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Harmonic Analysis

Let C^∞ be the smooth functions, infinitely differentiable.

$$C^\infty(\mathbb{Z}\ \mathbb{R}) = C^\infty(S^1)$$

For $f(x) \in C^\infty(\mathbb{Z}\ \mathbb{R})$ we can write

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}(n) e(nx)$$

where

$$e(u) = e^{2\pi i u},\ \widehat{f}(n) = \int_{\mathbb{Z}\ \mathbb{R}} f(x) e(-nx) dx$$

This sum is uniformly convergent.

For $\mathbb{R}$ we need a decay condition.

$$\mathcal{S}(\mathbb{R}) = C^\infty(\mathbb{R} - \text{compactified})$$

vanishes to infinite order at infinity.

$$\mathcal{S}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \left| \frac{x^m df(x)}{dx^n} \right| \to 0 \text{ as } |x| \to \infty, \text{ for all } m, n \right\}$$

Fourier Transform on $\mathbb{R}$:

$$\mathcal{F}f(x) = \int_{\mathbb{R}} f(r) e(-rx) dr$$

Fourier Inversion formula:

$$f(-x) = ((\mathcal{F} \circ \mathcal{F}) f)(x)$$

Poisson Summation Formula:

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \mathcal{F}f(n)$$

Proof:

Let

$$F(x) = \sum_{n \in \mathbb{Z}} f(n + x) \in C^\infty\left(\mathbb{Z}\ \mathbb{R}\right)$$

$$\begin{aligned} F(x) \quad &= \sum_{n \in \mathbb{Z}} \widehat{F}(n) e(nx) \\ \textstyle\sum_{n \in \mathbb{Z}} f(n) = F(0) \quad &= \sum_{n \in \mathbb{Z}} \widehat{F}(n) \end{aligned}$$

But

$$\begin{aligned}
\widehat{F}(n) &= \int_0^1 F(x)e(-nx)dx \\
&= \int_0^1 \sum_m f(m+x)e(-n(x+m))dx \\
&= \int_0^1 \sum_m f(-m+x)e(-n(x-m))dx \\
&= \sum_{m\in\mathbb{Z}} \int_0^1 f(x-m)e(-n(x-m))dx \\
&= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(x)e(-nx)dx \\
&= \int_{\mathbb{R}} f(x)e(-nx)dx \\
&= \mathcal{F}f(n)
\end{aligned}$$

$\mathcal{F}$ has order 4 and has order 2 on even functions. If f is even, $\mathcal{F}f$ is also even

$$\begin{aligned}
\mathcal{F}f(-x) &= \int f(r)e(rx)dr \\
&= \int f(-r)e(-rx)dr \\
&= \int f(r)e(-rx)dr \\
&= \mathcal{F}f(x)
\end{aligned}$$

Then

$$\mathcal{F}(f+\mathcal{F}) = \mathcal{F}f + \mathcal{F}^2 f = \mathcal{F}f + f$$

Hence for any even function f, $f+\mathcal{F}f$ is its own Fourier transform.
HW 1: Show $e^{-\pi x^2}$ is its own Fourier Transform. Note that

$$\int e^{-\pi x^2} dx = e^{-\pi 0^2} = 1$$

Take some $t > 0$. In the Poisson summation formula, send $x \to x/t$, $dx \to dx/|t|$. Then Poisson summation formula says

$$\sum e^{-\pi n^2 t^2} = \frac{1}{t} e^{-\pi n^2/t^2}$$

Let $f_t(x) = f(xt)$. Then

$$\begin{aligned}\widehat{f_t}(r) &= \textstyle\int_{\mathbb{R}} f(xt)e(-xr)dx \\ &= \frac{1}{|t|}\int_{\mathbb{R}} f(x)e\left(\frac{-x}{t}r\right)dx \\ &= \frac{1}{|t|}\widehat{f}\left(\frac{r}{t}\right)\end{aligned}$$

We then have the Jacobi Inversion Formula

$$\begin{aligned}\textstyle\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} &= \tfrac{1}{\sqrt{t}}\textstyle\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} \text{ for all } t>0 \\ \textstyle\sum e^{-\pi n^2/t} &\sim \sqrt{t} \text{ as } t\to\infty \\ \textstyle\sum e^{-\pi n^2/t^2} &\sim t\end{aligned}$$

Definition:

$$\begin{aligned}\Theta(z) &= \sum_{n\in\mathbb{Z}} e(n^2 z) \\ \Theta(it/2) &= \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 it/2} \\ &= \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}\end{aligned}$$

Take $z = x+iy$ with $x, y \in \mathbb{R}$, $y>0$.

$$\begin{aligned}\Theta(z) &= \sum e^{2\pi i n^2 x + 2\pi i n^2 iy} \\ &= \sum e^{2\pi i n^2 x - 2\pi n^2 y} \\ |\Theta(z)| &\le \sum e^{-2\pi n^2 y}\end{aligned}$$

This converges quickly to a holomorphic function on $\mathbb{H}$.
HW 2: Is Θ ever zero on $\mathbb{H}$? If not, does it have a minimum absolute value?

$$\Theta(it/2) = \sum e^{-\pi n^2 t} = t^{-1/2}\Theta(i/(2t))$$

from the Jacobi Inversion formula.

We take $t \to 2t$ and have

$$\Theta(it) = (2t)^{-1/2}\Theta(i/(4t))$$

The transformation law

$$\Theta(z) = \left(\frac{2z}{i}\right)^{-1/2}\Theta(-1/(4z))$$

is valid on all of $\mathbb{H}$.

We also have $\Theta(z+1) = \Theta(z)$.

$$\begin{aligned}\Theta\left(\tfrac{1}{2}+it\right) &= \textstyle\sum e^{2\pi i n^2 (1/2+it)} \\ &= \sum e^{\pi i n^2} e^{-2\pi n^2 t} \\ &= \sum (-1)^n e^{-2\pi n^2 t} \\ &= \sum_{n \text{ even}} e^{-2\pi n^2 t} - \sum_{n \text{ odd}} e^{-2\pi n^2 t}\end{aligned}$$

$$\begin{aligned}\Theta(1/2+it) + \Theta(it) &= 2 \sum_{n \text{ even}} e^{-2\pi n^2 t} \\ &= 2 \sum_{m \in \mathbb{Z}} e^{-8\pi m t} \\ &= 2\Theta(4it)\end{aligned}$$

$$W(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(2\pi n^2 x\right)$$

Conjecture (Weierstrass):

$W(x)$ is nowhere differentiable.

