A Denjoy-Wolff Theorem for Hilbert Metric Nonexpansive Maps on Polyhedral Domains

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Abstract

For a polyhedral domain $\Sigma \subset \mathbb{R}^n$, and a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ which does not have a fixed point in Σ, we prove that the omega limit set $\omega(x; T)$ of any point $x \in \Sigma$ is contained in a convex subset of the boundary $\partial\Sigma$. We also identify a class of order-preserving homogeneous of degree one maps on the interior of the standard cone $\mathbb{R}^n_+$ which demonstrate that there are Hilbert metric nonexpansive maps on an open simplex with omega limit sets that can contain any convex subset of the boundary.

1. *Introduction*

A bounded subset $\Sigma \subset \mathbb{R}^n$ is a *polyhedral domain* if Σ is the intersection of finitely many open half-spaces in $\mathbb{R}^n$. The usual Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$. We let $\mathrm{cl}(\Sigma)$ denote the Euclidean closure of Σ and $\partial\Sigma$ denote the boundary of $\mathrm{cl}(\Sigma)$. On any bounded convex domain Σ, one may define the Hilbert metric distance d between any two points as the logarithm of the cross ratio:

$$d(x, y) = \log\left(\frac{|x - \bar{y}||y - \bar{x}|}{|x - \bar{x}||y - \bar{y}|}\right) \tag{1·1}$$

where $\bar{x}$ is the unique point in $\partial\Sigma$ which lies on the ray from y passing through x, and $\bar{y} \in \partial\Sigma$ is the point on the ray from x through y. Note that the norm topology inherited from $\mathbb{R}^n$ and the Hilbert metric topology are equivalent inside Σ.

We consider maps $T : \Sigma \to \Sigma$ which are nonexpansive with respect to the Hilbert metric, that is $d(T(x), T(y)) \le d(x, y)$ for all $x, y \in \Sigma$. For any $x \in \Sigma$, the set of norm topology accumulation points of the sequence $T^k(x)$, $k \ge 0$, is called the *omega limit set of* x and is denoted $\omega(x; T)$. If T has a fixed point in Σ, then $\omega(x; T)$ is contained in Σ and each point $z \in \omega(x; T)$ is an accumulation point of $T^k(x)$ in the Hilbert metric topology as well as the norm topology ([**13**], Theorem 4.1; see also [**15**]). Conversely, if T has no fixed point in Σ, the iterates $T^k(x)$ converge to $\partial\Sigma$ ([**13**], Theorem 4.2). In this paper we aim to understand better the possible structure of the omega limit sets $\omega(x; T)$ when T does not have a fixed point. Work by Beardon ([**2**],[**3**]; see also [**8**] and [**9**]) has shown that if Σ is a strictly convex bounded domain and T has no fixed points in Σ, then the omega limit set $\omega(x; T)$ of any $x \in \Sigma$ consists of a single point on the boundary $\partial\Sigma$. This is precisely analogous to the classical Denjoy-Wolff theorem for analytic maps in the unit disc (see [**2**]). If Σ is not strictly convex, simple examples ([**11**], see also section 3 of this paper) show that the omega limit set can consist of more than one point. However, Karlsson and Noskov proved [**9**] that there is always some point $z \in \omega(x;T)$ such that for any other $\zeta \in \omega(x;T)$ the line segment connecting z to ζ is contained in the boundary $\partial\Sigma$. Both Karlsson and Nussbaum have conjectured that an omega limit set of a fixed point free Hilbert metric nonexpansive map should be contained in a convex subset of the boundary, even when Σ is not strictly convex. In section 2 of this paper we prove this conjecture when Σ is a polyhedral domain.

