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Take $G = SL(2,\mathbb{R})$, $\Gamma \subset G$. Look at $\Gamma\backslash\mathbb{H}$. Take

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

Take $f \in \mathcal{A}(\Gamma\backslash\mathbb{H})$, $f \in \mathcal{A}_s(\Gamma\backslash\mathbb{H})$.
Then

$$(\Delta + s(1-s))f = 0$$

Let $s = 1/2 + it$. Then $\lambda = \lambda(s) = s(1-s) = 1/4 + t^2$.

$$L^2(\Gamma\backslash\mathbb{H}) = \{f \in \mathcal{A}(\Gamma\backslash\mathbb{H}) : \int_{\Gamma\backslash\mathbb{H}} |f(z)|^2 d\mu z < \infty\}$$

Take $z \in \mathbb{H}$, $z = x + iy$, $y > 0$. Take $f \in \mathcal{B}(\Gamma\backslash\mathbb{H})$, bounded and smooth. Take $\lambda_0 = 0$, $u_0(z)$ constant. Then have

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

$\lambda_j = s_j(1-s_j)$, $(\Delta + \lambda_j)u_j = 0$, $u_j(z) \in \mathcal{A}_{s_j}(\Gamma\backslash\mathbb{H})$, $1/2 < s_j \leq 1$ or $s_j = 1/2 + it_j$, $t_j \in \mathbb{R}$.

$$\int_{\Gamma\backslash\mathbb{H}} |u_j(z)|^2 d\mu z = 1$$

$\{u_j(z) : j = 0, 1, \ldots\}$, an orthonormal basis of Maass forms.
Continuous spectrum
Take $\mathfrak{a}$, cusp of Γ.

$$E_{\mathfrak{a}}(z,s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma} (Im\gamma^{-1}z)^s$$

Take $\sigma_{\mathfrak{a}} \in G = SL(2,\mathbb{Z})$

$$\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = B = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma : \gamma\mathfrak{a} = \mathfrak{a}\}$$

For $f \in \mathcal{B}(\Gamma\backslash\mathbb{H})$,

$$\begin{aligned} f(z) &= \sum_j \langle f, u_j\rangle u_j(z) \\ &+ \frac{1}{4\pi}\sum_{\mathfrak{a}} \int_{-\infty}^{\infty} \langle f, E_{\mathfrak{a}}(*, 1/2+it)\rangle E_{\mathfrak{a}}(z, 1/2+it)dt \end{aligned}$$

Define $L : L^2 \to L^2$:

$$\begin{aligned}(Lf)(z) &= \int_{\Gamma\backslash\mathbb{H}} K(z,w)f(w)dw \\ &= \int_{\mathbb{H}} k(z,w)f(w)d\mu w \\ K(z,w) &= \sum_{\gamma\in\Gamma} k(z,\gamma w)\end{aligned}$$

where $k(z,w) = k(u(z,w))$, $k(u)$, nice, smooth, and compactly supported on $\mathbb{R}^+$.

$$\begin{aligned}K(z,w) &= \sum_j h(t_j)u_j(z)\overline{u}_j(w) \\ &+\sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\overline{E_{\mathfrak{c}}}(w,1/2+it)E_{\mathfrak{c}}(z,1/2+it)dt\end{aligned}$$

$$\langle K(\cdot,w),u_j(\cdot)\rangle = h(t_j)\overline{u}_j(w)$$

where $\lambda_j = s_j(1-s_j)$ and $s_j = 1/2+it_j$.

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$$(Lf)(z) = \int_{\mathbb{H}} k(z,w)f(w)d\mu w = \int_{\Gamma\ \mathbb{H}} K(z,w)f(w)d\mu w$$

$f \in \mathcal{A}(\Gamma\ \mathbb{H})$

$$K(z,w) = \sum_{\gamma\in\Gamma} k(z,\gamma w)$$

$k(z,w) = k(u(z,w))$, $k(u) \in C^\infty(\mathbb{R}_{\geq 0})$

$$\begin{aligned}K(z,w) &= \sum_{\gamma\in\Gamma} k(z,\gamma w) \\ &= \sum_j h(t_j)u_j(z)\overline{u_j}(w) \\ &+\sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)E_{\mathfrak{c}}(z,\frac{1}{2}+it)\overline{E_{\mathfrak{c}}}(w,\frac{1}{2}+it)dt\end{aligned}$$

$$\begin{aligned}TrK &= \int_{\Gamma\ \mathbb{H}} K(z,z)d\mu z \\ &= \sum_j h(t_j) + \sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\left(\int_{\Gamma\ \mathbb{H}} \left|E_{\mathfrak{c}}(z,\frac{1}{2}+it)\right|^2 d\mu z\right)dt\end{aligned}$$

This is not possible. Problem with the integral over $\Gamma\ \mathbb{H}$ because it is infinite unless we are in co-compact case.

Instead,

$$\begin{aligned}
Tr^Y K &= \int_{F(Y)} K(z,z) d\mu z \\
&= \sum_j h(t_j) \int_{F(Y)} |u_j(z)|^2 \\
&+ \sum_{\mathfrak{c}} \frac{1}{4\pi} \int_{-\infty}^{\infty} h(t) \left(\int_{F(Y)} \left| E_{\mathfrak{c}}(z, \frac{1}{2} + it) \right|^2 d\mu z \right) dt \\
&= A \log Y + T + O(Y^{-\epsilon}) \text{ (spectral trace)} \\
&= A_1 \log Y + T_1 + O(Y^{-\epsilon}) \text{ (geometric trace)}
\end{aligned}$$

$F(Y)$ is truncated fundamental domain.

Selberg trace formula is $T = T_1$. $A = A_1$ is tautology.

Take the Green function $G_s(z/w)$. This has logarthmic singularity on the diagonal. Therefore, we instead look at $K(z,w) = G_s(z/w) - G_a(z/w)$, $a > s > 1$.

Then

$$k(u) = G_s(u) - G_a(u)$$

$$h(r) = \frac{1}{(s-\frac{1}{2})^2 + r^2} - \frac{1}{(a-\frac{1}{2})^2 + r^2} \ll \frac{1}{(|r|+1)^4}$$

$$g(x) = \frac{1}{2s-1} e^{-|x|(s-1/2)} - \frac{1}{2a-1} e^{-|x|(a-1/2)}$$

Properties:

$$0 < k(u) \ll (u+1)^{-s}$$

$$0 < h(r) \ll (|r|+1)^{-4}$$

$$0 < g(x) \ll e^{-|r|/2}$$

Now, we truncate $E_{\mathfrak{c}}$ to $E_{\mathfrak{c}}^Y$ because we are already integrating over the truncated fundamental domain $F(Y)$. We then use Maass-Selberg Relations:

$$\begin{aligned}
&\langle E_{\mathfrak{a}}^Y(\cdot, s_1), E_{\mathfrak{b}}^Y(\cdot, s_2) \rangle \\
&= \frac{1}{s_1 - s_2} \phi_{\mathfrak{ab}}(s_2) Y^{s_1 - s_2} + \frac{1}{s_1 - s_2} \phi_{\mathfrak{ab}} Y^{s_2 - s_1} \\
&+ \frac{1}{s_1 + s_2 - 1} \delta_{\mathfrak{ab}} Y^{s_1 + s_2 - 1} \\
&- \frac{1}{s_1 + s_2 - 1} \Phi_{\mathfrak{a}}(s_1) \Phi_{\mathfrak{b}}(\overline{s}_2) Y^{1 - s_1 - s_2}
\end{aligned}$$

We apply for $s_1 = \overline{s}_2 = \sigma + iv$.

$$\begin{aligned}
&\langle \mathcal{E}^Y(\cdot, \sigma + it), {}^t\mathcal{E}^Y(\cdot, \sigma + it)\rangle \\
&\frac{1}{2iv}\left(\Phi(\overline{s})Y^{2iv} - \Phi(s)Y^{-2iv}\right) \\
&+\frac{1}{2\sigma - 1}\left(Y^{2\sigma-1} - \Phi(s)\Phi(\overline{s})Y^{1-2\sigma}\right)
\end{aligned}$$

Want to push σ to $1/2$. Limits exist because Φ is holomorphic on the line $Re(s) = 1/2$. We take $\sigma \to 1/2$, so $s = 1/2 + iv$.

$$\begin{aligned}
&\langle \mathcal{E}^Y(\cdot, \sigma + it), {}^t\mathcal{E}^Y(\cdot, \sigma + it)\rangle \\
&+\frac{1}{2s - 1}\left(\Phi(1-s)Y^{2s-1} - \Phi(s)Y^{1-2s}\right) \\
&+\frac{1}{2\sigma - 1}\left(Y^{2\sigma-1} - \Phi(s)\Phi(\overline{s})Y^{1-2\sigma}\right)
\end{aligned}$$

Note:

$$Y^{2\sigma-1} = 1 + (2\sigma - 1)\log Y + \cdots$$

$$\Phi(\sigma + iv)\Phi(\sigma - iv) = 1 + (2\sigma - 1)\frac{\Phi'(s)}{\Phi(s)} + \cdots$$

Choose $\sigma = 1/2$.

$$\begin{aligned}
&\langle \mathcal{E}^Y(\cdot, \frac{1}{2} + it), {}^t\mathcal{E}^Y(\cdot, \frac{1}{2} + it)\rangle \\
&+\frac{1}{2s - 1}\left(\Phi(1-s)Y^{2s-1} - \Phi(s)Y^{1-2s}\right) \\
&+2\log Y - \Phi'(s)\Phi(s)^{-1}
\end{aligned}$$

Therefore, when s is on the critical line,

$$\begin{aligned}
&\langle \mathcal{E}^Y(\cdot, s), {}^t\mathcal{E}^Y(\cdot, s)\rangle \\
&+\frac{1}{2s - 1}\left(\Phi(1-s)Y^{2s-1} - \Phi(s)Y^{1-2s}\right) \\
&+2\log Y - \Phi'(s)\Phi(s)^{-1}
\end{aligned}$$

By positivity, if we keep the truncated Eisenstein series, we can now integrate over F instead of $F(Y)$ and still maintain inequality for an upper bound of $Tr^Y K$.

$$\begin{aligned}
&\sum_{\mathfrak{c}} \int_F |E^Y_{\mathfrak{c}}(\cdot, \frac{1}{2} + iv)|^2 d\mu z \\
&= Tr\langle \mathcal{E}^Y(\cdot, \frac{1}{2} + iv), {}^t\mathcal{E}^Y(\cdot, \frac{1}{2} + iv)\rangle
\end{aligned}$$

$$
\begin{aligned}
Tr^Y K &= \int_{F(Y)} K(z,z) d\mu z \\
&= Tr \frac{1}{2\pi r} \left(\Phi\left(\frac{1}{2} - ir\right) Y^{2ir} - \Phi\left(\frac{1}{2} + ir\right) Y^{-2ir} \right) \\
&+ 2h \log Y - \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right)
\end{aligned}
$$

where $\phi(s) = det(\Phi(s))$.
For trace, we can assume $\Phi(s)$ is diagonal

$$
\frac{\Phi'(s)}{\Phi(s)} = \begin{pmatrix} \frac{\lambda_1'(s)}{\lambda_1(s)} & & \\ & \vdots & \\ & & \frac{\lambda_h'(s)}{\lambda_h(s)} \end{pmatrix}
$$

$$
Tr = \sum_j \frac{\lambda_j'}{\lambda_j}(s)
$$

$$
\begin{aligned}
&\frac{1}{4\pi} \int_{-\infty}^{\infty} Tr \frac{h(r)}{2\pi r} \left(\Phi\left(\frac{1}{2} - ir\right) Y^{2ir} - \Phi\left(\frac{1}{2} + ir\right) Y^{-2ir} \right) \\
&+ 2hh(r) \log Y - h(r) \frac{\phi'}{\phi}\left(\frac{1}{2} + ir\right) dr
\end{aligned}
$$

$$
\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(r)}{2ir} \left[\Phi\left(\frac{1}{2} - ir\right) Y^{2ir} - \Phi(1/2) + \Phi(1/2) - \Phi(1/2 + ir) Y^{-2ir} \right] dr
$$

We move integration in the first part to $i\epsilon + r$ and in the second part to $-i\epsilon + r$. Then this term contributes $h(0)\Phi(1/2)/2\pi$ with error $O(Y^{-2\epsilon})$.