Hardy showed that it is not differentiable at irrational x and some of the rationals.

Joel Gerber showed that

$$W'(\frac{p}{2q}) = -\pi \text{ if } p, q \text{ odd}$$

$$W'(x) \sim \sum \sin(2\pi n^2 x)$$

which is almost θ.

Let

$$\begin{aligned}\Gamma = \Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ 4|c \right\} \\ = \{\pm 1\} \cdot \Gamma_1(4) \\ \Gamma_1(4) \quad = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) | d \equiv 1(4) \right\} \\ = \text{ free group generated by } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and}\end{aligned}$$

These groups act on $\mathbb{H}$ by fractional liner transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

For example,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : z \mapsto z+1$$

$$\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} : z \mapsto \frac{-1}{4z}$$

$$\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -16 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

Hence $\Theta(z/(4z+1))$ is related to $\Theta(z)$.

Let

$$\epsilon_d = \begin{cases} 1 & d \equiv 1(4) \\ i & d \equiv -1(4) \end{cases}$$

$$\left(\frac{c}{d}\right) = \begin{cases} \text{usual Jacobi symbol} & \text{if } c,d > 0 \\ sgn(c)\left(\frac{c}{-d}\right) & \text{if } c \neq 0, d < 0 \\ 1 & \text{if } c = 0 \end{cases}$$

$$\Theta\left(\frac{az+b}{cz+d}\right) = \overline{\epsilon}_d \left(\frac{c}{d}\right)(cz+d)^{1/2}\Theta(z), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

$$SL(2,\mathbb{R}) = B(\mathbb{R})SO(2,\mathbb{R})$$

where

$$B(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0\right\}$$

We take $\widetilde{SL(2,\mathbb{R})}$ to be the double cover, which is a metaplectic group with elements (g,ϵ) with $g \in SL(2,\mathbb{R})$ and $\epsilon = \pm 1$. The group action is given by:

$$(g_1,\epsilon_1)\cdot(g_2,\epsilon_2) = (g_1g_2, \epsilon_1\epsilon_2\sigma(g_1,g_2))$$

where σ is a cocycle. A subgroup $H \leq G$ <u>splits</u> in $\widetilde{G}$ if

$$H \cong \widetilde{H} \subset \widetilde{G}$$

and there is

$$s : H \to \pm 1$$

such that $\{(h, s(u))\}$ is a subgroup of $\widetilde{G}$.

$$s(h_1h_2) = s(h_1)s(h_2) = \sigma(h_1,h_2)$$

$\Gamma_1(4)$ splits in $\widetilde{(}SL(2,\mathbb{R}))$ with

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c}{d}\right)$$

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$$\Theta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z),\ Im(z) > 0$$

$$e(u) = e^{2\pi i u}$$

$$\left(\frac{i}{2z}\right)^{1/2} \Theta\left(\frac{-1}{4z}\right) = \Theta(z)$$

Poisson Summation.

$\Theta(z+1) = \Theta(z)$

Examples of $SL(2,\mathbb{R})$ acting on upper half plane $\mathbb{H} = \{x+iy | y > 0\}$ by Fractional Linear Transformations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

$$\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ -16 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

Therefore, we expect a relation for

$$\Theta\left(\frac{z}{4z+1}\right) \text{ and } \Theta(z)$$

$$\Theta\left(\frac{-1}{4(z+1)}\right) = \left(\frac{2(z+1)}{i}\right)^{1/2} \Theta(z+1) = \left(\frac{2(z+1)}{i}\right)^{1/2} \Theta(z)$$

We now send $z \mapsto -1/4z$.

$$\begin{aligned} \Theta\left(\frac{-1}{4(-1/4z+1)}\right) &= \left(\frac{2(1-1/4z)}{i}\right)^{1/2} \Theta(-1/4z) \\ &= \left(\frac{2(1-1/4z)}{i}\right)^{1/2} \left(\frac{2z}{i}\right)^{1/2} \Theta(z) \\ &= (-4z+1)^{1/2}\Theta(z) \end{aligned}$$

Therefore,

$$\Theta\left(\frac{z}{-4z+1}\right) = (-4z+1)^{1/2}\Theta(z)$$

This is an example of

$$\Theta\left(\frac{az+b}{cz+d}\right) = \bar{\epsilon}_d \left(\frac{c}{d}\right) \sqrt{cz+d}\Theta(z)$$

for $\Gamma_0(4)$ For $\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$. Know for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

We have

$$\Gamma_0(4) = \pm\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \rangle$$

$$\begin{aligned} \Gamma_1(4) \quad &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : 4|c, d \equiv 1(4), ad - bc = 1 \right\} \\ &= \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \rangle \end{aligned}$$

For $\Gamma_1(4)$,

$$\Theta((az+b)/(cz+d)) = \left(\frac{c}{d}\right)(cz+d)^{1/2}\Theta(z)$$

Kubota Symbol

Let

$$X \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases}$$

on $SL(2, \mathbb{R})$.

Kubota's cocycle

$$\sigma(g_1, g_2) = \left(\frac{X(g_1 g_2)}{X(g_1)}, \frac{X(g_1 g_2)}{X(g_2)} \right)$$

where (x, y) is the Hilbert symbol that is 1 if either $x > 0$ or $y > 0$, and -1 is $x, y < 0$.

Then $SL(2, \mathbb{R}) \times \{\pm 1\}$ can be made into a central extension of $SL(2, \mathbb{R})$ by the multiplication law

$$(g_1, \epsilon_1) \cdot (g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 \sigma(g_1, g_2))$$

$\{(0, \pm 1)\}$ is the center of $SL(\tilde{2}, \mathbb{R})$

Example

Let

$$k_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

parametrizes $SO(2, \mathbb{R})$.

$$k_{\theta_1} k_{\theta_2} = k_{\theta_1 + \theta_2}$$

$$X(k_\theta) = \begin{cases} \sin\theta & \pi \nmid \theta \\ \cos\theta & \end{cases}$$

$$\sigma(k_{\theta_1}, k_{\theta_2}) \sim \left(\frac{\sin(\theta_1 + \theta_2)}{\sin\theta_1}, \frac{\sin(\theta_1 + \theta_2)}{\sin\theta_2} \right)$$

Property: If $n = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, then

$$\sigma(ng_1, g_2) = \sigma(g_1, g_2) = \sigma(g_1, g_2 n)$$

because $X(ng) = X(gn) = X(g)$.

