}(\pi)} \).

Method: Adapt Zeilberger’s and Vatter’s enumeration schemes, which compute \( |S_n(B)| \).

Remark: Barcucci, Del Lungo, Pergola, Pinzani (2001) adapted generating trees to this purpose.
What can you do with $F(n, B, q)$?

- $F(n, B, q)$ gives a natural q-analogue to $F(n, B, 1)$. Barcucci et al. (2001) discuss q-Catalan, q-Motzkin, and q-Schroeder numbers.

- $F(n, B, -1)$ gives the difference between the numbers of even permutations and odd permutations avoiding $B$. Simion and Schmidt (1985): Single 3-patterns.

- We can compute the statistical moments for $F(n, B, q)$. Average number of inversions, variance, skewness, ... Suggests asymptotic normality (or other asymptotic distributions).

- Is $F(n, B, q)$ log-concave? unimodal? symmetric?
What can you do with $F(n, B, q)$?

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Introduction and Statement of Goal

- Outline enumeration schemes
- Forming the $q$-analogue
- Preliminary results
Schemes follow a “divide and conquer” approach to build a recurrence.

1. Divide (partition) $S_n(B)$ according to prefix-pattern.
2. Conquer (count) using gap vectors and reversibly-deletable letters.
Using enumeration schemes to q-enumerate avoidance sets

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Dividing

Notation

For pattern $p \in S_k$, let $S_n(B)[p]$ be the set of all permutations in $S_n(B)$ whose first $k$ letters form pattern $p$.

For $C \in \binom{[n]}{k}$, let $S_n(B)[p; C]$ be the set of all permutations in $S_n(B)[p]$ whose first $k$ letters form the set $C$.

Example:

$S_5(123)[21; \{3, 5\}] = \{53142, 53214, 53241, 53412, 53421\}$. 
Dividing

Notation

For pattern \( p \in S_k \), let \( S_n(B)[p] \) be the set of all permutations in \( S_n(B) \) whose first \( k \) letters form pattern \( p \).

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Example:

\( S_5(123)[21; \{3, 5\}] = \{53142, 53214, 53241, 53412, 53421\} \).

We may partition \( S_n(B) \) into these \( S_n(B)[p] \).

\[
S_n(B) = S_n(B)[1] \\
= S_n(B)[12] \cup S_n(B)[21]
\]
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**Example:**

$S_5(123)[21; \{3, 5\}] = \{53142, 53214, 53241, 53412, 53421\}$.

We may partition $S_n(B)$ into these $S_n(B)[p]$.

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S_n(B) = S_n(B)[1] \\
= S_n(B)[12] \cup S_n(B)[21] \\
S_n(B)[12] = S_n(B)[123] \cup S_n(B)[132] \cup S_n(B)[231] \\
S_n(B)[21] = S_n(B)[213] \cup S_n(B)[312] \cup S_n(B)[321]
$$
For each prefix $p \in S_k$, one of 3 following events must be true:

(a) $S_n(B)[p] = \{p\}$ (only when $n = k$).

(b) For each $C \in \binom{[n]}{k}$, one of the following happens:
   - $S_n(B)[p; C]$ is in bijection with some $S_{n'}(B)[p'; C']$ for $n' < n$.
   - $S_n(B)[p; C]$ is empty

(c) $S_n(B)[p]$ must be partitioned further.
Conquering

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(a) $S_n(B)[p] = \{p\}$ (only when $n = k$).

(b) For each $C \in \binom{[n]}{k}$, one of the following happens:

- $S_n(B)[p; C]$ is in bijection with some $S_{n'}(B)[p'; C']$ for $n' < n$. (Reducibility)
- $S_n(B)[p; C]$ is empty (Gap Vector Criteria)

(c) $S_n(B)[p]$ must be partitioned further.