Applications of the Hilbert metric often involve maps defined on cones. In a Banach space X, a *closed cone* C is a closed convex set such that $\lambda C \subseteq C$ for all $\lambda \geq 0$ and $C \cap \{-C\} = \{0\}$. If C is the intersection of finitely many closed half-spaces in X, then we say that C is a *polyhedral cone*. A closed cone $C \subset X$ induces a partial ordering $\leq$ on X as follows:

$$x \leq y \text{ if and only if } \mathrm{y - x \in C.} \tag{1·2}$$

We say that a function $f : Y \to X$, defined on a subset $Y \subseteq X$, is *order-preserving* with respect to C if $f(x) \leq f(y)$ whenever $x \leq y$. Two elements $x, y \in C$ are *comparable* if there are constants $\beta > \alpha > 0$ such that $\alpha x \leq y \leq \beta x$. We write $x \sim y$ when x any y are comparable, and we note that $\sim$ is an equivalence relation on C ([**13**], p. 12). The equivalence classes under this equivalence relation are called the *parts* of C. If C has nonempty interior, then the interior $\mathrm{int}(C)$ is a part of C.

Suppose C is a closed cone with nonempty interior in $\mathbb{R}^n$. Any map $f : \mathrm{int}(C) \to \mathrm{int}(C)$ which is order-preserving with respect to C and is also homogeneous of degree one is known to be nonexpansive with respect to Hilbert's projective metric on $\mathrm{int}(C)$ ([**13**], Prop 1.5). On a projective subset $\Sigma = \{x \in \mathrm{int}(C) \mid q(x) = 1\}$ where q is a linear functional such that $q(x) > 0$ for all $x \in C\backslash\{0\}$, Hilbert's projective metric is a metric and is given by equation (1·1). By scaling f we obtain a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ with $T(x) = f(x)/q(f(x))$. See [**4**], [**7**], [**12**], [**14**] for applications of such maps. It is interesting to note that the problem treated in section 2 is non-trivial even in the linear case, see [**11**] and [**6**]. In applications, the domain Σ is typically not strictly convex, and in most cases, it is polyhedral. In particular, if C is the standard cone $\mathbb{R}^n_+$, then the projective subset Σ is a simplex.

Akian, Gaubert, Lemmens and Nussbaum ([**1**], Theorem 6.8) have shown that if C is a polyhedral cone in $\mathbb{R}^n$, $f : C \to C$ is a continuous order-preserving homogeneous of degree one map, and some $x \in \Sigma$ has iterates $f^k(x)$ which are bounded (in norm), then the omega limit set $\omega(x; f)$ is a periodic orbit of f. This implies that $\omega(x; f)$ is contained in a single part of C (see [**16**]), which is a stronger result than merely asserting that $\omega(x; f)$ is contained in a convex subset of $\partial\Sigma$. A similar result is obtained in Theorem 2 of [**11**] for linear maps f on a polyhedral cone, even when the iterates $f^k(x)$ are not bounded. In section 3 of this paper, we give an example of an order-preserving, homogenous of degree one map on the standard cone in $\mathbb{R}^n$ which, when scaled, can have omega limit sets that contain any convex subset of $\partial\Sigma$. This shows that order-preserving homogeneous of degree one maps can have more complicated dynamics than one might have expected from the results discussed in [**1**] and [**11**].

2. *The Main Theorem*

Our strategy to prove the main result of this paper is to first prove a theorem about horofunctions for finite dimensional Banach spaces. We then use the fact that a polyhedral domain Σ with the Hilbert metric can be embedded isometrically into a subset of $\mathbb{R}^{N \times N}$ with the sup-norm, where N is a finite number which depends on Σ. The details of this embedding are not difficult and we restate them here for convenience. For more information see [**15**].

There is a natural correspondence between $\mathbb{R}^n$ and the hyperplane $Q = \{(1, x) \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$. Under this correspondence, Σ can be thought of as a projective subset of the closed cone $C = \{\lambda x \mid \lambda \geq 0,\, x \in \mathrm{cl}(\Sigma)\} \subset \mathrm{R}^{n+1}$. Note that C has nonempty interior in $\mathbb{R}^{n+1}$, and since Σ is a polyhedral domain, C is a polyhedral cone. This means that there are linear functionals θ_i with $1 \leq i \leq N$ such that $C = \{x \in \mathbb{R}^{n+1} \mid \theta_i(x) \geq 0 \; \forall \, i = 1, ..., N\}$. See chapter 2 of [**17**] for more details. The Hilbert metric distance between any two points $x, y \in \Sigma$ is given by the following variant of equation (1·1):

$$d(x, y) = \max_{1 \leq i, j \leq N} \log \left(\frac{\theta_i(x)\theta_j(y)}{\theta_j(x)\theta_i(y)} \right).$$