Splitting function for a subgroup $H \subset SL(2,\mathbb{R})$, $s : H \to \pm 1$ such that $\{(h, s(h))\}$ is a subgroup of $SL(\tilde{2},\mathbb{R})$.

H splits if s exists. In other words $H \simeq$ subgroup of cover.

Lemma

Free subgroup of $\widetilde{G}$ always splits.

Proof: Let Γ be a free subgroup. Each $\gamma \in \Gamma$ is uniquely $\gamma_1, \ldots, \gamma_k$ with $\gamma_1, \ldots, \gamma_k \in$ general set of γ_i, $\gamma_i \gamma_{i+1} \neq e$

$$\Gamma = \coprod_{k=0}^{\infty} W_k = \text{ words of length } k$$

s is defined to be 1 on W_0

s is defined to on $W_1 =$ generators and inverses arbitarily (say 1)

Suppose s is defined on W_k. Let $\gamma \in W_{k+1}$.

$\gamma = g\gamma'$, $g =$ generator, $\gamma' \in W_k$.

$$s(\gamma) = s(g\gamma) = \sigma(g, \gamma') s(g) s(\gamma')$$

all defined.

This defines s on Γ. Since Γ is free, there are no relations to check. ■

If you do this for $\Gamma_1(4)$, set

$$\begin{aligned} s\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} &= 1 \\ s\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} &= 1 \\ s\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} &= 1 \\ s\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} &= 1 \end{aligned}$$

Then

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c}{d}\right)$$

$$\Theta(\gamma z) = \bar{\epsilon}_d \left(\frac{c}{d}\right) (cz+d)^{1/2} \Theta(z)$$

$$\Theta^2(\gamma z) = \epsilon_d^2 (cz+d) \Theta^2(z)$$

$$\epsilon_d^2 = \xi_4(d) = \begin{cases} 1 & d \equiv 1(4) \\ -1 & d \equiv -1(4) \end{cases}$$

on $\Gamma_1(4)$ this is equal to $(cz+d)\Theta(z)$. Let

$$\begin{aligned}\Gamma(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : a,d \equiv 1(N), b,c \equiv 0(N) \right\} \\ &= ker(SL(2,\mathbb{Z}) \mapsto SL(2,\mathbb{Z}/N\mathbb{Z}))\end{aligned}$$

hence normal. Principal congruence subgroup of level N.

A congruence subgroup Γ of $SL(2,\mathbb{Z})$ is one which contains at least one $\Gamma(N)$.

Definition: A holomorphic modular form for a congruence subgroup of $SL(2,\mathbb{Z})$ of weight $k \in \mathbb{Z}_{\geq 0}$ and character ξ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$f\left(\frac{az+b}{cz+d}\right) = \xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that f has "moderate growth" (to be clarified).

Notation:

$M_k(\Gamma,\xi)$= holomorphic modular forms, $M_k(\Gamma) = M_k(\Gamma, triv)$.

$$\xi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi(d)$$

where ψ is a Dirichlet character of $modN$.

Modular forms for $\Gamma(1) = SL(2,\mathbb{Z})$.

Lemma: k must be even because $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{Z})$. $f(z) = (-1)^k f(z) \Rightarrow$ k is even.

Example: Eisenstein series, $k > 2$

$$E_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\ \{(0,0)\}} (mz+n)^{-k}$$

This is modular because

$$\begin{aligned}\left(m\tfrac{az+b}{cz+d} + n\right)^{-k} &= (cz+d)^k\,(maz+mb+ncz+nd)^{-k} \\ &= (cz+d)^k\,((ma+nc)z+(mb+n))^{-k}\end{aligned}$$

$$[m,n]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [ma+nc, mb+nd]$$

is a bijection of $\mathbb{Z}^2 - \{(0,0)\}$, so

$$E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k E_k(z)$$

Let

$$j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) = cz+d$$

Lemma:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} * & * \\ ce + dg & cf + dh \end{pmatrix}$$

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z)$$

Check

$$(ce + dy)z + cf + dh = \left(c\frac{ez + f}{gz + h} + d \right)(gz + h)$$

$E_k E_l$ is modular of weight $k + l$.

$$M_k(SL(2, \mathbb{Z}))M_l(SL(2, \mathbb{Z})) \subset M_{k+l}(SL(2, \mathbb{Z}))$$

k	2	4	6	8	10	12
$\dim M_k(SL(2, \mathbb{Z}))$	0	1	1	1	1	2

E_4^3 and E_6^2 are distinct.
Ramanujan form

$$\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}$$

Δ's product converges to a nonzero function.

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Fractional linear transformations on $\mathbb{H} = \{x + iy : y > 0\}$.

$G = SL(2, \mathbb{R})$ acts by group action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}$$

Look at $\Gamma = SL(2, \mathbb{Z})$.

In particular,

$$T^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$S^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$T : z \mapsto z + 1$. $z = x + iy$, $Tz = (x + 1) + iy$. Given any $z \in \mathbb{H}$, there is $n \in \mathbb{Z}$ such that $T^n = \begin{pmatrix} 1 & n \\ 9 & 1 \end{pmatrix}$ maps z to $T^n z$ which has same imaginary part as z, but

$$|ReT^n z| \leq \frac{1}{2}$$

Take $S : z \mapsto -1/z$

$$x+iy \mapsto \frac{-1}{x+iy}\frac{x-iy}{x-iy} = \frac{-x+iy}{x^2+y^2} = \frac{-\overline{z}}{|z|^2}$$

Take $z = re^{i\theta}$ with $\theta \in (0,\pi)$.

$$Sz = \frac{-1}{re^{i\theta}} = \frac{-e^{-i\theta}}{r} = \frac{e^{i(\pi-\theta)}}{r}$$

Can arrange to have z or Sz have $|\cdot| \geq 1$

Fractional linear transformations send circles and lines to circles and lines.