We also have $\int h(r) dr = g(0)$. Then we have

$$
\begin{aligned}
Tr^Y K &\leq \sum_j h(t_j) + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\phi'}{\phi}(1/2 + ir) dr \\
&+ \frac{1}{4} h(0) Tr \Phi(1/2) + g(0) h \log Y + O(Y^{-\epsilon})
\end{aligned}
$$

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$$\begin{aligned} K(z,z) &= \sum_{\gamma\in\Gamma} k(z,\gamma z) \\ &= \sum_j h(t_j)|u_j(z)|^2 \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\mathfrak{a}} h(r)|E_{\mathfrak{a}}(z,1/2+ir)|^2 dr \end{aligned}$$

$Tr^Y K = \int_{F(Y)} K(z,z)$

$$\begin{aligned} &= \sum_j h(t_j)\left(\int_{F(Y)} |u_j(z)|^2 d\mu z\right) \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\mathfrak{c}} h(r)\int_{F(Y)} |E^Y_{\mathfrak{c}}(z,1/2+ir)|^2 d\mu z dr \end{aligned}$$

$$\begin{aligned} Tr^Y K &\le \sum_j h(t_j)\left(\int_F |u_j(z)|^2 d\mu z\right) \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\mathfrak{c}} h(r)\int_F |E^Y_{\mathfrak{c}}(z,1/2+ir)|^2 d\mu z dr \end{aligned}$$

$$\begin{aligned} &0 \le \sum_{\mathfrak{c}}\int_F |E^Y_{\mathfrak{c}}(z,1/2+ir)|^2 d\mu z \\ &= Tr\frac{1}{2ir}\left[\Phi(1/2-ir)Y^{2ir} - \Phi(1/2+ir)Y^{-2ir}\right] \\ &+ 2h\log Y - \frac{\phi'}{\phi}(1/2+ir) \end{aligned}$$

with $\phi(s) = det\Phi(s)$.

$$\frac{\phi'}{\phi}(1/2+it) \ge \text{ large constant}$$

$$\begin{aligned} Tr^Y K &\le \sum_j h(t_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{-\phi}{\phi}(1/2+ir)h(r)dr \\ &+ \frac{1}{4}h(0)Tr\Phi(1/2) + g(0)h\log Y + O(Y^{-\epsilon}) \end{aligned}$$

$$\textstyle\sum_{|t_j|<T}|u_j(z)|^2 + \sum_{\mathfrak{a}}\int_{-T}^{T}|E_{\mathfrak{a}}(z,1/2+it)|^2 dt \ll_\Gamma T^2 + y_\Gamma(z)T$$

$Y < Imz < 2Y$, $0 < x < 1$.

$$\sum_{|t_j| \leq T} 1 \ll T^2,\ T \geq 1$$

$$\sum_j h(t_j) \int_{F_u(Y)} |u_j(z)|^2 d\mu z \ll \sum_j h(t_j) \int_Y^\infty \frac{dy}{y^3} \ll \frac{1}{Y^\epsilon}$$

$$h(t_j) \ll 1/t_j^4$$

$$|u_j(z)|^2 \ll \frac{|s_j|}{y}$$

$s_j = 1/2 + it_j$

$$|u_j(z)|^2 \ll y^{2(1-s_j)},\ 1/2 < s_j \leq 1$$

We can get the lower bound using similar techniques estimating the contribution in the cuspidal zones. Then we have

$$\begin{aligned} Tr^Y K &= \int_{F(Y)} K(z,z) = \sum_j h(t_j) + \frac{1}{4\pi} \int_{-\infty}^\infty \frac{\phi'}{\phi}(1/2+ir)h(r)dr \\ &+ \frac{1}{4}h(0) + Tr\Phi(1/2) + g(0)h \log Y + O(Y^{-\epsilon}) \end{aligned}$$

This concludes the spectral trace formula. Now we look at the geometric trace.

$$\begin{aligned} K(z,z) &= \sum_{\gamma \in \Gamma} k(z, \gamma z) = \sum_{\mathcal{C}} K_{\mathcal{C}}(z,z) \\ K_{\mathcal{C}}(z,z) = \textstyle\sum_{\gamma \in \mathcal{C}} k(z,\gamma z) & \\ \mathcal{C} = [\gamma] &= \{\tau^{-1}\gamma\tau : \tau \in \Gamma\} \end{aligned}$$

If $\tau^{-1}\gamma\tau = \tau_1^{-1}\gamma\tau_1$, then $(\tau_1\tau^{-1})\gamma = \gamma(\tau_1\tau^{-1})$, so $\tau_1\tau^{-1} \in Z(\gamma) = \{\rho \in \Gamma : \rho\gamma = \gamma\rho\}$.

Then

$$\begin{aligned} K_{\mathcal{C}}(z,z) &= \sum_{\tau \in Z(\gamma)\backslash\Gamma} k(z, \tau^{-1}\gamma\tau z) \\ TrK_{\mathcal{C}} &= \int_{Z(\gamma)\backslash\mathbb{H}} k(z,\gamma z) d\mu z \\ &= \int_{g^{-1}Z(\gamma)g\backslash\mathbb{H}} k(z, g^{-1}\gamma g z) d\mu z,\ g \in G = SL_2(\mathbb{R}) \end{aligned}$$

$$G = SL(2,\mathbb{R}) = NAK$$

$$N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, A = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}, K = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -sin(\theta) & \cos(\theta) \end{pmatrix}$$

Lemma: If $ST = TS$, then S maps fixed points of T to itself.

Lemma: S commutes with T if and only if S has the same set of fixed points as T.

Proof: Suppose T is parabolic, say $Tz = z + u$. Suppose $ST = TS$, so $S = \alpha z + \beta$ because ∞ is a fixed point. Then $\alpha(z + u) + \beta = \alpha z + \beta + u$, so $\alpha u = u$, so $\alpha = 1$ unless $u = 0$, but $u \neq 0$ because T is not identity.

Now suppose T is hyperbolic, so $Tz = \lambda z$, $\lambda \neq 0, 1$.

$$\frac{\alpha\lambda z + \beta}{\gamma\lambda z + \delta} = \lambda\frac{\alpha z + \beta}{\gamma z + \delta}$$

Letting $z \to \infty$ implies $\gamma = 0$, which then implies $\beta = 0$, so $S = \mu z$.

The elliptic case is similar.

The sets of fixed points characterize the stabalizer.

Lemma: $Z(\gamma)$ is cyclic.

Proof: First consider γ parabolic. Write $\gamma z = z + u$. Any other element of the centralizer is also a translation, so $Z(\gamma) \subset \mathbb{R}$, discrete subgroup, so cyclic generated by the minimum element.

Now consider γ hyperbolic. Write $\gamma z = \lambda z$, $Z(\gamma) \subset \mathbb{R}^*$, so cyclic.

Elliptic case is similar.

The generators are called the primitive transformations.

Fundamental domain of hyperbolic is horizontal strip. Fundamental domain of parabolic is vertical strip. Fundamental domain of elliptic is a hyperbolic sector.

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$$K(z, w) = \sum_{\gamma\in\Gamma} k(z, \gamma w)$$

$k(u)$, $u(z, w) = \frac{|z-w|^2}{4Im(z)Im(w)}$

$k(z, \gamma w) = k(u(z, \gamma w))$.

For trace we look at

$$Tr^Y K = \int_{F(Y)} K(z, z)d\mu z = \begin{cases} A_1 \log Y + T_1 + o(1) \text{ as } Y \to \infty & \text{spectral} \\ A_1 \log Y + T_2 + o(1) \text{ as } Y \to \infty & \text{geometric} \end{cases}$$

$T_1 = T_2$.

$[\gamma] = \{\tau^{-1}\gamma\tau : \tau \in \Gamma\}$.

The centralizer $Z(\gamma) = \{\rho \in \Gamma : \rho\gamma = \gamma\rho\}$ is cyclic.

$$\begin{aligned} K(z,w) \qquad &= \sum_{\mathcal{C}} K_{\mathcal{C}}(z,w) \\ K_{\mathcal{C}}(z,w) = \textstyle\sum_{\gamma\in\mathcal{C}} k(z,\gamma w) & \\ Tr^Y K_{\mathcal{C}} \qquad &= \sum_{\gamma\in\mathcal{C}} \int_{F(Y)} k(z,\gamma z) d\mu z \\ &= \int_{Z(\gamma)\backslash\mathbb{H}} k(z,\gamma z) d\mu z \end{aligned}$$

$\mathcal{C} = \mathcal{C}_0^l$, $l \neq 0$, where $\mathcal{C}_c$ is primitive, $l \in \mathbb{Z}$ if infinite order. $0 \leq l < m$ if m is the order of $Z(\gamma)$.

Suppose $\mathcal{C}$ is parabolic. $SL(2,\mathbb{R}) = NAK$.

$$N \quad = \left\{ \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} 1, t \in \mathbb{R} \right\}$$

$\gamma = \gamma_0^l$. $\gamma_0 z = z+1$, $Z(\gamma) = Z(\gamma_0)$, $Z(\gamma_0)\backslash\mathbb{H}$ is the vertical strip from 0 to 1.

Suppose $\mathcal{C}$ is hyperbolic ($= P$ different notation).

$\gamma = \gamma_0^l$, from A: $\gamma_0 z = pz$, $p > 1$,

$$\gamma_0 = \begin{pmatrix} 0 & r_p \\ 0 & 0 \end{pmatrix} 1/r_p \in A$$

$p = NP$

$Z(\gamma_0)\backslash\mathbb{H}$ is a horizontal strip from i to pi.

Now suppose $\mathcal{C}$ is elliptic ($= E$, different notation)

$\gamma = \gamma_0^l$, γ_0 or order m, $0 < l < m$.

$\gamma_0 \in K$, $\gamma_0 = k(\theta)$, $0 \leq \theta < \pi$.

$Z(\gamma_0)\backslash\mathbb{H}$ is a sector with an angle $\theta = \pi/m$ at i.

In parabolic case,

$\mathcal{C}_{\mathfrak{a}}$ is the conjugacy class of the cusp $\mathfrak{a}$. $\Gamma_{\mathfrak{a}}$ is the stability group of the cusp $= \{\gamma_{\mathfrak{a}}^b : b \in \mathbb{Z}\}$.

γ is representative of $\mathcal{C}_{\mathfrak{a}}$, $Z(\gamma) = Z(\gamma_{\mathfrak{a}})$.

$$Tr^Y K_{\mathcal{C}} \quad = \int_{B\backslash\sigma_{\mathfrak{a}}\mathbb{H}(Y)} k(z, z+l) d\mu z$$

$$\{0 \leq x \leq 1, Y' \leq y \leq Y\} \subset \sigma_{\mathfrak{a}}\mathbb{H}(Y) \subset \{0 \leq x \leq 1 : y \leq Y\}$$

where $YY' = c_{\mathfrak{a}}^{-2}$.

Then

$$\begin{aligned} Tr^Y K_{\mathcal{C}} &\le \int_0^1 \int_0^Y k(z, z+l) d\mu z \\ &= \int_0^1 \int_0^Y k\left(\frac{l^2}{4y^2}\right) \frac{dxdy}{y^2} \\ &= \int_0^Y k\left(\frac{l^2}{4y^2}\right) \frac{dy}{y^2} \\ &= \frac{1}{|l|} \int_{l^2/4Y^2}^{\infty} k(u) \frac{du}{\sqrt{u}} \end{aligned}$$

Now sum over $l \in \mathbb{Z}_{\neq 0}$,

$$\begin{aligned} &\sum_{l \in \mathbb{Z}^*} Tr^Y K_{\mathcal{C}} \\ &\le 2 \sum_{l=1}^{\infty} \frac{1}{l} \int_{l^2/4Y^2}^{\infty} \frac{k(u)}{\sqrt{u}} du \\ &= 2 \int_{1/4Y^2}^{\infty} \frac{k(u)}{\sqrt{(u)}} \left(\sum_{1 \le l \le 2\sqrt{u}Y} \frac{1}{l} \right) \\ &= 2 \int_{1/4Y^2}^{\infty} \frac{k(u)}{\sqrt{u}} \left[\log\left(2\sqrt{u}Y\right) + \gamma + O\left(\frac{1}{\sqrt{u}Y}\right) \right] \end{aligned}$$

$k(u) \ll 1$.

$$\frac{1}{Y} \int_{1/4Y^2}^{\infty} \frac{k(u)}{u} du \ll \frac{\log Y}{Y}$$

$$\begin{aligned} &\sum_{l \in \mathbb{Z}^*} Tr^Y K_{\mathcal{C}} \\ &\le 2 \int_{1/4Y^2}^{\infty} \frac{k(u)}{\sqrt{u}} \left[\log\left(2\sqrt{u}Y\right) + \gamma \right] + O\left(\frac{\log Y}{Y}\right) \end{aligned}$$

Now we extend integration from 0 to ∞. This is okay because

$$\int_0^{1/4Y^2} \frac{du}{\sqrt{u}} \approx \frac{1}{Y}$$

$$\begin{aligned}
& \sum_{l\in\mathbb{Z}^*} Tr^Y K_{\mathcal{C}} \\
& \leq 2\int_0^\infty \frac{k(u)}{\sqrt{u}}\left[\log\left(2\sqrt{u}Y\right)+\gamma\right] + O\left(\frac{(\log Y)^2}{Y}\right) \\
& = A_2 \log Y + T_2 + O\left(\frac{(\log Y)^2}{Y}\right) \\
A_2 & = 2\int_0^\infty \frac{k(u)}{\sqrt{u}}du = q(0) = \frac{1}{2}g(0) \\
T_2 & = 2\int_0^\infty \frac{k(u)}{\sqrt{u}}\left\{\log\left(2\sqrt{u}\right)+\gamma\right\}du
\end{aligned}$$

A_1 was $(h/2)g(0)$. This agrees because we will next sum over the cusps (conjugacy classes).