Alternatively we may write:

$$d(x, y) = \max_{1 \leq i, j \leq N} \log \left(\frac{\theta_i(x)}{\theta_j(x)} \right) - \log \left(\frac{\theta_i(y)}{\theta_j(y)} \right).$$

If $\Phi : \Sigma \to \mathbb{R}^{N \times N}$ is given by $\Phi_{ij}(x) = \log(\theta_i(x)/\theta_j(x))$, we see immediately that Φ is one-to-one and $||\Phi(x) - \Phi(y)||_\infty := \max_{ij} |\Phi_{ij}(x) - \Phi_{ij}(y)| = d(x, y)$.

In addition to this embedding, we need to mention a few other preliminary details. Suppose that (Y, d_Y) is a proper metric space. By proper, we mean that every closed and bounded subset of Y is compact. Choose a fixed reference point z and a sequence x^k in Y. We say that if the sequence of functions $h_k(y) = d_Y(y, x^k) - d_Y(z, x^k)$ converges uniformly on all compact subsets of Y, then the limit function $h(y) = \lim_{k \to \infty} h_k(y)$ is a *horofunction*. For any constant $m \in \mathbb{R}$, the sublevel set $\{x \in Y \mid h(x) \leq m\}$ is called a *horoball*. For more information see [**2**], [**8**], [**10**].

Suppose that X is a finite dimensional Banach space with norm $|| \cdot ||$. Let $B^* = \{\phi \in X^* \mid \phi(x) \leq 1 \; \forall x \in X \text{ with } ||\mathrm{x}|| \leq 1\}$. In this framework, we prove the following lemma.

Lemma 2·1 *Let $y \in X$ be an element with $||y|| = 1$. For any $R > r > 0$ and any $z \in X$ with $||z|| \leq R$, if $||z - ry|| > R - \frac{3}{4}r$ then $||z - Ry|| > \frac{1}{4}R$.*

Proof. By the Hahn-Banach theorem there is some $\phi \in B^*$ such that $||z - ry|| = \phi(z - ry) > R - \frac{3}{4}r$. Then, $\phi(z) - \phi(ry) > R - \frac{3}{4}r$ so $\phi(ry) < \phi(z) - R + \frac{3}{4}r$. Since $\phi(z) \leq ||z|| \leq R$ it follows that $\phi(ry) < \frac{3}{4}r$ and hence $\phi(y) < \frac{3}{4}$. By scaling, $(R - r)\phi(y) = \phi(Ry - ry) < \frac{3}{4}(R - r)$. So

$$\phi(z - Ry) = \phi(z - ry) - \phi(Ry - ry) > R - \frac{3}{4}r - \frac{3}{4}(R - r) = \frac{1}{4}R.$$

Since $||z - Ry|| \geq \phi(z - Ry) > \frac{1}{4}R$, we are done. ☐

Theorem 2·1 *Suppose that X is a finite dimensional Banach space with norm $|| \cdot ||$ and $Y \subseteq X$. If $f : Y \to Y$ is norm nonexpansive and, for some $x \in X$, $\lim_{k \to \infty} ||f^k(x)|| = \infty$, then there is a horofunction h defined on X such that $h(f^k(x)) \to -\infty$ as $k \to \infty$.*

Proof. Let $x^k = f^k(x)$ for $k \geq 1$. Assume with out loss of generality that $x = 0$. Since $\lim_{k\to\infty} ||x^k|| = \infty$, an easy observation (see [**5**], [**8**]) indicates that we may choose a subsequence x^{k_i} with the property $||x^{k_i}|| > ||x^m||$ for all $m < k_i$. We will call this property (A). Since X is locally compact, the Ascoli-Arzela theorem implies that by taking a further subsequence we may assume the horofunction $h(y) = \lim_{j\to\infty} ||y - x^{k_j}|| - ||x^{k_j}||$ exists and is finite for all $y \in X$.