Part (b) is accomplished through reducibility via reversibly-deletable letters and gap vector criteria.
Consider $S_n(123)[21]$ (i.e. those 123-avoiding permutations beginning with a downstep). Suppose “2” were involved in a 123 pattern.
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Consider $S_n(123)[21]$ (i.e. those 123-avoiding permutations beginning with a downstep). Suppose “2” were involved in a 123 pattern. Then the “1” is necessarily involved in a 123 pattern as well. The “2” is reversibly-deletable.

For $i > j$, $S_n(123)[21; \{i,j\}] \leftrightarrow S_{n-1}(123)[1; \{j\}]$. “Reduction via reversibly-deletable letters.”
Now consider $S_n(123)[12]$. Then one of two cases occurs:

**Case 1:** “2” is played by $n$.

Here, “2” is reversibly deletable since it cannot be involved in any 123 patterns. Thus $S_n(123)[12; \{i, n\}] \leftrightarrow S_{n-1}(123)[1; i]$. 
Now consider $S_n(123)[12]$. Then one of two cases occurs: **Case 1**: “2” is played by $n$.

Here, “2” is reversibly deletable since it cannot be involved in any 123 patterns. Thus $S_n(123)[12; \{i, n\}] \leftrightarrow S_{n-1}(123)[1; i]$. 
Now consider $S_n(123) [12]$. Then one of two cases occurs:

**Case 2:** “2” is not played by $n$.

No permutation can start this way and still avoid 123. Thus $S_n(123)[12; i, j] = \emptyset$ for $i < j < n$. 
Now consider \( S_n(123)[12] \). Then one of two cases occurs:

**Case 2**: “2” is not played by \( n \).

No permutation can start this way and still avoid 123. Thus \( S_n(123)[12; i, j] = \emptyset \) for \( i < j < n \).

**Key Idea**: You may need to restrict vertical space above/below/between letters in a prefix. Gap vector criteria check this.
The recurrence for $S_n(123)$ from the scheme

Let $s_n(B)[p; C] = |S_n(B)[p; C]|$. Summarizing the previous observations, we get the recurrence:

$$s_n(123) = s_n(123)[1]$$

$$= \sum_{i=1}^{n} s_n(123)[1; \{i\}]$$
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$$s_n(123)[1, \{i\}] = \sum_{j=1}^{i-1} s_n(123)[21; \{i > j\}] + \sum_{j=i+1}^{n} s_n(123)[12; \{i < j\}]$$
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$$= \sum_{j=1}^{i-1} s_{n-1}(123)[1; \{j\}] + 0 + s_{n-1}(123)[1; \{i\}]$$
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Outline Revisited

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✓ Outline enumeration schemes
  ■ Forming the $q$-analogue
  ■ Preliminary results
Recall we wish to compute $F(n, B, q) = \sum_{\pi \in S_n(B)} q^{\text{inv} (\pi)}$.

Notation

$$F(n, B, q)[p] = \sum_{\pi \in S_n(B)[p]} q^{\text{inv} (\pi)} \text{ and analogously for } F(n, B, q)[p; C].$$

Partition $S_n(B)[p]$ as before:

$$S_n(B)[p] = \bigcup_{p'} S_n(B)[p']$$

$$s_n(B)[p] = \sum_{p'} s_n(B)[p']$$

$$F(n, B, q)[p] = \sum_{p'} F(n, B, q)[p']$$
q-Conquering

For each prefix $p \in S_k$, one of 3 following events must be true:

(a) $S_n(B)[p] = \{p\}$

(b) For each $C \in \binom{[n]}{k}$, one of the following happens:
   - $S_n(B)[p; C]$ is in bijection with another $S_{n'}(B)[p'; C']$ for smaller $n', p', \text{ and } C'$.
   - $S_n(B)[p; C]$ is empty.

(c) $S_n(B)[p]$ must be partitioned further.
q-Conquering

For each prefix $p \in S_k$, one of 3 following events must be true:

(a) $S_n(B)[p] = \{p\}$
    
    $s_n(B)[p] = 1$, but $F(n, B, q)[p] = q^{\text{inv}(p)}$.