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Integrals: $k(u), q(v), g(r), h(t)$:

$$\begin{aligned}
q(v) & = \int_v^\infty \frac{k(u)}{\sqrt{u-v}}du \\
g(r) & = 2q\left(\left(\sinh\frac{r}{2}\right)^2\right) \\
h(t) & = \int_{-\infty}^\infty g(r)e^{irt}dr \\
g(r) & = \frac{1}{2\pi}\int_{-\infty}^\infty h(t)e^{irt}dt \\
q(v) & = \frac{1}{2}q\left(2\log\left(\sqrt{v+1}+\sqrt{v}\right)\right) \\
k(u) & = -\frac{1}{\pi}\int_u^\infty \frac{q'(v)}{\sqrt{v-u}}dv \\
h(t) & = 4\pi\int_0^\infty F_s(u)k(u)du,\ s=\frac{1}{2}+it \\
k(u) & = \frac{1}{4\pi}\int_{-\infty}^\infty F_s(u)h(t)\tanh(\pi t)tdt
\end{aligned}$$

Spectral trace formula truncated at Y is:

$$\begin{aligned}
Tr^Y K = \textstyle\sum_j h(t_j) + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{\infty} h(r)\frac{-\phi'}{\phi}\left(\frac{1}{2}+ir\right)dr \\
&+\frac{1}{4}h(0)Tr\Phi(1/2) + g(0)h\log Y + O(Y^{-\epsilon}) \\
K(z,z) &= \sum_{\gamma\in\Gamma} k(u(z,\gamma z)) \\
Tr^Y K &= \sum_{\gamma\in\Gamma}\int_{F(Y)} k(u(z,\gamma z))d\mu z
\end{aligned}$$

$$\begin{aligned}
\mathcal{C} &= \text{conjugacy class in } \Gamma\subset SL_2(\mathbb{R}) \\
\gamma &= \text{a representative of } \mathcal{C} \\
Z(\gamma) &= \text{the centralizer, cyclic}
\end{aligned}$$

$$\begin{aligned}
Tr_{\mathcal{C}}^Y K &= \sum_{\gamma_1\in\mathcal{C}}\int_{F(Y)} k(u(z,\gamma_1 z)) \\
&= \int_{Z(\gamma)\backslash\mathbb{H}(Y)} k(u(z,\gamma z))d\mu z
\end{aligned}$$

Consider

$$\begin{aligned}
\gamma &\quad \text{-parabolic} \\
\mathfrak{a} &\quad \text{-corresponding cusp} \\
\Gamma_{\mathfrak{a}} &\quad \text{-the stability group of } \mathfrak{a}
\end{aligned}$$

Let $\gamma_{\mathfrak{a}}$ be parabolic, primitive with fixed point $\mathfrak{a}$. Every other one is the unique powers of $\gamma_{\mathfrak{a}}$.

$$\gamma = \gamma_{\mathfrak{a}}^l,\ l\in\mathbb{Z}, l\neq 0$$

$$\begin{aligned}
\textstyle\sum_{l\neq 0} Tr^Y K_{\mathcal{C}_{\mathfrak{a}}^l} &= L(Y) + O(Y^{-1}\log Y) \\
L(Y) &= g(0)(\log 2Y+\gamma) + \int_0^\infty \frac{k(u)}{\sqrt{u}}(\log u)du \\
&= g(0)(\log Y+\gamma) - \int_0^\infty \log(\sinh\frac{r}{2})g'(r)dr \\
&= g(0)\log\frac{Y}{2} + \frac{1}{4}h(0) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\psi(1+it)dt
\end{aligned}$$

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n+1}\right)$$

Now take $\mathcal{C}$- the identity class, $\mathcal{C} = \{1\}$

$$\begin{aligned}
K_{\mathcal{C}}(z,w) &= k(z,w) \\
TrK_{\mathcal{C}} = \textstyle\int_F k(z,z)dz = k(0)|F| & \\
k(0) &= \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\tanh(\pi r) r dr, \text{ because } F_s(0) = 1
\end{aligned}$$

Now we look at the hyperbolic case:
Take $\mathcal{C}$-hyperbolic class. There are two fixed points a, b.

$$\begin{aligned}
\mathcal{C} &= P^l,\ P\text{-primitive class, } l \neq 0 \\
\gamma_P \in P &= \text{ a representative} \\
\gamma = \gamma_P^l & \quad \text{a representative of } \mathcal{C} \\
Z(\gamma) &= Z(\gamma_P) \\
TrK_{\mathcal{C}} &= \int_{Z(\gamma_P)\backslash\mathbb{H}} k(u(z,\gamma z)) \\
Pz &= pz,\ p > 1 \\
TrP &= \sqrt{p} + \frac{1}{\sqrt{p}} \\
p &= NP
\end{aligned}$$

As a model, can send a to zero and b to infinity.
$u(ip, i) = \log p$ is the length of the geodesic.

$$\begin{aligned}
TrK_{\mathcal{C}} &= \int_1^p \int_{-\infty}^{\infty} k(u(z, p^l z)) d\mu z \\
&= \int_1^p \int_{-\infty}^{\infty} k\left(\left(\frac{d|z|}{y}\right)^2\right) \frac{dxdy}{y^2} \\
&= \int_1^p \frac{dy}{y} \int_{-\infty}^{\infty} k(d^2(x^2+1))dx \\
&= \frac{\log p}{d} \int_{d^2}^{\infty} \frac{k(u)}{\sqrt{u - d^2}} du \\
&= \frac{\log p}{d} q(d^2) \\
&= \left| p^{l/2} - p^{-l/2} \right|^{-1} g(l \log p) \log p
\end{aligned}$$

because

$$\begin{aligned} u(z, p^l z) &= \frac{|z - p^l z|^2}{4yp^l y} \\ &= \frac{|z|^2}{4y^2}|p^{l/2} - p^{-l/2}|^2 \\ d &= \frac{1}{2}|p^{l/2} - p^{-l/2}| \end{aligned}$$

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$\mathcal{R}$-an elliptic class, primitive of order m.

$\mathcal{C} = \mathcal{R}^l$, $0 \leq l < m$, $k(\theta) \in \mathcal{R}$, $\theta = \pi/m$

$$TrK_{\mathcal{C}} = \textstyle\int_{Z(\mathcal{R})\backslash\mathbb{H}} k(z, k(\theta l)z)d\mu z$$

polar coordinates: $z = k(\phi)e^{-r}i$, $0 \leq \phi < \pi$, $r > 0$

$d\mu z = (2\sinh r)drd\phi$.

If you take $g \in K$,

$$\begin{aligned} TrK_{\mathcal{C}} = \textstyle\int_{Z(\mathcal{R})\backslash\mathbb{H}} k(gz, k(\theta l)gz)d\mu z \\ &= \frac{1}{m}\int_{\mathbb{H}} k(z, k(\theta l)z)d\mu z \\ &= \frac{1}{m}\int_{\mathbb{H}} k(k(\phi)e^{-r}i, k(\theta l)k(\phi)e^{-r}i)(2\sinh r)drd\phi \\ &= \frac{\pi}{m}\int_0^\infty k(e^{-r}i, k(\theta l)e^{-r}i)(2\sinh r)dr \end{aligned}$$

$$k(u(e^{-r}i, k(\theta l)e^{-r}i))$$

$$u(e^{-r}i, k(\theta)e^{-r}i) = (\sinh r\sin(\theta))^2$$

$$TrK_{\mathcal{C}} = \frac{\pi}{m}\int_0^\infty k((\sinh r\sin(\theta l))^2)(2\sinh r)dr$$

$$\begin{aligned} u &= (\sinh r\sin(\theta l))^2 \\ \tfrac{du}{dr} &= 2\sin hr\cos hr(\sin(\theta l))^2 \\ \cosh r &= \sqrt{1 + (\sinh r)^2} \end{aligned}$$

$$TrK_{\mathcal{C}} = \frac{\pi}{m\sin(\theta l)}\int_0^\infty \frac{k(u)}{\sqrt{u^2 + \sin^2(\theta l)}}du$$

$$\begin{aligned} TrK_{\mathcal{C}} &= \frac{1}{m}\int_0^\infty \frac{g(r)\cosh(r/2)}{\cosh r - \cos(2\pi i/m)} \\ &= \frac{1}{2m\sin(\pi l/m)}\int_{-\infty}^{\infty} h(r)\frac{\cosh\left(\pi(1-2l/m)r\right)}{\cosh \pi r}dr \end{aligned}$$

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s) = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n+1}\right)$$

$$\begin{aligned} \textstyle\sum_{0\le l<m} TrK_{\mathcal{C}^l} &= \frac{1}{(2s-1)m}\sum_{0\le k<m}\psi\left(\frac{s+k}{m}\right)\left(\frac{2k-1}{m}-1\right) - (s=a) \\ &= \frac{\log m - \psi(s)}{m} + \frac{R_m(s)}{2s-1} - (s=a) \\ R_m(s) &= \frac{1}{m^2}\sum_{0\le k<m}(2s+2k-m)\psi\left(\frac{s+k}{m}\right) - (s=a) \end{aligned}$$

Theorem (Resolvent Trace Formula)
Let $a>1$, $Re(s)>1$. Then

$$\begin{aligned} &\sum_j\left[\frac{1}{(s-\frac{1}{2})^2+t_j^2} - (s=a)\right] \\ &+\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\left[\frac{1}{(s-\frac{1}{2})^2+r^2} - (s=a)\right]\frac{-\phi'}{\phi}(\frac{1}{2}+ir)dr} \\ &= \frac{1}{(2s-1)^2}\left(h - Tr\Phi(\frac{1}{2})\right) - \frac{h}{2s-1}\left(\psi(s+\frac{1}{2}) + \log 2\right) \\ &-\psi(s)\frac{|F|}{2\pi} + \frac{1}{2s-1}\sum_P\sum_{k=0}^{\infty}\frac{\log p}{p^{s+k}-1} \\ &+\frac{1}{2s-1}\sum_{\mathcal{R}}\sum_{0\le k<m}\frac{2k+1-m}{m^2}\psi(s+k) - (s=a) \end{aligned}$$

Corollary:

$$\textstyle\sum_P \frac{\log p}{p^s} \quad \sim \frac{1}{s-1},\ s\to 1+$$

$$\begin{aligned} Z(s) &= \prod_P\prod_{k=0}^{\infty}(1-p^{-s-k}),\ Re(s)>1 \\ Z(s) &= G(s)Z(1-s) \end{aligned}$$

$G(s)$ is infinite product of gamma factors of meromorphic order 2.

$$\frac{Z'}{Z}(s) \quad = \sum_P \sum_{k=0}^{\infty} \frac{\log p}{p^{s+k}-1}$$

so meromorphic continuation comes from trace formula.

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Contribution from the point spectrum, $\lambda_j = s_j(1-s_j)$, $s_j = 1/2+it_j$, $0 = \lambda_0 < \lambda_1 \leq \ldots$:

$$\sum_j \left[\frac{1}{(s-s_j)(s-(1-s_j))} - (s=a) \right]$$

The continuous spectrum contributes

$$+\frac{1}{4\pi} \int_{-\infty}^{\infty} \left[\frac{1}{(s-s_r)(s-(1-s_r))} - (s=a) \right] \frac{-\phi'}{\phi}(s_r)dr$$

$$= \frac{1}{(2s-1)^2}(h-\Phi(1/2)) - \frac{h}{2s-1}(\psi(s+1/2)+\log 2)$$

$s_r = 1/2+ir$.
$\phi(s) = det\Phi(s)$
$\psi(s) = \frac{\Gamma'}{\Gamma}(s) - \gamma - \sum_{n=0}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n+1}\right)$
Contribution of conjugacy classes is

$$-\psi(s)\frac{|F|}{2\pi}$$

Contribution of hyperbolic lcasses is

$$\frac{1}{2s-1} \sum_P \sum_{k=0}^{\infty} \frac{\log p}{p^{s+k}-1}$$

$p = NP$.
Finally,

$$+\frac{1}{2s-1} \sum_{\mathcal{R}} \sum_{0\leq k<m} \frac{2k+1-m}{m^2} \psi(s+k) - (s=a)$$

$$\sum_P \frac{\log p}{p^s} \sim \frac{1}{s-1},\ s \to 1^+$$

$$\begin{aligned}
Z(s) &= \prod_P \prod_{k=0}^{\infty} 1-(NP)^{-s-k},\ Res>1 \\
\tfrac{1}{2s-1}\tfrac{Z'}{Z}(s) &= \frac{1}{2s-1}\sum_P\sum_{k=0}^{\infty}\frac{\log p}{p^{s+k}} \\
\textstyle\sum_{\mathcal{C}=\mathcal{R}^l} TrK_{\mathcal{C}} &= \frac{1}{m}(\log m-\psi(s))+\frac{1}{2s-1}R_m(s) \\
R_m(s) &= \frac{1}{m^2}\sum_{0\le k<m}(2s+2(k-m)\psi(\frac{s+k}{m}))
\end{aligned}$$

Want to prove $|F|/2\pi+1/m$ is an integer. Have $|F|/2\pi+1/m = 2g-2+h+l \in \mathbb{Z}$ by Gauss-Bonnet.

$$Z(s)=\exp\left(\int_a^s \frac{Z'}{Z}(u)du\right)Z(a)$$

$Z(s)=\psi(s)Z(1-s)$.
$\Gamma = SL(2,\mathbb{Z})$, Binary quadratic forms,
$Q(x,y)=ax^2+bxy+cy^2$, $d=b^2-4ac$.
$(X,Y)=(\alpha x+\beta y, \gamma x+\delta y)$, $\begin{pmatrix} 0 & \alpha \\ \beta & \gamma \end{pmatrix}\delta\in\Gamma$
(a,b,c), the content of Q. Assume $d\neq e^2$.
$(a,b,c)=1$ primitive.
$d\equiv 0,1(4)$.
Take $d>0$.

$$P=\begin{pmatrix} 0 & \frac{1}{2}(t-du) \\ -cu & au \end{pmatrix}\frac{1}{2}(t+du)\in SL_2(\mathbb{Z})$$

$t^2-d^2u^2=4$, Pell's equation.

This preserves the quadratic form.
$\epsilon_d=\frac{1}{2}(t_d+\sqrt{d}u_d)>1$, u_d, t_d the smallest positive solution of Pell's equation.
$\frac{1}{2}(t+\sqrt{d}u)=\pm\epsilon_d^n$, $n\in\mathbb{Z}$.
$trP=t>2$, so P is hyperbolic with fixed points, $(-b-\sqrt{d})/2a$, $(-b+\sqrt{d})/2a$.
$\rho(z,Pz)=\log NP=\log p$
$[a,b,c]\to P=\begin{pmatrix} 0 & \frac{1}{2}(t_d-bu_d) \\ -cu_d & au_d \end{pmatrix}\frac{1}{2}(t_d+bu_d)\in SL_2(\mathbb{Z})$
$TrP=t_d$, $NP=(TrP+\sqrt{(TrP)^2-4}/2)^2=((t_d+\sqrt{t_d^2-4})/2)^2=((t_d+\sqrt{d}u_d)/2)^2=\epsilon_d^2$.
$\log NP = 2\log\epsilon_d$.