Reduction Algorithm

Seeks to apply S and T to maximize the imaginary part of z.

Step 1: Let $w = z$. Apply T to get $|ReT^n w| \leq 1/2$. Replace w with $T^n w$.

Step 2L Look at Sw.

Look at

$$\frac{a}{c} - \frac{1}{c(z+d)} = \frac{acz+ad-(ad-bc)}{c(cz+d)} = \frac{acz+bc}{c(cz+d)} = \frac{az+b}{cz+d}$$

$$Im\left(\frac{az+b}{cz+d}\right) = \frac{-\overline{cz+d}}{c|cz+d|^2} = \frac{Im(z)}{|cz+d|^2}$$

$$Im(Sw) = \frac{Imw}{|w|^2}$$

If $|w| \geq 1$, stop. Otherwise, replace w with Sw.

Repeat these steps.

Eventually this process does terminate making the height bigger.

We eventually want to end up in the fundamental domain

$$\mathcal{F} = \{z \in \mathbb{H} : |z| \geq 1,\ |Rez| \leq 1/2\}$$

This algorithm finds an explicit word γ in S and T such that $\gamma z \in \mathcal{F}$.

Overlaps

For what

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \text{ does } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathcal{F} \text{ intersect } \mathcal{F}?$$

Answer:

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ \pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ ST = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},\ TS = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

and a few more. This is a finite set.

Fixed Points

What $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can fix a point?

If $c = 0$, there is only a real solution. There is no solution in the upper half plane, so we assume $c \neq 0$.

$az + b = z(cz + d)$, $cz^2 + z(d - a) - b = 0$,

$$\begin{aligned} z &= \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c} \\ &= \frac{a - d \pm \sqrt{d^2 + a^2 + 2ad + 4bc - 4ad}}{2c} \\ &= \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c} \end{aligned}$$

This lies in $\mathbb{H}$ if and only if $|a+d| < 2$, so $a+d = -1$, $a+d = 0$ or $a+d = 1$.

$$\frac{Im(\sqrt{(a+d)^2 - 4})}{2c}$$

How can this be in $\mathcal{F}$? $c = \pm 1$. If not, $Imz \leq 2/2|c| \leq 1/2$, no good.

Using $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can assume $c = 1$.

We now real part has to be bounded by 1/2. Since $c = 1$, we need $|Re((a-d)/2)| \leq 1/2$, so $|a - d| \leq 1$. We also have $|a + d| \leq 1$. Only way this works if $a = -1$, $d = 0$ or $a = 0$, $d = -1$ or $a = d = 0$ or $a = 1$, $d = 0$ or $a = 0$, $d = 1$.

This gives all the possibilities:

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

The first fixes $(-1+\sqrt{-3})/2$. The second fixes $(1+\sqrt{-3})/2$. The third fixes i. The fourth fixes $(1+\sqrt{3})/2$. The fifth fixes $(-1+\sqrt{3})/2$. These are examples of elliptic elements.

$\mathcal{F}$ is a hyperbolic triangle. $(3, 3, \infty)$ triangle.

We have the Poincare Disk that maps $\mathcal{F}$ by $i\frac{z-i}{2\pi i}$.

Suppose z and $(az + b)/(cz + d) \in \mathcal{F}$.

$$Re\left(\frac{az+b}{cz+d}\right) = \frac{a}{c} - \frac{Re(z)}{|cz+d|^2}$$

This must be in $[-1/2, 1/2]$ as does $Re(z)$.

$$\left|\frac{az+b}{cz+d}\right| \geq 1 \Leftrightarrow |az + b| \geq |cz + d|$$

$$\frac{Im(z)}{|cz+d|^2} \geq \frac{\sqrt{3}}{2},\ |cz+d|^2 \leq \frac{2}{\sqrt{3}} Im(z)$$

$$|cz+d|^2 = |cx+ciy+d|^2 = (cx+d)^2 + c^2y^2 \leq \frac{2}{\sqrt{3}} y$$

$c^2y^2 \leq (2/\sqrt{3})y$, so $c^2 \leq 2/(\sqrt{3}y) \leq 1$, so $c = 0, \pm 1$. When $c = 0$,

$$\pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

$$Re \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} z = Rez + b \in [-1/2, 1/2]$$

implies $b \in \{-1, 0, 1\}$

Reduced to $c = 1$:

$$-\frac{1}{2} \leq a - \frac{Re(z)}{|z+d|^2} \leq \frac{1}{2}$$

$|az+b| \geq |z+d|$, $|z+d|^2 \leq (2/\sqrt{3}) Im(z)$.

$|z+d|^2 = (x+d)^2 + y^2 \leq (2/\sqrt{3})y$. Since $c = 1$, we have $c^2 \leq 2/(\sqrt{3}y)$ is nontrivial and $y \leq 2/\sqrt{3}$. We now have a smaller part of the fundamental domain that we are examining.

Claim: $|a| \leq 1$.

Proof: $y^2 \leq |z+d|^2 \leq (2/\sqrt{3})y$, so

$$y^{-2} \geq \frac{1}{|z+d|^2} \geq \frac{\sqrt{3}}{2y}$$

$$\frac{1}{2y^2} \geq \frac{|Re(z)|}{y^2} \geq \frac{|Re(z)|}{|z+d|^2}$$

$$\frac{\sqrt{3}2}{\leq} y \leq \frac{2}{\sqrt{3}}$$

so

$$3/8 \leq \frac{1}{2y^2} \leq \frac{2}{3}$$

Hence

$$a - \frac{Re(z)}{|z+d|^2} \in [-1/2, 1/2] \Rightarrow a \in [-7/6, 7/6]$$

Three possibilities:

$$\begin{pmatrix} -1 & b \\ 1 & d \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}, \begin{pmatrix} 1 & b \\ 1 & d \end{pmatrix}$$

Look at $\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}$:

$$\left|\frac{Re(z)}{|z+d|^2}\right| \leq \frac{1}{2}$$

$1 \geq |z+d|^2 = y^2 + (x+d)^2$, $|x+d| \leq 1$, $|x| \leq 1/2$ gives that $d = -1, 0, 1$.