(b) For each $C \in \binom{[n]}{k}$, one of the following happens:
    
    - $S_n(B)[p; C]$ is in bijection with another $S_{n'}(B)[p'; C']$ for smaller $n', p'$, and $C'$.
    
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     $s_n(B)[p; C] = s_{n'}(B)[p; C']$, but
     $F(n, B, q)[p, C] = q^{\text{something}} F(n', B, q)[p'; C']$

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\[ s_n(B)[p; C] = s_{n'}(B)[p; C'] \text{, but } F(n, B, q)[p, C] = q^{\text{something}} F(n', B, q)[p'; C'] \]

- $S_n(B)[p; C]$ is empty.

\[ s_n(B)[p; C] = 0, \text{ and also } F(n, B, q)[p; C] = 0. \]

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     $s_n(B)[p; C] = s_{n'}(B)[p; C']$, but
     
     $F(n, B, q)[p, C] = q^{\text{something}} F(n', B, q)[p'; C']$
   
   - $S_n(B)[p; C]$ is empty.
     
     $s_n(B)[p; C] = 0$, and also $F(n, B, q)[p; C] = 0$.
   
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   $s_n(B)[p] = \sum_{p'} s_n(B)[p']$ and

   $F(n, B, q)[p] = \sum_{p'} F(n, B, q)[p']$. 
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   $s_n(B)[p] = \sum_{p'} s_n(B)[p']$ and
   
   $F(n, B, q)[p] = \sum_{p'} F(n, B, q)[p']$.

Thus the only trouble spot is when considering reversibly-deletable elements.
Reversibly-Deletability, revisited

Consider $S_n(B)[p; C]$, and suppose that $p_t$, played by $a$, is reversibly-deletable, then removing $p_t$ will cause the loss of some number of inversions:
Reversibly-Deletability, revisited

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![Diagram showing reversibly-deletability](image)
Reversibly-Deletability, revisited

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Algorithm: Start with $a - 1$. Look at $p_1 p_2 \cdots p_{t-1}$ and add 1 for every $p_i > p_t$ and subtract 1 for every $p_i < p_t$. 
Multiple reversibly-deletable letters

Suppose there are multiple reversibly-deletable letters, $p_t$ for $t \in T$.

1. Compute the number of lost inversions for each $p_t$.
2. Total these lost inversions.
3. Subtract to this the number of inversions formed by the $p_t$ for $t \in T$. These were counted twice.
Multiple reversibly-deletable letters

Suppose there are multiple reversibly-deletable letters, $p_t$ for $t \in T$.

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Recall when counting $s_n(123) = |S_n(123)|$, we had:

$$s_n(123)[1, \{i\}] = \sum_{j=1}^{i-1} s_{n-1}(123)[1; \{j\}] + s_{n-1}(123)[1; \{i\}]$$
123-avoiding, revisited

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Now computing $F(n, 123, q) = \sum_{\pi \in S_n(123)} q^{\text{inv}(\pi)}$, we get:

$$F(n, 123, q)[1, \{i\}] = \sum_{j=1}^{i-1} q^{i-1} \cdot F(n-1, 123, q)[1; \{j\}] + q^{n-2} \cdot F(n-1, 123, q)[1; \{i\}]$$
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These $q$-analogues depend only on the schemes computed by Zeilberger’s VATTER package (or Vatter’s WILFPLUS package).

- Schemes list reversibly-deletable letters and gap vectors for each required prefix.
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- Schemes list reversibly-deletable letters and gap vectors for each required prefix.
- No extra information is required to compute $F(n, B, q)$ if we already have the scheme for computing $|S_n(B)|$. 

Comments

These $q$-analogues depend only on the schemes computed by Zeilberger’s \texttt{VATTER} package (or Vatter’s \texttt{WILFPLUS} package).