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Selberg Trace Formula, $\lambda_j = s_j(1-s_j)$, $s_j = 1/2 + t_j$

$$\begin{aligned}
&\sum_j h(t_j) + \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)\frac{-\phi}{\phi}(1/2+it)dr \\
&= \frac{|F|}{4\pi}\int_{-\infty}^{\infty} h(r)\tanh(\pi r) r dr \\
&+ \sum_P \sum_{l=1}^{\infty} \left(p^{l/2} - p^{-l/2}\right)^{-1} g(l\log p)\log p \\
&+ \sum_{\mathcal{R}} \sum_{1\le l<m} \left(2m\sin\frac{\pi l}{m}\right)^{-1} \int_{-\infty}^{\infty} h(r)\frac{\cosh \pi(1-2l/m)r}{\cosh \pi r} dr \\
&+ \frac{h(0)}{4} Tr(I - \Phi(1/2)) - hg(0)\log 2 - \frac{h}{2\pi}\int_{-\infty}^{\infty} h(r)\psi(1+ir)dr
\end{aligned}$$

where $h(r)$ is test function $h(-r) = h(r)$, holomorphic in $|Im(r)| \le 1/2 + \epsilon$.

$$h(r) \ll \frac{1}{(|r|-1)^{2+\epsilon}}$$

$$h(r) = \int_{-\infty}^{\infty} e^{irx} g(x) dx$$

$g(x) = 2(\cosh(x/2))e^{-2\delta\cosh x}$, $0 < \delta < 1$.

$$\begin{aligned}
h(t) &= \int_{-\infty}^{\infty} e^{irx} g(x) dx \\
&= 2\int_0^{\infty} \left[\cosh(sx) + \cosh(1-s)x\right] e^{-2\delta\cosh x} dx \\
&= 2K_s(2\delta) + 2K_{1-s}(2\delta) \\
&= \Gamma(s)\delta^{-s} + O(\delta^{-1/2}|\Gamma(s)|)
\end{aligned}$$

$1/2 \le Re(s) \le 1$.
$\sum_j h(t_j)$ contributes

$$\sum_{1/2<s_j\le 1} \Gamma(s_j)\delta^{-s_j} + O(\delta^{-1/2})$$

$$g(\log p) = \left(\sqrt{p} + \frac{1}{\sqrt{p}}\right) e^{-\delta(p+1/p)}$$

The sum over P contributes

$$\begin{aligned}
&\sum_{P} \frac{\sqrt{p^l}+\frac{1}{\sqrt{p^l}}}{\sqrt{p^l}-\frac{1}{\sqrt{p^l}}} e^{-\delta(p^l+\frac{1}{p^l})} \log p \\
&= \sum_{P} \sum_{l=1}^{\infty} \frac{p^l+1}{p^l-1} e^{-\delta p^l} \log p \left(1+O(\frac{1}{p^l}\right) \\
&= \sum_{P} e^{-\delta p} (\log p) \left(1+O(\frac{1}{p})\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{1/2<s_j\le 1} \Gamma(s_j)\delta^{-s_j} + O(\delta^{-1/2}) \\
&= \sum_{p} e^{-\delta p} \log p
\end{aligned}$$

$$\sum_{p\le x} \log p = \sum_{1/2<s_j\le 1} \frac{x^{s_j}}{s_j} + O(x^{3/4})$$

by Selberg.

Conjecture:

$$\sum_{p\le x} \log p = \sum_{1/2<s_j\le 1} \frac{x^{s_j}}{s_j} + O(x^{1/2+\epsilon})$$

$$\sum_{\epsilon_d<\sqrt{x}} 2h(d)\log\epsilon_d = x + O(x^{3/4})$$

Weyl's law:

$$\begin{aligned}
N_\Gamma(T) &= \#\{j : |t_j| \le T\} \ll T^2 \\
M_\Gamma(T) &= \frac{1}{4\pi}\int_{-T}^{T} \frac{-\phi}{\phi}(1/2+it)dt \ll T^2 \\
N_\Gamma(T) + M_\Gamma(T) &\sim \frac{T^2}{4\pi}|F|
\end{aligned}$$

$M_\Gamma(T) \ll T\log T$ for congruence groups

$$N_\Gamma(T) \sim \frac{T^2}{4\pi}|F|$$

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$s = 1/2 + it$, $t \in \mathbb{R}$, $g(x) = 2(\cosh(x/2))q(x)$ where $q(x)$ is a smooth function with $q(x) = 1$ for $-\log X \le q(x) \le \log X$ and $q(x) = 0$ for $x \ge \log(X+Y)$ and $x \le \log(X+Y)$, $X \ge Y \ge 1$

$$\frac{\log(X+Y) - \log X}{1} \approx \frac{Y}{X}\log X$$

$$\begin{aligned} h(t) &= \int_{-\infty}^{\infty} (e^{sx} + e^{(1-s)x})q(x)dx \\ &= \frac{X^s}{s} + O(Y + X^{1/2}),\ 1/2 < s \le 1 \end{aligned}$$

For $s = 1/2 + it$

$$\begin{aligned} h(t) &= \int_{0}^{\infty} e^{x/2}\cos(xt)q(x)dx \ll \frac{\sqrt{X}}{|s|}\min\left\{1, \frac{T^2}{|s|^2}\right\},\ T = \frac{X}{Y} \\ \textstyle\sum_j h(t_j) &= \sum_{1/2<s_j\le 1} \frac{X^{s_j}}{s_j} + O\left(Y + X^{1/2}T\right) \\ & \frac{|F|}{4\pi}\int_{-\infty}^{\infty} h(t)\tanh(\pi t)tdt \ll X^{1/2}T \end{aligned}$$

$$\begin{aligned} \textstyle\sum_P q(\log p)\log p &= \sum_{1/2<s_j\le 1} \frac{X^{s_j}}{s_j} + O(Y + \sqrt{X}T),\ T = X/Y \\ \textstyle\sum_{p\le X}\log p &= \sum_{1/2<s_j\le 1} \frac{X^{s_j}}{s_j} + O(Y + \sqrt{X}T) \end{aligned}$$

Theorem (Selberg)

$$\sum_{p\le X}\log p = \sum_{1/2<s_j\le 1}\frac{X^{s_j}}{s_j} + O(X^{3/4})$$

$$\begin{aligned} & \sum_{t_j} e^{-\delta t_j^2} + \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{-\phi'}{\phi}(1/2+it)e^{-\delta t^2}dt,\ 0 < \delta < 1 \\ &= \frac{|F|}{4\pi\delta} + \frac{h\log\delta}{4\sqrt{\pi\delta}} - \frac{\gamma h}{4\sqrt{\pi\delta}} + O(1) \end{aligned}$$

Corollary (Weyl's Law)

$$\begin{aligned} & \sum_{|t_j|\le T} 1 + \frac{1}{4\pi}\int_{-T}^{T}\frac{-\phi'}{\phi}(1/2+it)dt \\ &= \frac{|F|}{4\pi}T^2 - \frac{h}{\pi}T\log T + c_T T + O(T/\log T) \end{aligned}$$

$\lambda_j = s_j(1-s_j) = 1/4 + t_j^2$. Multiplicity $m(\lambda_j) \ll t_j/\log t_j \approx \sqrt{\lambda_j}/\log\lambda_j$.

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$$\begin{aligned}
M_\Gamma(T) &= \frac{1}{4\pi}\int_{-T}^{T} \frac{-\phi'}{\phi}(1/2+it)dt \\
N_\Gamma(T) + M_\Gamma(T) &\sim \frac{|F|}{4\pi}T^2 \\
\phi(s) &= \det\Phi(s) \\
\Phi(s) &= (\phi_{\mathfrak{a}\mathfrak{b}}(s)) \\
\phi_{\mathfrak{a}\mathfrak{b}}(s) &= \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{c>0} c^{-2s}S_{\mathfrak{a}\mathfrak{b}}(0,0;c) \\
\begin{pmatrix} 0 & * \\ * & c \end{pmatrix} * &\in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \\
\Gamma &= SL(2,\mathbb{Z}) \\
S_{\mathfrak{a}\mathfrak{b}}(0,0;c) &= S_{\infty,\infty}(0,0;c) = \sum_{d(\mod c)} e\left(\frac{0d+0\overline{d}}{c}\right) = {}_{d(}
\end{aligned}$$

$\phi_{\infty,\infty}(s) = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\sum_{c=1}^{\infty}\frac{\phi(s)}{c^{2s}} = \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}$

$$\phi(s) = \left(\sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\right)^h\sum_{n=1}^{\infty} a_n b_n^{-2s}$$

$0 < b_1 < b_2 < \cdots$, $a_n \in \mathbb{R}$, $a_1 \neq 0$, $Res \geq 1+\epsilon$, bounded.

$$b_1^{2s}\phi(s) \ll |s|^{-h/2},\ Res \geq 1+\epsilon$$

$\phi(s)$ does not vanish in $Res \geq \sigma_0$ for some σ_0.

$\phi(s)$ has only a finite number of poles in $Re(s) \geq 1/2$ at $s = s_j$ with $1/2 < s_j \leq 1$.

$$\phi^*(s) = b_1^{2s-1}\phi(s)\prod_{1/2<s_j\leq 1}\frac{s-s_j}{s-(1-s_j)}$$

holomorphic in $Re(s) \geq 1/2$

$phi^*(s) \ll |s|^{-h/2}$, $Re(s) \geq 1+\epsilon$, $\phi^*(s)\phi^*(1-s) = 1$.

$|\phi^*(s)| = 1$ for $Re(s) = 1/2$, $|\phi^*(s)| \leq 1$ for $Re(s) \geq 1/2$

$f(z)$ continuous in $|z| \leq 1$, holomorphic in $|z| < 1$, $f(0) \neq 0$, $f(z) \neq 0$, $|z| = 1$

$$\int_0^1 \log f(e(\theta))d\theta = \log|f(0)| - \sum_j \log|z_j|$$

z_j zeros of $f(z)$ on $|z| < 1$.

$$f(z) = \phi^*(\frac{1}{1-z})z^m$$

$m = 0, 1, \ldots$

By transormation $z = (s-1)/s$, Jensen's formula implies

$$\sum_j \log \left| \frac{z_j - 1}{z_j} \right| = \log |f(0)|$$

z_j is zero of $\phi^*(s)$ in $Re(s) > 1/2$, $z_j \neq 1$.

$$\sum_j \log \left| \frac{1 - s_j}{s_j} \right| < \infty$$

s_j poles of $\phi^*(s)$ in $Re(s) < 1/2$, $s_j \neq 0$.

$s_j = \beta_j + i\gamma_j$, $\beta_j < 1/2$, poles of $\phi^*(s)$

$$2 \log |(1-s)/s| = \log \left(1 + \frac{1 - 2\beta}{|s|^2} \right) = \frac{1 - 2\beta}{|s|^2} + O(1/|s|^4)$$

$$\sum_j \frac{1 - 2\beta_j}{|s_j|^2} < \infty$$

$$\phi^*(s) = \prod_j \frac{s + 1 + \overline{s_j}}{s - s_j} e^{g(s)}$$

Hadamard Product

s_j zeros of $\phi^*(s)$, $1 - \overline{s_j}$ ploes of $\phi^*(s)$.

$g(s)$ polynomial and bounded, so $g(s)$ is constant.

The product is bounded because $\sum_j (1 - 2\beta_j)/(|s_j|^2)$ converges.

$$-\frac{\phi^{*\prime}}{\phi^*}(s) = \sum_j \left(\frac{1}{s - s_j} - \frac{1}{s - 1 + \overline{s}_j} \right)$$

$$-\frac{\phi^{*\prime}}{\phi^*}(s) = \sum_{s_j} \left(\frac{1}{s - s_j} - \frac{1}{s - 1 + \overline{s}_j} \right) + 2 \log b_1$$

s_j all poles of $\phi(s)$.

$s_j = \beta_j + i\gamma_j$, $s = 1/2 + it$,

$$\frac{1 - 2\beta_j}{(1/2 - \beta_j)^2 + (t - \gamma_j)^2} \geq 0$$

so $-\phi'/\phi(1/2 + it) > 2 \log b_1$

$$N_\Gamma(T) = \#\{\text{poles of } \phi(s) \text{ in } \sigma < 1/2, |t| \leq T\} + O(T)$$

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$$\begin{aligned}
\Phi(s) &= (\phi_{\mathfrak{ab}}(s)) \\
\phi_{\mathfrak{aa}}(s) = \sqrt{\pi}\tfrac{\Gamma(s-1/2)}{\Gamma(s)}\textstyle\sum_c \tfrac{1}{c^{2s}} S_{\mathfrak{aa}}(0,0;c) & \\
\begin{pmatrix} 0 & * \\ * & c \end{pmatrix} * &\in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}} \\
\Gamma &= \Gamma(N) \\
\phi_{\infty\infty}(s) &= \sqrt{\pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{1}{N^{2s}}\sum_{c\equiv 0 \quad \bmod N^2}\frac{1}{c^{2s}}\phi(s) \\
&= \sqrt{pi}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}\frac{\phi(N)}{N^{4s}}\prod_{p|N}\left(1-\frac{1}{p^{2s}}\right)^{-1}
\end{aligned}$$

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{a}}, s) \quad = \sum_{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}}\left(Im\sigma_{\mathfrak{a}}^{-1}\gamma z\right)^s$$

D. Hejhol, $\Gamma = \Gamma_0(p)$

$$\Phi(s) = \begin{pmatrix} 0 & \phi_{\infty\infty}(s) \\ \phi_{\infty 0}(s) & \phi_{0\infty}(s) \end{pmatrix} \phi_{00}(s) = \phi(s)N_p(s)$$

$$N_p(s) = \frac{1}{p^{2s}-1}\begin{pmatrix} 0 & p-1 \\ p^s - p^{1-s} & p^s - p^{1-s} \end{pmatrix} p-1$$

$$\Phi(s)\Phi(1-s) = I$$

$$\Phi(s) = M(s)^{-1}M(1-s)$$

$$M(s) = \pi^{-s}\Gamma(s)\zeta(2s)\begin{pmatrix} 0 & 1 \\ p^s & p^s \end{pmatrix} 1$$

$\Gamma_0(N)$, N squarefree, $\Phi(s) = \phi(s)\bigotimes_{p|N} N_p(s)$.

Small Eigenvalues

$\lambda_j = s_j(1-s_j) < 1/4$, $1/2 < s_j < 1$.

Let $X = \Gamma\backslash\mathbb{H}$, smooth, compact (only hyperbolic and identity). $|F| = 4\pi(g-1)$, g is genus.