Since the unit ball in X is compact, there is a point $\bar{y} \in X$ with $||\bar{y}|| = 1$ which is an accumulation point of the sequence $x^{k_i}/||x^{k_i}||$ $(i \geq 1)$. By taking a further refinement of the sequence k_i, we may assume that:

$$\left|\left|\frac{x^{k_i}}{||x^{k_i}||} - \bar{y}\right|\right| \leq 2^{-i} \quad \text{for all } i \geq 1.$$

Thus,

$$||x^{k_i} - (||x^{k_i}||\bar{y})|| \leq 2^{-i}||x^{k_i}||, \quad \forall\, i \geq 1.$$

If we denote $||x^{k_i}||\bar{y}$ by y^i we get:

$$||x^{k_i} - y^i|| \leq 2^{-i}||x^{k_i}||, \quad \forall\, i \geq 1. \tag{2·1}$$

Fix some $i \geq 1$. Note that $||x^{k_j-m}|| < ||x^{k_j}||$ by property (A). Also

$$\begin{aligned}||x^{k_j-m} - y^j|| &\leq ||x^{k_j-m} - x^{k_j}|| + ||x^{k_j} - y^j||\\ &\leq \sum_{l=1}^{m} ||x^{k_j-l} - x^{k_j-l+1}|| + 2^{-j}||x^{k_j}||\\ &\leq m||x - f(x)|| + 2^{-j}||x^{k_j}||, \end{aligned} \tag{2·2}$$

by (2·1) and the nonexpansiveness of f. If j is large enough, equation (2·2) implies that:

$$||x^{k_j-m} - y^j|| \leq m||x - f(x)|| + 2^{-j}||x^{k_j}|| \leq \frac{1}{4}||x^{k_j}||.$$

This allows us to use Lemma 2·1 to obtain:

$$||x^{k_j-m} - y^i|| \leq ||x^{k_j}|| - \frac{3}{4}||x^{k_i}||. \tag{2·3}$$

We will now use equations (2·1) and (2·3) to estimate the horofunction h. For any $i \geq 1$ and $m \geq 0$ we have:

$$h(x^{k_i+m}) = \lim_{j\to\infty} ||x^{k_i+m} - x^{k_j}|| - ||x^{k_j}|| \leq \liminf_{j\to\infty} ||x^{k_i} - x^{k_j-m}|| - ||x^{k_j}||,$$

by the nonexpansiveness of f. Since

$$||x^{k_i} - x^{k_j-m}|| \leq ||x^{k_j-m} - y^i|| + ||y^i - x^{k_i}||,$$

equations (2·1) and (2·3) imply that:

$$||x^{k_i} - x^{k_j-m}|| \leq ||x^{k_j}|| - \frac{3}{4}||x^{k_i}|| + 2^{-i}||x^{k_i}||.$$

Thus, we conclude that:

$$h(x^{k_i+m}) \leq -\frac{3}{4}||x^{k_i}|| + 2^{-i}||x^{k_i}|| \leq -\frac{1}{4}||x^{k_i}||$$

for all $i \geq 1$ and $m \geq 0$. Since $||x^{k_i}|| \to \infty$ this immediately implies that $h(x^k) \to -\infty$. ☐

The result of Theorem 2·1 is actually true for any metric space which can be embedded isometrically into a subset of a finite dimensional Banach space. We use this strategy to prove the following theorem, which is the primary goal of this paper.

Theorem 2·2 *If Σ is a bounded polyhedral domain, equipped with the Hilbert metric d, and $T : \Sigma \to \Sigma$ is a Hilbert metric nonexpansive map with no fixed point in Σ, then the omega limit set $\omega(x;T)$ of any point $x \in \Sigma$ is contained in a convex subset of $\partial\Sigma$. In fact,* $\mathrm{co}(\bigcup_{\mathrm{y}\in\Sigma} \omega(\mathrm{y};\mathrm{T})) \subset \partial\Sigma$.