Look at $\begin{pmatrix} 1 & b \\ 1 & d \end{pmatrix}$.

$$1 - \frac{Re(z)}{|z+d|^2} \in [-1/2, 1/2]$$

$|z+b| \geq |z+d|$, $|z+d|^2 \leq (2/sqrt(3))y$

$$\frac{Re(z)}{|z+d|^2} \in [1/2, 3/2]$$

so $Re(z) > 0$. This implies $d - b = 1$ and $d > b$. Then $|d| \leq 1$ so the three possibilities are:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$$

Checking the last by brute force shows that it can't happen.

Can check $\begin{pmatrix} -1 & b \\ 1 & d \end{pmatrix}$ the same way through symmetry.

Let $\Delta = \langle S, T \rangle$. If the algorithm terminates, then Δ-translates of $\mathcal{F}$ tile $\mathbb{H}$. Suppose $\gamma \in \Gamma$. Consider $z \in \mathcal{F}$, $\gamma z \in \mathbb{H}$. Then $\delta \in \Delta$ such that $\delta(\gamma z) \in \mathcal{F}$ by the algorithm. Hence $\delta\gamma$ has a fixed point. Therefore, $\delta\gamma$ is in the list $\subset \Delta$ and $\gamma \in \delta^{-1}\Delta = \Delta$. This implies that $\Gamma \subset \Delta$, so $\Gamma = \Delta$. Finally, $SL(2,\mathbb{Z})$ is generated by S and T.

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Why does the reduction algorithm halt?

Imaginary part goes up each time step 2 is executed.

$$Im\frac{az+b}{cz+d} = \frac{Imz}{|cz+d|^2}$$

The values of $cz+d$, $\{cz+d : c, d \in \mathbb{Z}\}$ is a lattice in $\mathbb{C} \cong \mathbb{R}^2$. The values of norm in a lattice are discrete.

Since there are only finitely many (c,d) with $|cz+d|$ bounded by a fixed amount and since each loop of the algorithm increases $Im\frac{az+b}{cz+d}$ if an donly if it decreases $|cz+d|$, so it cannot go on forever. This proves that the algorithm terminates. This implies that

$$SL(2,\mathbb{Z}) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \rangle$$

$SL(2,\mathbb{Z})$ applied to the fundamental domain tiles the entire plane.

There is a graph structure to this action. Hexagonal referring to group relations.

$$(ST)^6 = I \text{ ?}$$

$$\Gamma_1(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : 4|c, d \equiv 1(4), \Rightarrow a \equiv 1(4) \right\}$$

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : 4|c \right\}$$

$$\frac{\Gamma_0(4)}{\Gamma(2)} = \pm\Gamma_1(4)$$

where

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : a \equiv d \equiv 1(2), 2|b, c \right\}$$

$$\begin{aligned}
&\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma(2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}, \text{ conjugate} \\
&= \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix} : a \equiv d \equiv 1(2), b, c \text{ even} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : 4|c, a, d \text{ odd} \right\} \\
&= \Gamma_0(4)
\end{aligned}$$

Hence $\Gamma_0(4)$ and $\Gamma(2)$ are conjugate. $P\Gamma(2) = \Gamma(2)/\pm 1$, $P\Gamma(2)$ is conjugate to $\Gamma_1(4)$.

To compute a fundamental domain for $P\Gamma(2)$, a normal subgroup of $PSL(2,\mathbb{Z})$. Find coset representations $\gamma_1, \ldots, \gamma_k$ and take

$$\bigcup_{i \le k} \gamma_i F$$

This is a fundamental domain for $P\Gamma(2)$. Let

$$\pi : PSL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z}/2\mathbb{Z})$$

$P\Gamma(2) = ker\pi$. Coset representations: Take an element of $\pi^{-1}(M)$ for all $M \in PSL(2,\mathbb{Z}/2\mathbb{Z})$.

$PSL(2,\mathbb{Z}/2\mathbb{Z})$	π^{-1}
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

Fundamental domain for $\Gamma(2)$, hyperbolic square, no internal corners.

Let $F_2 = \langle a, b\rangle$ Free group on a and b. The words form a 4-regular tree isomorphic to F_2.

Hyperbolic volume

<u>Theorem</u> Let $dA = dxdy/y^2$. Then dA is presented by fractional linear transformations.

<u>Fact</u> $SL(2,\mathbb{R})$ is given by

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$,
$c \neq 0$:

$$\begin{pmatrix} 1 & -\frac{a}{c} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -c^{-1} \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$$

Check

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : z \mapsto z + t$$

dx, dy, and y unchanged.

$$\left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right) : z \mapsto a^2 z$$

$x \mapsto a^2x$, $y \mapsto a^2y$, $dx \mapsto a^2dx$, $dy \mapsto a^2dy$, $dxdy/y^2 \mapsto dxdy/y^2$.

$$\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right) : x+iy \mapsto \frac{-x+iy}{x^2+y^2}$$

$$Re = \frac{-x}{x^2+y^2},\ Im = \frac{y}{x^2+y^2}$$

Jacobian

$$\left(\begin{array}{cc} \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2} \\ \frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2} \end{array}\right)$$

The determinant is

$$(x^2+y^2)^{-4}\left[(x^2-y^2)^2+4x^2y^2\right] = (x^2+y^2)^{-2}$$

$$\frac{dxdy}{y^2} \to (x^2+y^2)^{-2}\frac{dxdy}{\frac{y^2}{(x^2+y^2)^2}} = \frac{dxdy}{y^2}$$

<u>Lemma</u>
$Vol(\mathcal{F}) = \pi/3$

$$\int_{-1/2}^{1/2} dx \int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} = \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{3}$$

Check volume:

$$\int_0^{1/2} dx \int_{\sqrt{1/4-(1/2-x)^2}^{\infty}\frac{dy}{y^2}} = \pi/2$$

Eisenstein series
$k > 2$, k even

$$\begin{aligned} G_k(z) &= \sum_{(m,n)\in\mathbb{Z}^2_{\neq(0,0)}} \frac{1}{(mz+n)^{-k}} \in M_k(SL(2,\mathbb{Z})) \\ dimM_k &= \left\{\begin{array}{ll} \lfloor\frac{k}{12}\rfloor & k \equiv 2(12) \\ \lfloor\frac{k}{12}\rfloor + 1 & k \not\equiv 2(12) \end{array}\right. \end{aligned}$$