$\lambda_j \sim 4\pi|F|^{-1}j$.

$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$.

λ_{2g-3} can be arbirarily small. $\lambda_{4g-2} \geq 1/4$. $\lambda_1 \leq 2\frac{g+1}{g-1} \leq 6$ for $g \geq 2$.

$$\frac{1}{(4\pi\sinh d/2)^2} \leq \lambda_1 \leq \frac{1}{4} + \frac{4\pi^2}{d^2}$$

where d is the diameter of X.

$$\lambda_1 \leq \frac{8\pi(g+1)}{|F|}$$

by Zognoff.

Selberg Conjecture: $\lambda_1 \geq 1/4$ for congruence groups.

$$\#\{j : 1/4 < \lambda_j < 1/4 + c/(\log q)^2\} \approx \frac{|F|}{(\log q)^3}$$

where $\Gamma = \Gamma_0(q)$.

Theorem Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index. Let q be the maximal width of cusps. Then

$$\lambda_1 > \frac{3}{2}\left(\frac{\pi}{q}\right)^2$$

Corollary $\lambda_1 > (3/2)\pi^2$ for $SL_2(\mathbb{Z})$. $\lambda_1 > 1/4$ for $\Gamma_0(q)$, $q \leq 7$.

Proof (Roelcke).

Let $u(z)$ be the eigenfunction for $\lambda > 0$, $\|u\| = 1$.

$$\lambda = \int_F |y\nabla u|^2 d\mu z$$

$F = \Gamma\backslash\mathbb{H}$. $F = \cup_\nu \gamma_\nu F'$, $F'' = F' \cup \omega F'$.

$F_1 = \cup_{0\leq b<B}(F'+b)$, $F_2 = \cup_{0\leq b<B}(F''+b)$

$$\begin{aligned}
2\lambda B &= \sum_{\nu} \int_{\gamma_\nu F_2} |y \nabla u(z)|^2 d\mu z \\
&= \sum_{\nu} \int_{F_2} \left|y \nabla u(\gamma_\nu z)\right|^2 d\mu z \\
&\geq \sum_{\nu} \int_{\sqrt{3}2}^{\infty} \int_{0}^{B} |y \nabla u(\gamma_\nu z)|^2 d\mu z \\
u(\gamma_\nu z) &= \sum_{n \neq 0} c_{\nu u}(y) e(\frac{nx}{q_\nu}) \\
\nabla u(\gamma_\nu z) &= \left[\sum_{n \neq 0}^{\infty} \frac{2\pi n i}{q_\nu} e\left(\frac{nx}{q_\nu}\right), \sum_{n \neq 0} c_{\nu n}(y)' e\left(\frac{nx}{q_\nu}\right)\right] \\
2\lambda B &\geq B \sum_{\nu} \int_{\sqrt{3}/2}^{\infty} \sum_{n \neq 0} \left|\frac{2\pi n}{q_\nu} c_{\nu n}(y)\right|^2 dy \\
&\geq \left(\frac{2\pi}{q}\right)^2 B \sum_{\nu} \int_{\sqrt{3}/2}^{\infty} \sum_{n \neq 0} |c_{\nu n}(y)|^2 dy \\
&= \left(\frac{2\pi}{q}\right)^2 \sum_{\nu} \int_{\sqrt{3}/2}^{\infty} \int_{0}^{B} \sum_{n \neq 0} |u(\gamma_\nu z)|^2 dy \\
&\geq 3 \left(\frac{\pi}{q}\right)^2 \int_{\gamma_\nu F_1} |u(z)|^2 d\mu z \\
2\lambda B &\geq \frac{3\pi^2}{q^2} B
\end{aligned}$$

$$\lambda \geq \tfrac{3}{2} \left(\tfrac{\pi}{q}\right)^2$$

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Take F to be a polygon with an even number of sides, $\mathcal{C}_i$ a side of F, $\mathcal{D}$ another side of F. Take f such that $f(z) = 0$ on $\mathcal{C}$ (Dirichlet condition) and

$$\frac{\partial}{\partial \mathfrak{a}} f(z) = 0$$

on $\mathcal{D}$ (Neumann's condition)

$f(z) = f(w)$ for $z \equiv w$ on ∂F.

$$\lambda = \min \frac{\langle -\Delta g, g\rangle}{\langle g, g\rangle}$$

$$\langle -\Delta f, g\rangle = \int_F \nabla f \cdot \overline{\nabla g} d\mu z$$

$$\nabla f = y\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

$g(z) = v$, distance

$$\cosh v = \frac{|z-a||z-b|}{(b-a)Imz}$$

$$d\mu z = (\cosh v) du dv$$

$$\nabla g = \text{ a unit vector}$$

$$\frac{\langle -\Delta g, g\rangle}{\langle g, g\rangle} = \frac{volF}{\|g\|^2}$$

See some other notes for a description of F. $volF = 2\pi - \pi/3 - \pi/2 - \pi/2 - 0 = (2/3)\pi$.

$$\begin{aligned}
\|g\|^2 &= \int_F v^2 (\cosh v) du dv \\
&\geq \int_0^{u_0}\int_0^{v_0} v^2 (\cosh v) du dv \\
&\approx u_0 v_0^2 \sinh v_0
\end{aligned}$$

$$\cosh u_0 = \frac{|x+ir+1||x+ir-1|}{2r}$$
$$\sinh u_0 = r$$

noting $x^2 - r^2 = 1$.

$$\begin{aligned}
\cosh v_0 &= \min_y \frac{|-\frac{1}{2}+iy-x+r||-1/2+iy-x-r|}{2ry} \\
&= \frac{1}{2r}\min_y\left\{y^2 + (x+1/2-r)^2 + (x+1/2+r)^2 + \frac{(1+1/2-r)^2(x+1/2+r)^2}{y^2}\right\}^{1/2} \\
&= \frac{1}{2r}\left(x+1/2-r+x+1/2+r\right) \\
&= \frac{x+1/2}{r} \sim \frac{3}{2r}
\end{aligned}$$

as $x \to 1$

$$v_0 \quad \sim \log(3/r)$$

so

$$\begin{aligned}
\|g\|^2 &\geq \frac{3}{2}(\log(3/r))^2 \to \infty \text{ as } r \to 0 \\
\tfrac{\langle \Delta g, g\rangle}{\langle g, g\rangle} &\to 0
\end{aligned}$$

as $x \to 1$, $r \to 0$.

$$\begin{aligned}
L_s(m,n) &= \sum_{c>0} c^{-2s} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \\
\begin{pmatrix} 0 & * \\ * & c \end{pmatrix} * &\in \sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{b}} \\
Z_s(m,n) &= \frac{1}{2\sqrt{|mn|}} \sum_{c} c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \begin{cases} J_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) & mn > 0 \\ I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & mn < 0 \end{cases}
\end{aligned}$$

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$\Gamma, \mathfrak{a}, \mathfrak{b}$, $s = 1/2 + it$, $mn \neq 0$

$$Z_s(m,n) = \frac{1}{2\sqrt{|mn|}} \sum_{c>0} \frac{1}{c^{2s}} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) \begin{cases} J_{2s-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) & \text{if } mn > 0 \\ I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) & \text{if } mn < 0 \end{cases}$$

Res > 1.
$\begin{pmatrix} 0 & * \\ * & c \end{pmatrix} * \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$

Theorem $Z_s(m,n)$ has analytic continuation to $s \in \mathbb{C}$.

$$\begin{aligned}
Z_s(m,n) - Z_{1-s}(m,n) &= \frac{1}{2\pi|n|} \delta_{\mathfrak{a}\mathfrak{b}} \delta_{mn} \sin\left(\pi(s - 1/2)\right) \\
&\quad - \frac{1}{2s-1} \sum_{\mathfrak{c}} \phi_{\mathfrak{a}\mathfrak{c}}(m, 1-s) \phi_{\mathfrak{b}\mathfrak{c}}(n,s)
\end{aligned}$$

$Z_s(m,n)$ has simple poles at $s = s_j$ and $s = 1 - s_j$ with residue

$$-\frac{1}{2s_j - 1} \sum_{s_k = s_j} \overline{\phi}_{\mathfrak{a}j}(m) \rho_{\mathfrak{b}j}(n)$$

if $s_j \neq 1/2$. If instead, $Z_s(m,n)$ has double pole at $s = 1/2$ with Taylor expansion

$$\begin{aligned}
&-\frac{1}{(s-1/2)^2} \sum_{s_k = 1/2} \overline{\rho}_{\mathfrak{a}k}(m) \rho_{\mathfrak{b}k}(n) \\
&-\frac{1}{4(s-1/2)} \sum_{\mathfrak{c}} \overline{\phi}_{\mathfrak{a}\mathfrak{c}}(m,s) \phi_{\mathfrak{b}\mathfrak{c}(n,s)} + \cdots
\end{aligned}$$

$\{u_j(z)\}$, an orthonormal basis of the point spectrum.

$$u_j(\sigma_{\mathfrak{a}} z) = \cdots + \sum_{m \neq 0} \rho_{\mathfrak{a}j}(m) W_{s_j}(mz)$$

$$E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,s)=\cdots+\sum_{m\neq 0}\phi_{\mathfrak{ac}}(m,s)W_s(mz)$$

Test function

$$f(s)=4\pi\frac{s-1/2}{\sin\pi s}h(i(s-1/2))$$

same conditions on h as in Selberg trace formula. $s=1/2+it$

$$f(s)=\frac{4\pi t}{\cosh(\pi t)}h(t)\ll(|t|+1)^{-1-\epsilon}|\cosh\pi t|^{-1}$$

$$f(s)=-f(1-s)$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(1-\epsilon)}Z_s(m,n)f(s)ds+\frac{1}{2\pi i}\int_{\epsilon}Z_s(m,n)f(s)ds\\&=\frac{1}{2\pi|n|}\delta_{\mathfrak{ab}}\delta_{mn}\frac{1}{2\pi i}\int_{(1-\epsilon)}\sin(\pi(s-1/2))f(s)ds\\&-\frac{1}{2\pi i}\int_{(1-\epsilon)}\sum_{\mathfrak{c}}\phi_{\mathfrak{ac}}(m,1-s)\phi_{\mathfrak{bc}}(m,s)\frac{f(s)}{2s-1}ds\\&=\frac{1}{2\pi|n|}\delta_{\mathfrak{ab}}\delta_{mn}h_0\\&-\frac{1}{2\pi i}\int_{(1-\epsilon)}\sum_{\mathfrak{c}}\phi_{\mathfrak{ac}}(m,1-s)\phi_{\mathfrak{bc}}(m,s)\frac{f(s)}{2s-1}ds\end{aligned}$$

where

$$h_0=\frac{1}{\pi}\int_{-\infty}^{\infty}t\tanh(\pi t)h(t)dt$$

$$\begin{aligned}&=\frac{2}{2\pi i}\int_{(1-s)}Z_s(m,n)f(s)ds+\sum_{s_j}\overline{\rho}_{\mathfrak{a}j}(m)\rho_{\mathfrak{b}j}(n)\frac{h(t_j)}{\cosh(\pi t_j)}\\&=\frac{1}{2\pi|n|}\delta_{\mathfrak{ab}}\delta_{mn}h_0\\&-\frac{1}{2\pi i}\int_{(1/2)}\sum_{\mathfrak{c}}\phi_{\mathfrak{ac}}(m,1/2+it)\phi_{\mathfrak{bc}}(m,1/2+it)\frac{f(1/2+it)}{2it}dt\end{aligned}$$

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(1/2)}\sum_{\mathfrak{c}}\phi_{\mathfrak{ac}}(m,1/2+it)\phi_{\mathfrak{bc}}(m,1/2+it)\frac{f(1/2+it)}{2it}dt\\&=\frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\mathfrak{c}}\overline{\phi}_{\mathfrak{ac}}(m,1/2+it)\phi_{\mathfrak{bc}}(n,1/2+it)\frac{h(t)}{\cosh(\pi t)}dt\end{aligned}$$

$$\begin{aligned}
& \frac{2}{2\pi i}\int_{(1-\epsilon)} Z_s(m,n)f(s)ds \\
& = \frac{1}{\sqrt{|mn|}}\sum_c c^{-1}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)h^{\pm}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \\
h^+(x) & = \frac{1}{2\pi i}\int_{(1-\epsilon)} J_{2s-1}(x)f(x)ds \\
h^-(x) & = \frac{1}{2\pi i}\int_{(1-\epsilon)} I_{2s-1}(x)f(x)ds \\
h^+(x) & = 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{h(t)t}{\cosh{(\pi t)}}dt \\
h^-(x) & = \frac{4}{\pi}\int_{-\infty}^{\infty} K_{2it}(x)h(t)\sinh{(\pi t)}tdt
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{a}j}(n) & := \left(\frac{4\pi|n|}{\cosh \pi t_j}\right)^{1/2}\phi_{\mathfrak{a}j}(n) \\
\eta_{\mathfrak{a}\mathfrak{c}}(n,t) & := \left(\frac{4\pi|n|}{\cosh \pi t}\right)^{1/2}\phi_{\mathfrak{a}\mathfrak{c}}(n,1/2+it)
\end{aligned}$$

<u>Theorem</u> Let $\Gamma, \mathfrak{a}, \mathfrak{b}$, $mn \neq 0$, $h(t)$ test function. Then

$$\begin{aligned}
& \sum_j h(t_j)\overline{\nu}_{\mathfrak{a}j}(m)\nu_{\mathfrak{b}j}(n) \\
& +\frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\mathfrak{c}} h(t)\overline{\eta}_{\mathfrak{a}\mathfrak{c}}(m,t)\eta_{\mathfrak{b}\mathfrak{c}}(n,t)dt \\
& = \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}h_0 + \sum_c c^{-1}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)h^{\pm}\left(\frac{4\pi\sqrt{|mn|}}{c}\right)
\end{aligned}$$

where we use h^+ if $mn > 0$ and h^- if $mn < 0$.