Proof. As shown above, there is an isometric embedding $\Phi : \Sigma \to \mathbb{R}^{N\times N}$ of Σ with the Hilbert metric into a subset of $\mathbb{R}^{N\times N}$ with the sup-norm, $||\cdot||_\infty$. Under this embedding, the induced map $f(v) = \Phi \circ T \circ \Phi^{-1}(v)$ is a sup-norm nonexpansive map defined on a subset of $\mathbb{R}^{N\times N}$. Assume without loss of generality that $0 = \Phi(x)$. Since T has no fixed point in Σ, we must have $d(x, T^k(x)) \to \infty$ as $k \to \infty$ ([**13**], Theorem 4.2). Thus $||f^k(0)||_\infty \to \infty$ as $k \to \infty$. Theorem 2·1 now implies that there is a horofunction h on $\mathbb{R}^{N\times N}$ such that $h(f^k(0)) \to -\infty$. Let k_j be the subsequence of integers so that $h(v) = \lim_{j\to\infty} ||v - f^{k_j}(0)||_\infty - ||f^{k_j}(0)||_\infty$. Because of the isometry between (Σ, d) and $(\mathbb{R}^{N\times N}, ||\cdot||_\infty)$, the horofunction $\hat{h}$ defined on (Σ, d) by $\hat{h}(y) = \lim_{j\to\infty} d(y, T^{k_j}(x)) - d(T^{k_j}(x), x)$ exists and $\hat{h}(y) = h(\Phi(y))$ for all $y \in \Sigma$. This implies that $\hat{h}(T^k(x)) = h(f^k(0))$ for all $k \geq 0$ and therefore $\lim_{k\to\infty} \hat{h}(T^k(x)) = -\infty$. Since the balls in Hilbert metric are convex (see [**13**] for example) it follows from the definition of $\hat{h}$ that the horoballs $H_m = \{y \in \Sigma \mid \hat{h}(y) \leq -m\}$ are convex. Let $\mathrm{cl(H_m)}$ denote the norm closure of H_m. Because $\omega(x;T) \subset \mathrm{cl(H_m)}$ for all $m \in \mathbb{N}$ it follows that $\omega(x;T) \subset \bigcap_{m\geq 0} \mathrm{cl(H_m)}$ which is a convex subset of $\partial\Sigma$.

To complete the proof, observe that for any $y \in \Sigma$, $d(T^k(x), T^k(y)) \leq d(x, y)$ for all $k > 0$. Thus, $\hat{h}(T^k(y)) \leq \hat{h}(T^k(x)) + d(x, y)$, $\forall k > 0$. Therefore $\lim_{k\to\infty} \hat{h}(T^k(y)) = -\infty$ for all $y \in \Sigma$, and thus $\omega(y;T) \subset \bigcap_{m\geq 0} \mathrm{cl(H_m)}$ for all $y \in \Sigma$. Since $\bigcap_{m\geq 0} \mathrm{cl(H_m)} \subset \partial\Sigma$ is convex, we conclude that $\mathrm{co}(\bigcup_{\mathrm{y}\in\Sigma} \omega(\mathrm{y};\mathrm{T})) \subset \partial\Sigma$. ☐

We remarked above that Theorem 2·1 can be applied to any metric space which is isometric to a subset of a finite dimensional Banach space. In addition to the Hilbert metric on a polyhedral domain, this is also true for any part of a polyhedral cone equipped with Thompson's metric (for a definition of Thompson's metric see [**13**], p. 13). Thus, repeating the argument given in the proof of Theorem 2·2 will also prove the following:

Theorem 2·3 *Suppose that $C \subset \mathbb{R}^n$ is a polyhedral cone and $C_u \subset C$ is a part of C. Let $f : C_u \to C_u$ be a Thompson metric nonexpansive map. For any $x \in C_u$, if $\omega(x;f) \neq \emptyset$, then $\omega(x;f)$ is contained in a convex subset of the boundary ∂C_u.*

3. *An Example*

Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0 \ \forall 1 \leq i \leq n\}$. We refer to $\mathbb{R}^n_+$ as the standard cone in $\mathbb{R}^n$. Let $\Sigma = \{x \in \mathrm{int}(\mathbb{R}^n_+) \mid ||x||_1 = 1\}$, where $||x||_1$ is understood to be the norm $||x||_1 = \sum_{i=1}^n |x_i|$. The set Σ is equipped with the Hilbert metric d. Note also that Σ is an $(n-1)$-dimensional simplex.