- Schemes list reversibly-deletable letters and gap vectors for each required prefix.
- No extra information is required to compute $F(n, B, q)$ if we already have the scheme for computing $\lvert S_n(B) \rvert$.
- \textit{Any} scheme to compute $\lvert S_n(B) \rvert$ can be re-interpreted to compute $F(n, B, q)$ instead.
Outline, Revisited again

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The adaptation to compute $F(n, B, q)$ via schemes has been implemented in the Maple package $q\text{VATTER}$.

$F(n, B, q)$ has been generated for many pattern sets $B$, for $n \leq 15$.

Data is posted at http://math.rutgers.edu/~baxter/ID/directory.html
q-analogues of known results

Barcucci et al. used generating trees and considered $S_n(321)$, $S_n(321, 3\overline{1}42)$, $S_n(4231, 4132)$ to obtain natural $q$-Catalan numbers, $q$-Motzkin, and $q$-Schroeder numbers (respectively).

Simion and Schmidt (1985):
$S_n(213, 312)$, $S_n(321, 231)$, $S_n(132, 213)$ are all counted by $2^{n-1}$, but the proofs imply

- $F(n, \{213, 312\}, q) = \prod_{i=1}^{n-1} (1 + q^i)$
- $F(n, \{321, 231\}, q) = (1 + q)^{n-1}$
- $F(n, \{132, 213\}, q) = \sum_{k=0}^{n-1} q^k F(k, \{132, 213\}, q)$
Symmetry Results

\[ F(n, B, -1) \] says how many more even permutations avoid \( B \) than odd permutations.

Sometimes strange behavior appears:

\[ \{F(n, 1243, -1)\}_{n \geq 1} = 1, 0, 0, 1, 1, 1, 3, 3, 4, 4, 3, 3, \ldots \]
Symmetry Results

\( F(n, B, -1) \) says how many more even permutations avoid \( B \) than odd permutations.

Sometimes strange behavior appears:

\[
\{ F(n, 1243, -1) \}_{n \geq 1} = 1, 0, 0, 1, 1, 1, 3, 3, 4, 4, 3, 3, -16, -16, -77, -77, -164, -164, 115, \ldots
\]

Sign changes occur at \( n = 14, 20, 28, \ldots \). Why?
More symmetry results

Sometimes $F(n, B, -1)$ is very well-behaved:

**Theorem**

- $F(n, \{213, 312\}, -1) = 0$ ($n \geq 2$)
- $F(n, \{321, 231\}, -1) = 0$ ($n \geq 2$)

**Conjecture**

- $F(n, \{3412, 4312\}, -1) = 0$ \((confirmed for n \leq 25)\)
- $F(n, \{1234, 1243\}, -1) = 0$ \((confirmed for n \leq 25)\)
- $F(n, \{1243, 1342\}, -1) = 0$ \((confirmed for n \leq 25)\)
- $F(n, \{1423, 1432\}, -1) = 0$ \((confirmed for n \leq 25)\)

and many more...
Even more symmetry conjectures

Conjecture

\[ F(n, \{4231, 213\}, -1) = 1 \text{ for } n \geq 3 \]
\[ F(n, \{123, 1432\}, -1) = \pm F_{n-2} \text{ for Fibonacci numbers } F_n \]
Future Directions

- When is \( F(n, B, -1) = 0? F(n, B, -1) > 0? \left| F(n, B, -1) \right| \leq M? \)
- For what \( B \) is \( F(n, B, q) \) log-concave? Unimodal? Symmetric? Asymptotically normal?
- Extend these methods to Pudwell’s schemes for pattern-avoiding words or for barred-pattern avoidance.
- Can we use similar methods for other permutation statistics? e.g. can we compute

\[
\sum_{\pi \in S_n(B)} q^{\text{des}(\pi)} \text{ or } \sum_{\pi \in S_n(B)} q^{(32-1)(\pi)}
\]