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Γ, $\mathfrak{a}, \mathfrak{b}$ cusps, $\{u_j(z)\}$ an orthonormal basis in the point spectrum. $E_{\mathfrak{a}}(z,s)$, $mn \neq 0$, $m, n \in \mathbb{Z}$

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;c) = \sum_{\substack{\begin{pmatrix} 0 & a \\ * & c \end{pmatrix}_{d \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \\ d \mod c}} e\left(\frac{ma+nd}{c}\right)$$

$c > 0$

$$u_j(\sigma_{\mathfrak{a}} z) \qquad = \cdots + \sum_{n\neq 0} \rho_{j\mathfrak{a}}(m) W_{s_j}(mz)$$

$$E_{\mathfrak{c}}(\sigma_{\mathfrak{a}} z) = \cdots + \textstyle\sum_{n\neq 0} \cdots$$

$$\begin{aligned} \nu_{j_{\mathfrak{a}}}(n) &= \left(\frac{4\pi|n|}{\cosh \pi s_j}\right)^{1/2} \phi_{j\mathfrak{a}}(m) \\ \eta_{\mathfrak{ac}}(n,t) &= \left(\frac{4\pi|n|}{\cosh\left(\pi(1/2+it)\right)}\right)^{1/2} \phi_{\mathfrak{ac}}(n, 1/2+it) \end{aligned}$$

Theorem

$$\begin{aligned} &\sum_j h(t_j)\overline{\nu}_{j\mathfrak{a}}(m)\nu_{j\mathfrak{b}}(n) \\ &+\sum_{\mathfrak{c}} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t)\eta_{\mathfrak{ac}}(m,t)\eta_{\mathfrak{bc}}(n,t)dt \\ &= \delta_{\mathfrak{ab}}\delta_{mn}h_0 + \sum_c c^{-1} S_{\mathfrak{ab}}(m,n;c) h^{\pm}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \\ h^+(x) &= 2i\int_{-\infty}^{\infty} J_{2it}(x)\frac{h(t)t}{\cosh \pi t}dt \\ h^-(x) &= \frac{4}{\pi}\int_{-\infty}^{\infty} K_{2it}(x)h(t)\sinh(\pi t)tdt \\ h_0 &= \frac{1}{\pi}\int_{-\infty}^{\infty} t\tanh(\pi t)h(t)dt \end{aligned}$$

Use h^+ if $mn > 0$ and use h^- if $mn < 0$.

$$\begin{aligned} h(t) &= h(-t) \\ h(t) &\ \text{holomorphic in } |Imt| \leq 1/2+\epsilon \\ h(t) &\ll \frac{1}{(|t|+1)^{2+\epsilon}} \end{aligned}$$

$$\sum_{|t_j|\leq T} |\nu_{\mathfrak{a}j}(m)|^2 + \cdots = \frac{2}{\pi}T^2 + O(|m|T)$$

constant depends on Γ
Want to look at transforms $h \to h^+$, $h \to h^-$.

$$\mathcal{L}^2(\mathbb{R}^+, \frac{dx}{x}) = \{f : \mathbb{R}^+ \to \mathbb{C} : \int_0^{\infty} |f(x)|^2 \frac{dx}{x} < \infty\}$$

We first look at $h \to h^-$.

$$\begin{aligned}
L_f(t) &= \int_0^\infty K_{it}(y) f(y) \frac{dy}{y} \\
f(y) &= \int_{-\infty}^\infty L_f(t) K_{it}(y) \frac{\sinh(\pi t)}{\pi^2} dt \\
& \int_{-\infty}^\infty |f(y)| \left(\frac{1}{\sqrt{y}} + \frac{|\ln y|}{y} \right) dy < \infty \\
& f : (0, \infty) \to \mathbb{C} \\
& f(0) = 0 \\
& f^j(x) \ll (x+1)^{-2-\epsilon} \\
f(x) &= \frac{4}{\pi} \int_{-\infty}^\infty K_{2it}(x) K_f(t) \sinh(\pi t) dt \\
K_f(t) &= \frac{4}{\pi} \cosh(\pi t) \int_0^\infty K_{2it}(x) f(x) \frac{dx}{x}
\end{aligned}$$

<u>Theorem</u> Let $mn < 0$ and $f(x)$ satisfy above conditions. Then

$$\begin{aligned}
& \sum_c c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m, n; c) f\left(\frac{4\pi\sqrt{|mn|}}{c} \right) \\
&= \sum_j K_f(t_j) \overline{\nu}_{\mathfrak{a}_j}(m) \nu_{\mathfrak{b}_j}(n) + \cdots
\end{aligned}$$

Now to $h \to h^+$:

$$\begin{aligned}
h^+(x) &= 4 \int_0^\infty B_{2it}(x) h(t) \tanh(\pi t) t dt \\
B_\nu(x) &= \frac{1}{2 \sin \pi\nu/2} \left[J_{-\nu}(x) - J_\nu(x) \right] \\
&= -\frac{1}{2 \sinh(\pi t)} \left[J_{-2it}(x) - J_{2it}(x) \right] \\
T_f(t) &= \int_0^\infty B_{2it}(x) f(x) \frac{dx}{x} \\
f^\infty(x) &= 4 \int_0^\infty B_{2it}(x) T_f(t) \tanh(\pi t) t dt \\
f^\infty &= \frac{2}{\pi} \int_0^\infty T_f(t) \tanh(\pi t) t dt
\end{aligned}$$

Apply $h(t) = T_f(t)$.

<u>Theorem</u> Let $mn > 0$ and $f(x)$ satisfy the same conditions. Then

$$\begin{aligned}&\sum_c c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) f^\infty\left(\frac{4\pi\sqrt{mn}}{c}\right) + \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} f^\infty \\ &= \sum_j T_f(t_j)\overline{\nu}_{\mathfrak{a}_j}(m)\nu_{\mathfrak{b}_j}(n) + \cdots\end{aligned}$$

Define f^0 by

$$f(x) = f^\infty(x) + f^0(x)$$

Take l and odd integer, $l \geq 1$,

$$J_l(x) = i^{l+1} B_l(x)$$

$$\int_0^\infty J_\mu(x) J_\nu(x) \frac{dx}{x} = \frac{2\sin(\pi(\mu-\nu)/2)}{\pi(\mu-\nu)(\mu+\nu)}$$

$$2l\int_0^\infty J_l(x) J_m(x)\frac{dx}{x} = \delta_{lm}$$

$$\int_0^\infty J_l(x) B_{2it}(x)\frac{dx}{x} = 0$$

$$\begin{aligned} J_l(x) &\in \mathcal{L}^2(\mathbb{R}^+, dx/x) \\ J_l(x) &\ll \min\left\{\frac{1}{\sqrt{x}}, x^l\right\}\end{aligned}$$

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Honkel's transform

$$\int_0^\infty \frac{|f(x)|}{\sqrt{x}} dx < \infty$$

$J_\nu(x)$-Bessel function

$$H_f(y) = \int_0^\infty f(x) J_\nu(xy) dx$$

Honkel's inversion

$$f(x) = \int_0^\infty H_f(y) J_\nu(xy) dy$$

$$J_\nu(x) \ll \frac{1}{\sqrt{x}}$$

as $x \to \infty$ $\nu = 2it$, $\nu = l$, l odd, $l \geq 1$

$$J_\nu(x) \quad = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1+\nu)} \left(\frac{x}{2}\right)^{\nu+2k}$$

Recurrence formulas:

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)$$

$$J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z)$$

$$(z^\nu J_\nu(z))' = z^\nu J_{\nu-1}(z)$$

The Neumann series $J_l(x)$, l odd, $l \geq 1$

$$J_l(x) \in \mathcal{L}^2(\mathbb{R}^+, \frac{dx}{x})$$

$$N_f(l) = \int_0^\infty f(x) J_l(x) \frac{dx}{x}$$

$$f^0(x) := \sum_{1 \leq l(\text{odd})} 2l N_f(l) J_l(x)$$

Neumann series.

$$2l \int_0^\infty J_l(x) J_m(x) \frac{dx}{x} = \begin{cases} 1 & l = m \\ 0 & l \neq m \end{cases}$$

The Titchmarsh integral

$$B_\nu(x) \quad = \frac{1}{\sin \pi\nu/2} \left[J_{-\nu}(x) - J_\nu(x)\right]$$

$$B_{2it}(x) \quad = \frac{i}{2\sinh(\pi t)} \left[J_{2it}(x) - J_{-2it}(x)\right] \in \mathcal{L}^2(\mathbb{R}^+, dx/x)$$

$$\int_0^\infty J_\nu(x) J_\mu(x) \frac{dx}{x} = \frac{2}{\pi} \frac{\sin\left(\pi(\mu-\nu)/2\right)}{(\mu-\nu)(\mu+\nu)}$$

so

$$\int_0^\infty J_l(x) B_{2it}(x) \frac{dx}{x} = 0$$

for l odd.

$$T_f(t) = \int_0^\infty f(x) B_{2it}(x) \frac{dx}{x}$$

$$f^\infty(x) = \int_0^\infty T_f(t) B_{2it}(x) \tanh(\pi t) t dt$$

$$f(x) = f^0(x) + f^\infty(x)$$

$$\begin{aligned}&\frac{2\nu}{xy}\frac{d}{du}J_\nu uxJ_\nu(uy)\\&= uJ_{\nu-1}(ux)J_{\nu-1}(uy) - uJ_{\nu+1}(ux)J_{\nu+1}(uy)\end{aligned}$$

Integrate over $0 < u < 1$, and sum over $\nu = l = 1, 3, 5, 7, \ldots$

$$\sum_{0<l \text{ odd}} 2lJ_l(x)J_l(y) = xy\int_0^1 uJ_0(ux)J_0(uy)du$$

Multiply by $f(y)/y$ and integrate over y.

$$\sum_{0<l \text{ odd}} 2lJ_l(x)N_f(l) = x\int_0^1 \left(\int_0^\infty f(y)J_0(uy)dy\right) uJ_0(ux)du$$

$$\begin{aligned}&\sum_j h(t_j)\overline{\nu}_{\mathfrak{a}j}(m)\overline{\nu}_{\mathfrak{b}j}(n) + \cdots\\&= \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn}h_0 + \sum_c c^{-1}S_{\mathfrak{a}\mathfrak{b}}(m,n;c)h^+\left(\frac{4\pi\sqrt{mn}}{c}\right), \, mn > 0\\ h^+(x) &= 2i\int_0^\infty J_{2it}(x)\frac{h(t)t}{\cosh(\pi t)}dt\\ h_0 &= \frac{1}{\pi}\int_{-\infty}^\infty t\tanh(\pi t)h(t)dt\\ h^+(x) &= 4\int_0^\infty B_{2it}(x)h(t)\tanh(\pi t)tdt\end{aligned}$$

Petersson's Formula:

$$\begin{aligned}&j_\gamma(z)^{-k}f(\gamma z) = f(z), \, \gamma \in \Gamma, \, k \text{ even}, \, k \geq 2\\&dim\mathcal{M}_k(\Gamma) < \infty\\&S_k(\Gamma) \subset \mathcal{M}_k(\Gamma)\\&j_{\sigma_\mathfrak{a}}(z)^{-k}(\sigma_\mathfrak{a} z) = \sum_{n=1}^\infty \widehat{f}_\mathfrak{a}(n)e(nz)\\&\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} y^k f(z)\overline{g}(z)d\mu z\end{aligned}$$

Poincare Series

$$P_{\mathfrak{a}m}(z) \quad = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} e(m\sigma_{\mathfrak{a}}^{-1}\gamma z) \subset S_k(\Gamma)$$

$m \geq 1$, cusp form.

$f \in S_k(\Gamma)$

$$\langle f, P_{\mathfrak{a}m}\rangle = \frac{(k-2)!}{(4\pi m)^{k-1}}$$

Let $\{f_j(t)\}$ be an orthonormal basis of $S_k(\Gamma)$

$$P_{\mathfrak{a}m}(z) = \frac{(k-2)!}{(4\pi m)^{k-1}} \sum_j \overline{\widehat{f}_{\mathfrak{a}j}(m)} f_j(z)$$

$$\begin{aligned} & j_{\sigma_{\mathfrak{b}}}(z)^{-k} P_{\mathfrak{a}m}(\sigma_{\mathfrak{b}} z) \\ & = \frac{(k-2)!}{(4\pi m)^{k-1}} \sum_j \overline{\widehat{f}_{\mathfrak{a}j}(m)} j_{\sigma_{\mathfrak{b}}}(z)^{-k} f_j(\sigma_{\mathfrak{b}} z) \\ & = \sum_{n=1}^{\infty} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \widehat{P}_{\mathfrak{a}\mathfrak{b}}(m,n) e(nz) \\ \widehat{P}_{\mathfrak{a}\mathfrak{b}}(m,n) \quad & = \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} + 2\pi i^k \sum_c c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \end{aligned}$$

<u>Theorem</u> Let m, n prime integers, $k > 0$, k even. Then

$$\begin{aligned} & \frac{(k-2)!}{(4\pi\sqrt{mn})^{k-1}} \sum_j \overline{\widehat{f}_{\mathfrak{a}j}}(m) \widehat{f}_{\mathfrak{b}j}(n) \\ & = \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} + 2\pi i^k \sum_c c^{-1} S_{\mathfrak{a}\mathfrak{b}}(m,n;c) J_{k-1}(\frac{4\pi}{c}\sqrt{mn}) \end{aligned}$$

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L-functions

Γ, $f(z)$-holomorphic

$$j_\gamma(z)^{-k} f(\gamma z) = f(z)$$

$\gamma \in \Gamma$, $\gamma = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} d$, $j_\gamma(z) = cz + d$ $k > 0$, k even.