The cone $\mathbb{R}^n_+$ induces a partial ordering $\leq$ on $\mathbb{R}^n$ given by:

$$x \leq y \text{ if and only if } x_i \leq y_i \text{ for each } i \in \{1, ..., n\}.$$

Let $e = (1, 1, ..., 1) \in \mathbb{R}^n$ be the vector with every entry equal to one. We say that a map

$g : \mathbb{R}^n \to \mathbb{R}^n$ is *additively homogeneous* if $g(x+\lambda e) = g(x) + \lambda e$ for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is both additively homogeneous and order-preserving then g is said to be *topical*. Much is known about topical maps, see [**7**] for example.

We define a map $L : \mathrm{int}(\mathbb{R}^n_+) \to \mathbb{R}^n$ by $L_i(x) = \log(x_i)$. The inverse map $E : \mathbb{R}^n \to \mathrm{int}(\mathbb{R}^n_+)$ is given by $E_i(y) = \exp(y_i)$. Note that if g is topical, then $E \circ g \circ L$ is an order-preserving homogeneous of degree one map which takes $\mathrm{int}(\mathbb{R}^n_+)$ into itself. Let $V = \{x \in \mathbb{R}^n \mid x_1 = 0\}$.

Lemma 3·1 *For any sequence* $\{x^k\}_{k\geq 0} \subset V$ *such that* $x^{k+1} \geq x^k$ *for all* $k \geq 0$ *and* $x^{k+1} - x^k \leq x^k - x^{k-1}$ *for all* $k \geq 1$*, there is a topical map* $g : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $g^k(x^0) = x^k$ *for each* $k \geq 1$*.*

Proof. For each $2 \leq j \leq n$ there is an order-preserving Lipschitz function $\alpha_j : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $\mathrm{Lip}(\alpha_j) \leq 1$ such that $\alpha_j(x^k_j) = x^{k+1}_j$. Indeed, by constructing each α_j piecewise linear we see immediately that this is the case. Let $G : V \to V$ be the map $G_j(x) = \alpha_j(x_j)$ for each $2 \leq j \leq n$. For any $x \in \mathbb{R}^n$, let $g(x) = G(x - x_1 e) + x_1 e$. It is easy to see that g is additively homogeneous since

$$g(x + \lambda e) = G(x + \lambda e - (x_1 + \lambda)e) + (x_1 + \lambda)e =$$

$$= G(x - x_1 e) + (x_1 + \lambda)e = g(x) + \lambda e.$$

Suppose that $x, y \in \mathbb{R}^n$ and $x \leq y$. If $x_j - x_1 \leq y_j - y_1$, then because α_j is order-preserving,

$$g_j(x) = \alpha_j(x_j - x_1) + x_1 \leq \alpha_j(y_j - y_1) + y_1 = g_j(y).$$

If $x_j - x_1 > y_j - y_1$, then because $\mathrm{Lip}(\alpha_j) \leq 1$,

$$0 \leq \alpha_j(x_j - x_1) - \alpha_j(y_j - y_1) \leq (x_j - x_1) - (y_j - y_1) \leq y_1 - x_1.$$

Therefore

$$g_j(x) = \alpha_j(x_j - x_1) + x_1 \leq \alpha_j(y_j - y_1) + y_1 = g_j(y).$$

Thus $g : \mathbb{R}^n \to \mathbb{R}^n$ is a topical map such that $g^k(x^0) = G^k(x^0) = x^k$. $\square$

Let x^k be a sequence in V which satisfies the hypotheses of Lemma 3·1 and assume that $x^0 = 0$. Let $a^k = x^k - x^{k-1}$ for each $k \geq 1$. Note that the sequence a^k satisfies $a^{k+1} \leq a^k$ and $a^k \geq 0$ for all $k \geq 1$. Lemma 3·1 implies that there is a toical map g on $\mathbb{R}^n$ such that $x^k = g^k(0)$. Since g is topical, the map $f = E \circ g \circ L$ is an order-preserving and homogeneous of degree one map, and $f : \mathrm{int}(\mathbb{R}^n_+) \to \mathrm{int}(\mathbb{R}^n_+)$.