Graded Ring $\bigoplus M_k$ generated by G_4 and G_6 using $M_{k_1}M_{k_2} \subset M_{k_1+k_2}$.
Fourier series:

$$\begin{aligned}
&\sum_{m=0} + \sum_{m\neq 0} \\
&= 2\sum_{n=1}^{\infty} n^{-k} + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} (mz+n)^{-k} \\
&= 2\zeta(k) + \frac{2(2\pi i)^k}{\Gamma(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e(nz)
\end{aligned}$$

$$\sigma_s(n) = \sum_{d|n} d^s$$

By Poisson this is

$$\begin{aligned}
&\sum_{\nu\in\mathbb{Z}} \int_{n\in\mathbb{R}} (mx+n+miy)^{-k} e(-n\nu)dn \\
&n \mapsto n - mx \\
&= \sum_{\nu\in\mathbb{Z}} e(\nu mx) \int_{n\in\mathbb{R}} (n+miy)^{-k} e(-n\nu)dn \\
&= \sum_{\nu\in\mathbb{Z}} e(\nu mx)(my)^{-k} \int_{n\in\mathbb{R}} (n+i)^{-k} e(-nmy\nu)dn
\end{aligned}$$

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Eisenstein series, k even.

$$\begin{aligned}
G_k(z) &= \sum_{(m,n)\neq(0,0)} (mz+n)^{-k},\ k>2 \\
&= \sum_{m=0} n^{-k} + \sum_{m\neq 0} (mz+n)^{-k} \\
&= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} (mz+n)^{-k}
\end{aligned}$$

Poisson Sum

$\sum_{n\in\mathbb{Z}}(mz+n)^{-k} = \sum_{r\in\mathbb{Z}}\int_{\mathbb{R}}(mz+u)^{-k}e(-ru)du$

Manipulate

$$\begin{aligned}\int_{\mathbb{R}}(mz+u)^{-k}e(-ru)du \qquad &= e(mrx)\int_{\mathbb{R}}(miy+u)^{-k}e(-ru)du\\ &= e(mrx)\int_{Imw=my}w^{-k}e(-r(w-miy))dw\\ &= e(mrz)\int_{Imw=my}w^{-k}e(-rw)dw\end{aligned}$$

If $r \leq 0$, the contour shift $y \to \infty$ shows this is zero. Thus, we may assume $r > 0$. Shift contour down to $Im \ll 0$. Get

$$\begin{aligned}\int &= -2\pi i Res_{w=0}w^{-k}e^{-2\pi i r w}\\ &= -2\pi i Res_{w=0}w^{-k}\sum_{n=0}^{\infty}\frac{(-2\pi i r w)^n}{n!}\\ &= \frac{-(2\pi i)^k r^{k-1}(-1)^{k-1}}{\Gamma(k)} = \frac{(2\pi i)^k r^{k-1}}{\Gamma(k)}\end{aligned}$$

Then

$$\begin{aligned}G_k(z) &= 2\zeta(k) + 2\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}e(mrz)\frac{(2\pi i)^k}{\Gamma(k)}r^{k-1}\\ &= 2\zeta(k) + 2\sum_{n=1}^{\infty}e(nz)\frac{(2\pi i)^k}{\Gamma(k)}\sigma_{k-1}(n)\\ &\text{where } \sigma_z(n) := \sum_{d|n}d^z\end{aligned}$$

Cool Fact:

If you can compute the n^{th} Fourier coefficient of any specified modular form in time $O((\log n)^e)$ for some $e > 0$, then you can break RSA in polynomial time.

Proof: Let $n = pq$, where p, q are prime. Let $k = 4$. Compute $\sigma_3(pq) = 1 + p^3 + q^3 + (pq)^3 = n^3 + 1 + p^3 + q^3$, so we know $p^3 + q^3$ and we know $p^3q^3 = n$. Solve for p^3 and q^3 using quadratic formula.

Normalize:

$$\begin{aligned}E_k &:= \frac{G_k}{2\zeta(k)}\\ &= 1 + \frac{(2\pi i)^k}{\Gamma(k)\zeta(k)}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz)\\ &= \sum_{(m,n)=1,m>0}(mz+n)^{-k}\end{aligned}$$

Example:

$$\begin{aligned}
E_4 &= 1 + \frac{(2\pi i)^4}{\Gamma(4)\zeta(4)} \sum_{n=1}^{\infty} \sigma_3(n) e(nz) \\
&= 1 + \frac{16\pi^4 \cdot 90}{6\pi^4} \sum_{n=1}^{\infty} \sigma_3(n) e(nz) \\
&= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) e(nz)
\end{aligned}$$

$$\begin{aligned}
E_6 &= 1 - \frac{2^6\pi^6 \cdot 945}{120\pi^6} \sum_{n=1}^{\infty} \sigma_5(n) e(nz) \\
&= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) e(nz)
\end{aligned}$$

$$\begin{aligned}
E_8 &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) e(nz) \\
E_{10} &= 1 - 66 \sum_{n=1}^{\infty} \sigma_9(n) e(nz) \\
E_{12} &= 1 + \frac{16380}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) e(nz)
\end{aligned}$$

k	$\mathrm{Dim} M_k(SL(2,\mathbb{Z}))$
0	1
2	0
4	1
6	1
8	1
10	1
12	2
14	1

$$\begin{aligned}
E_4(z) &= \sum a_n q^n \\
&a_0 = 1,\, a_n = 240\sigma_3(n) \\
E_4(z)^2 &= \sum b_n q^n \\
b_n &= \sum_{m=0}^{n} a_m a_{n-m} \\
b_n &= 2a_0 a_n + \sum_{m=1}^{n-1} a_m a_{n-m} \\
480\sigma_7(n) &= 2\cdot 240\sigma_3(n) + (240)^2 \sum \sigma_3(m)\sigma_3(n-m) \\
&\Rightarrow \sigma_7(n) = \sigma_3(n) + 120\sum \sigma_3(m)\sigma_3(n-m)
\end{aligned}$$

$$E_6E_8 = E_{14} = E_4E_{10}$$

Lemma

$$E_k(i) = 0 \Leftrightarrow 0 = G_6(i) = \sum (m+in)^{-6}$$

Lemma
E_6 and E_4 are algebraically independent.
Proof
Suppose $\sum_{6k+4l=const} c_{k,l}E_6^kE_4^l = 0$. WLOG, some $c_{0,l} \neq 0$. Look at this evaluated at i. Then $c_{0,const}E_4^l = 0$, which implies, $c_{0,const} = 0$, contradiction.