$f(z)dz^{k/2}$, $\mathfrak{a}$ is a cusp, $\sigma_{\mathfrak{a}}$ is a scaling matrix. $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$.

$$j_{\sigma_{\mathfrak{a}}}(z)^{-k} f(\sigma_{\mathfrak{a}} z) = f|_{\sigma_{\mathfrak{a}}}(z)$$

$$j_{\sigma_{\mathfrak{a}}}(z)^{-k} f(\sigma_{\mathfrak{a}} z) = \sum_n \widehat{f}_{\mathfrak{a}}(n) e(nz)$$

$$f_{\mathfrak{a}}(z) = j_{\sigma_{\mathfrak{a}}}(z)^{-k} f(\sigma_{\mathfrak{a}} z) = \sum_{n=0}^{\infty} \widehat{f}_{\mathfrak{a}}(n) e(nz)$$

$$dim \mathfrak{M}_k(\Gamma) = (k-1)(q-1) + \sum_j \left[\frac{k}{2}(1 - \frac{1}{m_j}) \right] + \frac{kh}{2}$$

Theorem. Let f be a cusp form. Then

$$\sum_{1 \le n \le N} |f_{\mathfrak{a}}(n)|^2 \ll N^k$$

where the implied constant depends on f.

Corollary

$$f_{\mathfrak{a}}(n) \ll n^{k/2}$$

(Hecke)

Proof. Consider

$$F(z) = y^{k/2} f(z)$$

$$|F(z)| \le c$$

on $\mathbb{H}$

$$\begin{aligned} \textstyle\int_0^1 |f_{\mathfrak{a}}(z)|^2 dx &= \sum_{n=1}^{\infty} |\widehat{f}_{\mathfrak{a}}(n)|^2 exp(-4\pi n y) \\ &= \int_0^1 y^{-k} |F(\sigma_{\mathfrak{a}} z)|^2 dx \\ &\le c y^{-k} \end{aligned}$$

Take $y = 1/n$ to get theorem.

Theorem: Let $\psi(n) = e(n\theta)$. Then

$$\sum_{1 \le n \le N} \psi(n) \widehat{f}_{\mathfrak{a}}(n) \ll N^{k/2} \log N$$

The implied constant depends on f but not on θ.

Proof.

$$\begin{aligned}
&\left|\int_0^1 f_n(z) \sum_{1\le n\le N} e(n(\theta+\overline{z}))dx\right| \\
&\le \int_0^1 y^{-k/2}|F(\sigma_{\mathfrak{a}}z)|| \sum_{1\le n\le N} e(n(\theta+\overline{z}))|dx \\
&\le cy^{-k/2}\int_0^1 \sum_{1\le n\le N} e(n(\theta+\overline{z}))dx \\
&\le cy^{-k/2}\int_0^1 \left|\frac{e(N\overline{z})-1}{e(\overline{z})-1}\right| dx \\
&\ll y^{-k/2}e^{2\pi Ny}\int_0^1 \frac{dx}{|e(\overline{z})-1|} \\
&\ll y^{-k/2}e^{2\pi Ny}\log(1/y)
\end{aligned}$$

L-series

Hecke correspondence $\mathfrak{M}_k^0(\Gamma)$ ∞-cusp of Γ $\sigma_\infty = \begin{pmatrix} 0 & 1 \\ & \end{pmatrix} 1$, $\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix} 1 \in \Gamma$ $\Gamma_\infty = \{\begin{pmatrix} 0 & 1 \\ b & \end{pmatrix} 1 : b \in \mathbb{Z}\}$ $f(z) = \sum_{n=1}^\infty a_n e(nz)$ $L_f(s) = \sum_1^\infty a_n n^{-s}$, $Re(s) > (k+1)/2$

$$\Lambda_f(s) = \int_0^\infty f(iy)y^{s-1}dy = (2\pi)^{-s}\Gamma(s)L_f(s)$$

Suppose $\Gamma = SL_2(\mathbb{Z})$, $\begin{pmatrix} 0 & \\ -1 & 1 \end{pmatrix} \in \Gamma$

$$\begin{aligned}
&= \int_0^1 + \int_1^\infty \\
&= \int_0^1 f(i/y)i^k y^{-k} y^{s-1}dy \\
&+ \int_1^\infty f(iy)y^{s-1}dy \\
&= \int_1^\infty f(iy)(i^k y^{k-s} + y^s)\frac{dy}{y}
\end{aligned}$$

entire

Theorem.

$$\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s)$$

entire function and

$$\Lambda_f(s) = i^k\Lambda_f(k-s)$$

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Γ, $\mathfrak{a} = \infty$,

$$\Gamma_\infty = \{\left(\begin{array}{cc} 0 & 1 \\ b & \end{array}\right) 1, b \in \mathbb{Z}\}$$

$\{u_j(z)\}$ cusp forms $(\Delta + \lambda_j)u_j = 0$, $\lambda_j = s_j(1 - s_j)$

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) W_{s_j}(nz)$$

$$W_s(z) = 2\sqrt{|y|} K_{s-1/2}(2\pi|n|y) e(nx)$$

$z \in \mathbb{C}\backslash\widehat{\mathbb{R}}$ $W_s(z) \sim e(z)$ as $y \to \infty$.

$$L_j(s) = \sum_{n \neq 0} \rho_j(n)|n|^{-s}$$

$$M_j(s) = \int_0^\infty u_j(iy) y^{s-1} dy$$

$$\int_0^\infty K_\nu(y) y^{s-1} dy = 2^{s-2} \Gamma\left(\frac{s+\nu}{2}\right) \Gamma\left(\frac{s-\nu}{2}\right)$$

$$\begin{aligned} M_j(s) &= 2^{-1/2} \pi^{-s-1/2} \Gamma\left(\frac{s+s_j}{2}\right) \Gamma\left(\frac{s+1-s_j}{2}\right) L_j(s) \\ &= \int_0^1 + \int_1^\infty \end{aligned}$$

$\left(\begin{array}{cc} 0 & 0 \\ -1 & 1 \end{array}\right) 0 \in \Gamma$, $u_j(-1/z) = u_j(z)$, $u_j(i/y) = u_j(iy)$

$$M_j(s) \quad = \int_1^\infty u_j(iy)(y^{-s} + y^s) \frac{dy}{y}$$

<u>Theorem</u> $M_j(s)$ is entire, $M_j(s) = M_j(-s)$.
Define

$$\widetilde{L}_j(s) = \sum_{n \neq 0} \rho_j(n) n |n|^{-s}$$

$$u_x = \frac{\partial}{\partial x} u = 2\pi i \sum_{n \neq 0} \rho_j(n) n W_{s_j}(2\pi n z)$$

$$u_x(i/y) = (iy)^2 u_x(iy)$$

$$(T_{-1}f)(z) = f(-\overline{z}) = f(-x + iy)$$

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

$$\begin{aligned} T_{-1}\Delta \quad & = \Delta T_{-1}, \\ T_{-1}u_j \quad & = \lambda u_j,\ \lambda = \pm 1 \\ \text{If } \lambda = 1, \quad & \rho_j(-n) = \rho_j(n) \\ \text{If } \lambda = -1, \quad & \rho_j(-n) = -\rho_j(n) \end{aligned}$$

Take $\Gamma = \Gamma_0(N)$, $\mathfrak{a} = \infty$, $\psi \mod N$, multiplicative character, $\psi(-1) = (-1)^k$, $k \geq 1$.

$$f(z) \in S_k(\Gamma_0(N), \psi).$$

$\gamma = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} d$, $c \equiv 0 \mod N$, $ad \equiv 1 \mod N$

$\psi(\gamma) = \psi(d) = \overline{\psi}(a)$.

$$f(\gamma z) = \psi(\gamma) j_\gamma(z)^k f(z),\ j_\gamma(z) = cz + d$$

Theorem

Let $\psi \mod s$ primitive, $s|N$. Let $\xi \mod r$ primitive. Let $f(z)$ have the Fourier series expansion

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz).$$

Let

$$g(z) = \sum_{n=1}^{\infty} \xi(n) a_n e(nz)$$

Then $f \in S_k(\Gamma_0(N), \psi) \Rightarrow g \in S_k(\Gamma_0(M), \psi\xi^2)$, where $M = [r^2, rs, N]$.

Proof:

$$\begin{aligned}
G(\xi) &= \sum_{u \mod r} \xi(n)e(u/r) \neq 0,\ |G(\xi)| = \sqrt{r} \\
G(\bar{\xi})\xi(n) &= \sum_{u \mod r} \bar{\xi}(u)e(un/r) \\
G(\bar{\xi})g(z) &= \sum_{u \mod r} \bar{\xi}(u)f(\begin{pmatrix} 0 & 1 \\ u/r & \end{pmatrix} 1z) \\
\text{Take} \quad & \gamma \in \Gamma_0(M),\ \gamma = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} d \\
G(\bar{\xi})g(\gamma z) &= \sum_{u \mod r} \bar{\xi}(u)f(\begin{pmatrix} 0 & 1 \\ u/r & 0 \end{pmatrix} 1\gamma z) \\
\tau &= \begin{pmatrix} 0 & 1 \\ u/r & \end{pmatrix} 1 \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} d \begin{pmatrix} 0 & 1 \\ -d^2u/r & \end{pmatrix} 1 \\
&= \begin{pmatrix} 0 & a + u/rc \\ b - (bcdu)/r - (cd^2u^2)/r^2 & c \end{pmatrix} d - (cd^2u/r) \in \Gamma_0(N) \\
G(\bar{\xi})g(\gamma z) &= \sum_{u \mod r} \bar{\xi}(u)f(\tau \begin{pmatrix} 0 & 1 \\ d^2u/r & \end{pmatrix} 1z) \\
&= \sum_{u \mod r} \bar{\xi}(u)f(\begin{pmatrix} 0 & 1 \\ d^2u/r & \end{pmatrix} 1z)\psi(d)(cz+d)^{-k} \\
&= \sum_{u \mod r} \bar{\xi}(u)f(\begin{pmatrix} 0 & 1 \\ u/r & \end{pmatrix} 1z)\psi(d)(cz+d)^{-k}\xi^2(d)
\end{aligned}$$

Now look at cusp at zero. <u>Theorem</u>.

$$\sigma_N = \begin{pmatrix} 0 & 0 \\ -1/\sqrt{N} & \sqrt{N} \end{pmatrix} 0$$

$$j_{\sigma_N}(z)^{-k} f(\sigma_N z) = \sum_{n=1}^{\infty} b_n e(nz)$$

Assume $(r, N) = 1$. Then $M = r^2 N$.

$$j_{r^2N}(z)^{-k} g(\sigma_{r^2N} z) = \psi(r)\xi(N)\epsilon_r^2 \sum_{n=1}^{\infty} \bar{\xi}(n) b_n e(nz)$$

$$\epsilon_r = G(\xi)/\sqrt{r}$$

<u>Theorem.</u>

$$\Lambda(s, f, \xi) = (2\pi)^{-s} r^s N^{s/2} \Gamma(s) L_f(s, \xi)$$

$$L_f(s,\xi) = \sum_1^\infty a_n \xi(n) n^{-s}$$

Then

$$\Lambda(s,f,\xi) = i^k \psi(r)\xi(N)\epsilon_r^2 \Lambda(k-s,\tilde{f},\bar{\xi})$$

where

$$\begin{aligned} \tilde{f}(z) &= j_{\sigma_N}(z)^{-k} f(\sigma_N z) \\ &= (\sqrt{N}z)^{-k} f(\frac{-1}{Nz}) \end{aligned}$$

$\tilde{f}(z) = \epsilon_f f(z)$, $|\epsilon_f| = 1$.

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Γ, $\mathfrak{a} \sim \infty$, $\Gamma_{\mathfrak{a}} = \begin{pmatrix} 0 & 1 \\ b & \end{pmatrix} 1$, $b \in \mathbb{Z}$.

$$u_j(z) = \sum_{n \neq 0} \rho_j(n) W_{s_j}(nz)$$

$$L_j(s,x) = \sum_{n \neq 0} \rho_j(n)|n|^{-s} e(nx)$$

$e(z) = e^{2\pi i z}$

$\gamma = \begin{pmatrix} 0 & a \\ b & c \end{pmatrix} d$, $\gamma\infty = a/c$.

$z = x + iy$,

$$W_s(z) = 2\sqrt{|y|} K_{s-1/2}(2\pi|y|)e(x)$$

Take the Mellin transform of the Fourier expansion.

$$\frac{1}{2}\pi^{-s-1/2}\Gamma(\frac{s+s_j}{2})\Gamma(\frac{s+1-s_j}{2})L_j(s,x)$$

Take $x = -d/c$, $\gamma z = a/c + i/c^2 y$, $u_j(\gamma z) = u_j(z)$.