Let $T(x) = f(x)/||f(x)||_1$. Note that $T : \Sigma \to \Sigma$ and T is nonexpansive in Hilbert's metric on Σ (see Prop. 1.5 in [**13**]). If $\xi = (1/n, 1/n, ...1/n) \in \Sigma$, then the iterates of ξ under the map f are $f^k(\xi) = \frac{1}{n} E \circ f^k \circ L(n\xi) = \frac{1}{n} E \circ f^k(0) = \frac{1}{n} E(x^k)$. Therefore the normalized iterates $\xi^k = T^k(\xi)$ will be:

$$\xi^k = \frac{1}{||E(x^k)||_1} E(x^k) = \frac{1}{||E(x^k)||_1} E(\sum_{i=1}^{k} a^i).$$

Thus we get the following formula for the behavior of the iterates ξ^k:

$$\xi^k = \frac{1}{||E(x^k)||_1} \Pi_{i=1}^{k} E(a^i)$$

where $\Pi_{i=1}^{k} E(a^i)$ is understood to be the entrywise product of the vectors $E(a^i)$. If we rename $E(a^i) = b^i$, then

$$\xi^k = \frac{\Pi_{i=1}^{k} b^i}{||\Pi_{i=1}^{k} b^i||_1}. \tag{3·1}$$

The sequence a^i could be any sequence of vectors in V with $a^i \geq 0$ and $a^{i+1} \leq a^i$, for all $i \geq 1$, in order for the corresponding sequence $x^k = \sum_{i=1}^{k} a^i$ to satisfy the hypotheses of Lemma 3·1. Therefore, if the sequence b^i is any sequence of vectors with $b_1^i = 1$, $b^i \geq e$, and $b^{i+1} \leq b^i$ for all $i \geq 1$, then there is a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ such that $T^k(\xi)$ is given by equation (3·1). Note that ξ^k will have $\xi_1^k \to 0$ as $k \to \infty$ if and only if $||\Pi_{i=1}^{k} b^i||_1 \to \infty$. Also note that since each sequence b_j^i with $i \geq 1$ is nonincreasing and bounded below, the vectors b^i converge to a vector b^∞ with $b_j^\infty \geq 1$ for each $2 \leq j \leq n$ and $b_1^\infty = 1$.

Theorem 3·1 *For any convex subset S of $\partial\Sigma$ there is a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ and a point $\xi \in \Sigma$ such that the omega limit set $\omega(\xi; T)$ contains S.*

Proof. Since S is convex, and $S \subset \partial\Sigma$, there is some coordinate j such that $x_j = 0$ for all $x \in S$. Assume without loss of generality that $j = 1$ and let $S_1 = \{x \in \mathrm{cl}(\Sigma) \mid \mathrm{x}_1 = 0\}$. Thus $S \subseteq S_1$. We will choose a sequence b^i with $b_1^i = 1$, $b^i \geq e$, and $b^{i+1} \leq b^i$ for all $i \geq 1$, for which there is a Hilbert metric nonexpansive map $T : \Sigma \to \Sigma$ satisfying equation (3·1). We wish to choose the b^i in such a way that subsequences of ξ^k converge to a countable dense collection of points in S_1. Suppose that the vectors $b^1, ..., b^N$ are fixed for some $N > 0$. Observe that for any two vectors $\zeta, \eta \in \mathbb{R}_+^n$ with $\zeta \leq \eta$ and $\zeta_1 = \eta_1$, we may choose a finite sequence of vectors b^{N+i}, $1 \leq i \leq m$ such that $b_1^{N+i} = 1$, $b^{N+i} \leq b^{N+i-1}$, and $b^{N+i} \geq e$ for all $1 \leq i \leq m$, and such that the entrywise product of ζ with $\Pi_{i=1}^{m} b^{N+i}$ equals η. For example, by choosing m large enough, let each $b_j^{N+i} = (\eta_j/\zeta_j)^{1/m}$. Now, suppose that $s \in S_1$ and $\zeta \in \Sigma$ is arbitrarily close to s. Suppose also that s' is any other point in S_1. We may choose an η with $\zeta \leq \eta$ and $\zeta_1 = \eta_1$ such that $\eta/||\eta||_1$ is arbitrarily close to s'. This implies that if ξ^N is arbitrarily close to some $s \in S_1$, and s' is any other point in S_1, we may find an $m > 0$ such that ξ^{N+m} is the entrywise product of ξ^N with $\Pi_{i=1}^{m} b^{N+i}$ scaled to have norm one, and ξ^{N+m} is arbitarily close to s'. Repeating this process, the sequence ξ^k can accumulate at any countable collection of points $\{s^i \mid i \geq 1\}$ contained in S_1. In particular, by choosing a countable dense subset of S_1 and using the fact that $\omega(\xi; T)$ is closed, we may ensure that the omega limit set $\omega(\xi; T)$ contains all of S_1. Thus $S \subseteq \omega(\xi; T)$. □