Hence,

$$\mathcal{M}(SL(2,\mathbb{Z})) = \bigoplus \mathcal{M}_k(SL(2,\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$

graded ring

$$\begin{aligned}
dim\mathcal{M}_k(SL(2,\mathbb{Z})) &= \#\{(r,s)\in\mathbb{Z}^2_{\geq 0} : 6r+4s = k\} \\
&= \lfloor\frac{k}{12}\rfloor + \begin{cases} 0 & k\equiv 2(12) \\ 1 & \end{cases}
\end{aligned}$$

Proof: If we look at $6r + 4s = k + 12$, there should be one solution beyond what there was for k. Continue.

$E_6^2 - E_4^3$ is a cusp form which is a constant multiplied by Δ, where Δ is Ramanujan's cusp form of weight 12.

$$\Delta = \sum_{n=1}^{\infty} \tau(n)e(nz)$$

$\tau(1) = 1$, $\tau(2) = 24$, $\tau(3) = 252$.

$$\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24}$$

Conjecture by Ramanujan: $\tau(n)$ never zero. (Still open)

$$|\tau(p)| \leq 2p^{11/2} \text{ (Deligne, 1970s)}$$

Mordell:

$$\tau(n)\tau(m) \quad = \sum_{d|(n,m)} d^{11}\tau(nm/d^2)$$

Next: Poincare Series

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Recall $SL(2,\mathbb{Z})$ acting on $\mathbb{H}$. Fundamental domain volume $\pi/3$ but noncompact. There do exist interesting discrete subgroups $\Gamma \subset SL(2,\mathbb{R})$ such that $\Gamma\backslash\mathbb{H}$ is compact.

Example Quaternionic groups. (Division algebra). Let $p \equiv 1(4)$ be a prime. Let n be such that $\left(\frac{n}{p}\right) = -1$. Let

$$\Gamma = \Gamma(n,p) = \left\{ \begin{pmatrix} a + b\sqrt{n} & (c + d\sqrt{n})\sqrt{p} \\ (c - d\sqrt{n})\sqrt{p} & a - b\sqrt{n} \end{pmatrix} \in SL(2,\mathbb{R}), a,b,c,d \in \mathbb{Z} \right\}$$

Proof that it is discrete: If we consider four open intervals I_1, I_2, I_3, I_4 and ask how many $a,b,c,d \in \mathbb{Z}$ satisfy $a + b\sqrt{n} \in I_1$, $(c + d\sqrt{n})\sqrt{p} \in I_2$, $(c - d\sqrt{n})\sqrt{p} \in I_3$, $a - b\sqrt{n} \in I_4$. We see $2a = (a + b\sqrt{n}) + (a - b\sqrt{n}) \subset I_1 + I_4 =$ bounded interval, so a is bounded.

$2c\sqrt{p} \subset I_2 + I_3$ implies c is bounded.

$2b\sqrt{n} \subset I_1 + (-I_4)$ implies b is bounded and likewise d is bounded.

So $\Gamma \cap I_1 \times I_2 \times I_3 \times I_4$ is finite.

Why is it a group?

$$\begin{pmatrix} a + b\sqrt{n} & (c + d\sqrt{n})\sqrt{p} \\ (c - d\sqrt{n})\sqrt{p} & a - b\sqrt{n} \end{pmatrix} = aI_2 + bi + cj + dk$$

where

$$i = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & -\sqrt{n} \end{pmatrix}, j = \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & \sqrt{np} \\ -\sqrt{np} & 0 \end{pmatrix}$$

x	I	i	j	k
I	I	i	j	k
i	i	nI	k	nj
j	j	k	pI	-pi
k	k	-nj	pi	-npI

Poincare series. $\Gamma = SL(2,\mathbb{Z})$

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subset \Gamma$$

Lemma If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ sends ∞ to ∞, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty$.
Proof:

$$\frac{a\infty + b}{c\infty + d} = \frac{a}{c} = \infty \Rightarrow c = 0$$

Poincare Series:

$$P_m(z) \quad := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e\left(m \frac{az+b}{cz+d} \right) (cz+d)^{-k}$$

Proposition: The cosets for $\Gamma_\infty \backslash \Gamma$ are indexed as $\Gamma_\infty \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ where $c = 1, 2, 3, \ldots$, $(c,d) = 1$ and $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ are any a, b you want so that $ad - bc = 1$, i.e. a, b exist and are chosen by some fixed function of c and d. To put another way, let $\gamma_{c,d}$ be a fixed $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

$$\Gamma = \Gamma_\infty \coprod \coprod_{c=1,(d,c)=1}^{\infty} \Gamma_\infty \gamma_{c,d}$$

Proof

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+nc & a+nb \\ c & d \end{pmatrix}$$

Look at elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ with $c = 0$. These are exactly Γ_∞. Hence we are reduced to proving

$$\Gamma_\infty \backslash \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \neq 0 \right\} = \coprod_{c \in \mathbb{Z}_{>0},(c,d)=1} \Gamma_\infty \gamma_{c,d}$$

If $\Delta = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$. It suffices to show that

$$\Delta \backslash \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c > 0 \right\} = \coprod_{c \in \mathbb{Z}_{>0},(c,d)=1} \Delta \gamma_{c,d}$$

This statement will follow if we show for $c > 0$ and $(c,d) = 1$ fixed,

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\} = \Delta \gamma_{c,d}$$

Suppose

$$\begin{aligned}
&\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \\
&\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\
&= \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a'd-b'c & b+na \\ 0 & ad-bc=1 \end{pmatrix} \in \Delta \\
&\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\end{aligned}$$

<u>Lemma</u>
$(\Gamma_\infty \backslash \Gamma / \Gamma_\infty)$

$$\Gamma = \Gamma_\infty \coprod \coprod_{c\in\mathbb{Z}_{>0}, d\in(\mathbb{Z}/c\mathbb{Z})^*} \Gamma_\infty \gamma_{c,d} \Gamma_\infty$$