Then we have

$$\begin{aligned} &\int_0^\infty u_j(z) y^{s-1} dy \\ &= \int_0^{1/c} + \int_{1/c}^\infty \\ &= \int_0^{1/c} u_j(a/c + i/c^2 y) y^{s-1} dy + \int_{1/c}^\infty u_j(-d/c + iy) y^{s-1} dy \\ &= c^{-s} \int_1^\infty [u_j(a/c + iy/c) y^{-s} + u_j(-d/c + iy/c) y^s] \frac{dy}{y} \end{aligned}$$

Put

$$\Lambda_j(s,-d/c) = \left(\frac{c}{\pi}\right)^s \Gamma(\frac{s+s_j}{2})\Gamma(\frac{s+1-s_j}{2}) L_j(s,-d/c)$$

Theorem $\Lambda_j(s,-d/c)$ is entire and it satisfies the functional equation

$$\Lambda_j(s,-d/c) = \Lambda_j(-s,a/c)$$

$$\begin{aligned}
\tilde{L}_j(s,x) &= \sum_{n\neq 0} \rho_j(n)\frac{n}{|n|}|n|^{-s}e(nx) \\
\tfrac{1}{2\pi i}\tfrac{\partial}{\partial x}u_j(z) &= \sum_{n\neq 0} \rho_j(n) n W_{s_j}(nz) \\
& \frac{1}{2}\pi^{-s-1/2}\Gamma(\frac{s+s_j}{2})\Gamma(\frac{s+1-s_j}{2})\tilde{L}_j(s,x) \\
&= \int_0^\infty \frac{1}{2\pi i}\frac{\partial}{\partial x}u_j(z)y^{s-1}dy \\
&= \int_0^{1/c} + \int_{1/c}^\infty
\end{aligned}$$

$$\gamma z = \frac{a}{c} - \frac{cz+d}{c((cx+d)^2+c^2y^2)}$$

$$= c^{-s}\int_1^\infty [\frac{\partial}{\partial x}u_j(-d/c+iy/c)y^{s-1} + \frac{\partial}{\partial x}u_j((a+iy)/c)y^s]dy$$

Theorem 2

$\tilde{\Lambda}_j(s,-d/c)$ is entire and it satisfies the functional equation

$$\tilde{\Lambda}_j(s,-d/c) = \tilde{\Lambda}_j(2-s,a/c)$$

$$\tilde{\Lambda}_j(s,-d/c) = \pi^{-s}\Gamma(\frac{s+s_j}{2})\Gamma(\frac{s+1-s_j}{2})\tilde{L}_j(s,-d/c)$$

Poisson's Formula

$$\begin{aligned}
& \left(\frac{c}{4\pi}\right)^s L_j(s,-d/c) = \psi(-s)\left(\frac{c}{4\pi}\right)^{-s} L_j(-s,a/c) \\
\psi(s) &= 2^{4s}\frac{\Gamma(\frac{s+s_j}{2})\Gamma(\frac{s+1-s_j}{2})}{\Gamma(\frac{-s+s_j}{2})\Gamma(\frac{-s+1-s_j}{2})} \\
& \Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi \\
& \Gamma(z/2)\Gamma((1+z)/2) = \sqrt{\pi}2^{1-z}\Gamma(z) \\
\psi(s) &= \frac{1}{\pi}4^s\Gamma(s+s_j)\Gamma(s+1-s_j)[\cos(\pi(s_j-1/2)) - \cos(\pi(s-1/2))].
\end{aligned}$$

$$\begin{aligned}
&\sum_{n\neq 0}\phi_j(n)e(-dn/c)f\left(\frac{4\pi\sqrt{|n|}}{c}\right)\\
&= \frac{1}{2\pi i}\int_{(\sigma)} L_j(s,-d/c)\widehat{f}(2s)\left(\frac{c}{4\pi}\right)^{2s}ds\\
\widehat{f}(s) &= \int_0^\infty f(y)y^{s-1}dy\\
f(y) &= \frac{1}{2\pi i}\int_{(\sigma)}\widehat{f}(s)y^{-s}ds\\
f(y) &\in C_0^\infty(\mathbb{R}^+)\\
g(y) &= \frac{1}{\pi i}\int_{(\sigma)}\widehat{f}(2s)\psi(-s)y^{-2s}ds
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(\sigma)} L_j(s,-d/c)\widehat{f}(2s)\left(\frac{c}{4\pi}\right)^{2s}ds\\
&= \sum_{n\neq 0}\rho_j(n)e(na/c)g(4\pi\sqrt{|n|}/c)\\
g(y) &= \int_0^\infty f(x)\left(\frac{1}{\pi i}\int_{(\sigma)}(xy)^{-2s}\psi(s)ds\right)dx\\
&= \int_0^\infty f(x/y)\left(\frac{1}{\pi i}\int_{(\sigma)}x^{-2s}\psi(s)ds\right)\frac{dx}{x}\\
&= \int_0^\infty f(x/y)h(x)dx
\end{aligned}$$

Barnes' functions

$$\begin{aligned}
h(x) &= J(x,s_j)+I(x,s_j)\\
J(x,s_j) &= \frac{J_{1-2s_j}(x)-J_{2s_j-1}(x)}{\sin\pi(s_j-1/2)}\\
I(x,s_j) &= \frac{I_{1-2s_j}(x)-I_{2s_j-1}(x)}{\sin\pi(s_j-1/2)}
\end{aligned}$$

Theorem

$$\sum_{n\neq 0}\rho_j(n)e(-dn/c)f(4\pi\sqrt{|n|}/c) = \sum_{n\neq 0}\rho_j(n)e(an/c)g(4\pi\sqrt{|n|}/c)$$

where

$$g(y) = \int_0^\infty f(x/y)h(x)dx$$

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Γ, $\{u_j\}$ orthonormal basis of Maass cusp forms.

$$\langle u_j, u_k\rangle = \int_{\Gamma\backslash\mathbb{H}} u_j(z)\overline{u}_k(z)d\mu z$$

$$d\mu z = \frac{dxdy}{y^2}$$

$$\mu_j z = |u_j(z)|^2 d\mu z$$

$$(\Delta + \lambda_j)u_j = 0,\ \lambda_j = s_j(1-s_j)$$

$\langle u_j, u_j\rangle = 1$,

<u>Conjecture (Sarnak-Rudnik)</u> $\mu_j z \to \frac{3}{\pi} d\mu z$.

This means that for any test function $h(z)$ on $\Gamma\backslash\mathbb{H}$, smooth, compactly supported, we have

$$\int_{\Gamma\backslash\mathbb{H}} h(z)d\mu_j z \to \int_{\Gamma\backslash\mathbb{H}} h(z)d\mu z$$

as $j \to \infty$.

Holowinsky and S?

$$f(z) \in S_k(\Gamma),\ \Gamma = SL_2(\mathbb{Z})$$

$$F(z) = y^{k/2}f(z)$$

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathbb{H}} y^k f(z)\overline{g}(z)d\mu z$$

$$dim S_k(\Gamma) \sim k/12,\ k \text{ even}$$

$$\mu_f z := |F_f(z)|^2 d\mu z \to \frac{3}{\pi}d\mu z \text{ (Conjecture)}$$

$$\int h(z)d\mu_f z \to \int h(z)d\mu z$$

as $k \to \infty$.

Take $f(z) = \Delta(z)^{k/12}$, $12|k$

$$\int h(z)|\Delta(z)|^{k/6}y^{k/2}d\mu z \to \int h(z)d\mu z$$

as $k \to \infty$.

$$T_n f = \lambda_f(n)f,\ n = 1, 2, 3, \ldots$$

$$\langle hF_k, F_k\rangle \to \frac{3}{\pi}\langle h, 1\rangle$$

as $k \to \infty$.

$$\begin{aligned} h(z) &\quad -\text{Maass-Hecke cusp form} \\ h(z) &\quad -E(z, \frac{1}{2}+it) \text{ the Eisenstein series} \\ h(z) &\quad -\text{constant (OK)} \end{aligned}$$

$\langle hF_k, F_k\rangle \to 0$ as $k \to \infty$, h a cusp form.
$\langle E(\cdot|\frac{1}{2}+it)F_k, F_k\rangle \to 0$ as $k \to \infty$ for any fixed $t \in \mathbb{R}$.

$$P_m(z|\psi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(Im\gamma z)e(m\gamma z),\ m = 0, 1, 2, \dots$$

$\psi(y) \in \mathcal{C}_0^\infty\mathbb{R}^+$, $\psi(y) = y^s$ $m = 0$:

$$P_0(z|\psi) = E(z|\psi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(Im\gamma z)$$

incomplete Eisenstein series.

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Quantum Unique Ergodicity

$\Gamma = SL_2(\mathbb{Z})$, $X = \Gamma\backslash\mathbb{H}$, $d\mu = y^{-2}dxdy$, $f \in S_k(\Gamma)$, $k \geq 2$, even. $dim(S_k(\Gamma)) \asymp k$ as $k \to \infty$. $F_k(z) = y^{k/2}f(z)$. $d\mu_k = y^k|f(z)|^2 d\mu \to \frac{3}{\pi}d\mu$.
$\langle F_k, F_k\rangle = \int_X y^k|f(z)|^2 \frac{dxdy}{y^2} = 1$
$\langle hF_k, F_k\rangle \to \frac{3}{\pi}\langle h, 1\rangle$, $f(z)$ Hecke cusp form, h smooth and bounded on X.
$T_n f = \lambda_f(n)f$, $n = 1, 2, \dots$.

$$\begin{aligned} h &= \sum_{\phi_j} \langle \phi, h\rangle \phi_j(z) \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty} \langle E(\cdot, \frac{1}{2}+it), h\rangle E(z, \frac{1}{2}+it)dt \end{aligned}$$

Fix a Maass cusp form $\phi(z)$. $\langle \phi F_k, F_k\rangle \to 0$.
$\langle E(\cdot, s)F_k, F_k\rangle \to 0$, $s = 1/2 + it$.
$\underline{L\text{-functions.}}$

$$\begin{aligned} L(f, s) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},\ Re(s) > 1 \\ &= \prod_p \left(1 - \alpha_1(p)p^{-s}\right)^{-1} \cdots \left(1 - \alpha_d(p)p^{-s}\right)^{-1}, \end{aligned}$$

Euler product, $d = 1, 2, \ldots$, $|\alpha_j(p)| \leq 1$, $1 \leq j \leq d$.

$$\gamma(f,s) \quad = \pi^{-sd/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right)$$

the gamma factor, $Re(\kappa_j) \geq 0$.

$q(f)$- conductor of f, a positive integer.

$\Lambda(f,s) = q(f)\gamma(f,s)L(f,s)$, complete L-function.

$\Lambda(f,s) = \epsilon_f \Lambda(\tilde{f}, 1-s)$ functional equation, $|\epsilon_f| = 1$, the root number.

$q_\infty(s) = \prod_{j=1}^{d} (|s+\kappa_j| + 3)$

$\mathfrak{q}(f,s) = q(f)q_\infty(s)$, analytic conductor.

$L(f,s) \ll_\epsilon \mathfrak{q}(f,s)^{1/4+\epsilon}$, convexity bound, $Re(s) = 1/2$.

$L(f,s) \ll q(f,s)^{\theta+\epsilon}$, $0 \leq \theta < 1/4$, subconvexity.

Take f, a Hecke cusp form. $T_n f = \lambda_f(n) f$, $n = 1, 2, \ldots$.

$$\lambda_f(mn) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right),$$

$\lambda_f(1) = 1$. $\lambda_f(p) = \alpha_f(p) + \beta_f(p)$, $\beta_f(p) = \overline{\alpha}_f(p)$, $|\alpha_f(p)| = 1$.

$|\lambda_f(p)| \leq 2$.

$$\lambda_f(p^k) = \alpha_f^k(p) + \beta_f^k(p)$$

$$\begin{aligned} L(s,f) \quad &= \sum \frac{\lambda_f(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \\ &= \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \end{aligned}$$

$$\gamma(s,f) = \pi^{-s}\Gamma\left(\tfrac{s+k-1}{2}\right)\Gamma\left(\tfrac{s+k}{2}\right)$$
$$\Lambda(s,f) = \gamma(s,f)L(s,f) = \epsilon(f)\Lambda(1-s,f)$$

$\epsilon(f) = i^k$.

Conductor of $f \asymp |s|^2 k^2$, $L(s,f) \ll |s|^{1/2} k^{1/2}$, convexity.

Sato-Tate Distribution: $\lambda_f(p) = 2\cos(2\pi\theta_f(p))$, $0 \leq \theta_f(p) < \pi$.

Conjecture (Sato-Tate): If f is not CM, then

$$\frac{1}{\pi(x)} \sum_{p \leq x} g(\theta_f(p)) = \int_0^\infty g(\theta) \frac{2(\sin\theta)^2}{\pi} d\theta + o(1)$$

Symmetric power L-function.

$$L(s, sym^m f) = \prod_p \prod_{j=0}^{m} \left(1 - \alpha_f(p)^j \beta_f(p)^{m-j} p^{-s}\right)^{-1}$$

Symmetric square L-function, $m = 2$.

$$\begin{aligned} L(s, sym^2 f) &= \prod_p \left(1 - \alpha_f(p)p^{-s}\right)^{-1} (1 - p^{-s})^{-1} \left(1 - \beta_f^2(p)p^{-s}\right)^{-1} \\ \Lambda(s, sym^2 f) &= \gamma(s, sym^2 f)L(s, sym^2 f) = \Lambda(1 - s, sym^2 f) \\ \gamma(s, sym^2 f) &= \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right) \end{aligned}$$

$L(s, sym^2 f)$ is entire, proved by G. Shimura. Conductor of $sym^2 f \asymp k^2|s|^3$.
$L(s, sym^2 f) \ll |s|^{3/4}k^{1/2}$, convexity bound.
Rankin-Selberg L-function

$$\begin{aligned} L(s, f \times f) &= \zeta(2s) \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)}{n^s} \\ &= \zeta(s)L(s, sym^2 f) \\ &= \prod_p \left(1 - \alpha_f^2(p)p^{-s}\right)^{-1} \left(1 - p^{-s}\right)^{-2} \left(1 - \beta_f^2(p)p^{-s}\right)^{-1} \end{aligned}$$

Triple L-functions

$$\begin{aligned} L(s, f \times f \times \phi) &= \sum_{n=1}^{\infty} \frac{\lambda_f^2(n)\lambda_\phi(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha_f^2(p)\alpha_\phi(p)}{p^s}\right)^{-1} \left(1 - \frac{\alpha_\phi(p)}{p^s}\right)^{-2} \left(1 - \frac{\beta_f^2(p)\alpha_\phi(p)}{p^s}\right)^{-1} \\ &\left(1 - \frac{\alpha_f^2(p)\beta_\phi(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_\phi(p)}{p^s}\right)^{-2} \left(1 - \frac{\beta_f^2(p)\beta_\phi(p)}{p^s}\right)^{-1} \\ &= L(s, \phi)L(s, sym^2 f \times \phi) \end{aligned}$$

degree 8. We choose ϕ to be a Maass Hecke cusp form.

$$\gamma(s, f \times f \times \phi) = \prod_{\pm} \pi^{-4s}\Gamma\left(\frac{s+k-1 \pm it_\phi}{2}\right)\Gamma\left(\frac{s+k \pm it_\phi}{2}\right)\Gamma\left(\frac{s \pm it_\phi}{2}\right)\Gamma\left(\frac{s+1 \pm it_\phi}{2}\right)$$

Conductor of $f \times f \times \phi \asymp k^4|s|^8$. $L(s, f \times f \times \phi) \ll k$ convexity bound.