This result shows that the restrictions on the omega limit sets of certain Hilbert metric nonexpansive maps coming from order-preserving homogeneous maps found in Theorem 6.8 of [**1**] and Theorem 2 of [**11**] are stronger than can be expected in general. In particular, the examples above show that the omega limit sets of Hilbert metric nonexpansive maps on Σ need not be contained in a single part of the cone. We can say a little more about the omega limit sets of the maps constructed in this example.

Lemma 3·2 *For any map $T : \Sigma \to \Sigma$ constructed as in the proof of Theorem* 3·1*, the omega limit set $\omega(\xi; T)$ is connected.*

Proof. For the maps constructed above, we have $\xi^k = T^k(\xi)$ given by equation (3·1), and in particular, $\lim_{i\to\infty} b^i = b^\infty$ for those maps. Note that:

$$||\xi^{k+1} - \xi^k||_\infty = \max_{1\le j\le n} \left| \frac{b_j^{k+1}\xi_j^k}{c_{k+1}} - \xi_j^k \right|,$$

where $c_{k+1} = \sum_{i=1}^n b_i^{k+1}\xi_i^k$. Since $\xi_j^k \to 0$ for any j with $b_j^\infty < \max_i b_i^\infty$ it follows that $\lim_{k\to\infty} c_k = \max_i b_i^\infty$. Thus

$$\lim_{k\to\infty} ||\xi^{k+1} - \xi^k||_\infty = \lim_{k\to\infty} \max_{1\le j\le n} \left| \frac{b_j^{k+1}\xi_j^k}{c_{k+1}} - \xi_j^k \right| \le \lim_{k\to\infty} \max_{1\le j\le n} \left| \frac{b_j^{k+1}}{c_{k+1}} - 1 \right| = 0.$$

We have shown that $||\xi^{k+1} - \xi^k||_\infty \to 0$, and we will now use this to show that $\omega(\xi;T)$ is connected. Indeed, suppose that $\omega(\xi;T)$ has a connected component $A \subseteq \omega(\xi;T)$. Since $\omega(x;T)$ is compact, A is also compact, as is $B = \omega(\xi;T)\backslash A$, assuming that B is nonempty. Let U_A be a neighborhood of A and U_B a neighborhood of B such that there is a constant $\delta > 0$ for which $||a-b||_\infty \ge \delta$ for all $a \in U_A$ and $b \in U_B$. For any k large enough, $\xi^k \in U_A \cup U_B$. However, if $M > 0$ is large enough, $||\xi^{k+1} - \xi^k||_\infty < \delta$, and therefore if $\xi^k \in U_A$, so is $T(\xi^k)$. Thus $B = \emptyset$, and $\omega(\xi;T)$ is connected. $\square$

We know that omega limit sets of a Hilbert metric nonexpansive map which are contained in $\partial\Sigma$ need not be connected in general. In all the examples of which we are aware, however, the omega limit set is either finite or connected. One might therefore ask if there are Hilbert metric nonexpansive maps with disconnected omega limit sets such that the connected components are not singletons.

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