<u>Proof</u>

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & * \\ c & cn+d \end{pmatrix}$$

$\Gamma_\infty \gamma_{c,d} = \Gamma_\infty \gamma_{c,d+nc}$

$$\begin{aligned}
P_m(z) \quad &= e(mz) + \sum_{c=1}^{\infty} \sum_{(d,c)=1} e\left(m\frac{az+b}{cz+d}\right)(cz+d)^{-k} \\
&\text{when } m = 0 \\
&(0z+1)^{-k} + \sum_{c=1,(c,d)=1}^{\infty} (cz+d)^{-k} = E_k
\end{aligned}$$

<u>Theorem</u> $m > 0$, $k > 2$ even

$$\begin{aligned}
P_m(z) &= \sum_{n=1}^{\infty} \widehat{P}_m(n) e(nz) \\
&\text{where} \\
\widehat{P}_m(n) = \delta_{m=n} + \textstyle\sum_{c=1}^{\infty} \frac{Kl(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right)\left(2\pi i^{-k}(\frac{n}{m})^{\frac{k-1}{2}}\right) & \\
Kl(m,n;c) &= \text{Kloosterman sum} \\
&= \sum_{h\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{mh+nh^{-1}}{c}\right)
\end{aligned}$$

In particular, $P_m(z)$ is a cusp form.
<u>Proof</u> Suffices to show for $c > 0$

$$\begin{aligned} & \sum_{(c,d)=1} e\left(m\frac{az+b}{cz+d}\right)(cz+d)^{-k} \\ & = 2\pi i^k \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \frac{Kl(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{nm}}{c}\right) \\ \text{First sum} & = \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\sum_{r\in\mathbb{Z}} e\left(m\gamma_{c,d+rc}\right)(cz+d+rc)^{-k} \end{aligned}$$

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$$

$$\gamma_{c,d+rc}z = \frac{az+(b+ra)}{cz+(d+rc)} = \frac{a}{c} + \frac{1}{c(cz+d+rz)} = \gamma_{c,d}(z+r)$$

$$\sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\sum_{r\in\mathbb{Z}} e\left(m\gamma_{c,d+rc}\right)(cz+d+rc)^{-k}$$

$$= \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\sum_{r\in\mathbb{Z}} e\left(\frac{ma}{c} + \frac{m}{c(c(z+r)+d)}\right)(c(z+r)+d)^{-k}$$

$$= \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{md^{-1}}{c}\right)\sum_{r\in\mathbb{Z}} e\left(\frac{m}{c(c(z+r)+d)}\right)(c(z+r)+d)^{-k}$$

$$= \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{md^{-1}}{c}\right)\sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} e(-ju)e\left(\frac{m}{c(c(z+u)+d)}\right)(c(z+u)+d)^{-k}du$$

Let $w = c(z+u)+d$, $dw = cdu$

$$= \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} \frac{e(md^{-1})}{c}\sum_{j\in\mathbb{Z}} \int_{Imw=cy>0} e\left(-j\left(\frac{w-d}{c} - z\right)\right) e\left(\frac{m}{cw}\right) w^{-k}dw$$

$$= \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*}\sum_{j\in\mathbb{Z}} \frac{e\left(\frac{md^{-1}+jd}{c}\right)}{c} e(jz) \int_{Imw=cy>0} e(\frac{-jw}{c})e(\frac{m}{cw})w^{-k}dw$$

Switch $j = n$

$$= \sum_{n\in\mathbb{Z}} \frac{Kl(m,n;c)}{c} e(nz) \int_{Imw=cy>0} e\left(\frac{-nw}{c} + \frac{m}{cw}\right) w^{-k}dw$$

$$\int_{Imw=c} w^{-k} e(-\mu_1 w - \mu_2 w^{-1})dw = 2\pi\left(\frac{\mu_1}{\mu_2}\right)^{\frac{k-1}{2}} i^k J_{k-1}\left(4\pi\sqrt{\mu_1\mu_2}\right)$$

Recall $dA = \frac{dx}{dy}y^2$ is invariant under $SL(2,\mathbb{R})$.
Inner product of weight k forms f, g

$$\begin{aligned}
\langle f, g\rangle &:= \int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} Im(z)^k dA \\
&= \int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} f(x+iy)\overline{g(x+iy)} y^{k-2} dx dy
\end{aligned}$$

Compute

$$\begin{aligned}
\langle P_m, f\rangle &= \int_{\Gamma\backslash\mathbb{H}} \overline{f(z)} \left(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(m\gamma z)(cz+d)^{-k}\right) y^k \frac{dx}{dy} y^2 \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\mathcal{F}} \overline{f(z)} e\left(m\frac{az+b}{cz+d}\right)(cz+d)^{-k} y^{k-2} dx dy \\
&= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{\gamma\mathcal{F}} \overline{f(w)} e(mw) Im(w)^k dA \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} \overline{f(w)} e(mw) Im(w)^k dA \\
&= \int_0^\infty \int_{-1/2}^{1/2} \sum \overline{c}_n \overline{e(nw)} e(mw) Im(w)^k dA \\
&\quad \text{where } f(w) = \sum c_n e(nw) \\
&= c_m \int_0^\infty e^{-4\pi m y} y^{k-2} dy \\
&= c_m (4\pi m)^{1-k}\Gamma(k-1)
\end{aligned}$$

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Let $k > 2$, even, $m > 0$.

$$\begin{aligned}
P_m(z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e\left(m\frac{az+b}{cz+d}\right)(cz+d)^{-k} \\
&= \sum_{n\geq 1} \widehat{P}_m(n) e(nz) \\
&\quad \text{where} \\
\widehat{P}_m(n) &= \left(\delta_{m=n} + 2\pi i^{-k}\sum_{c=1}^{\infty} \frac{Kl(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right)\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \\
Kl(m,n;c) &:= \sum_{x\overline{x}\equiv 1 (\mod p)} e\left(\frac{mx+n\overline{x}}{c}\right)
\end{aligned}$$

Let $f = \sum_{n=1}^{\infty} \widehat{f}(n)e(nz)$ be a cusp form for $SL(2,\mathbb{Z})$ of weight k.

$$\langle f, g\rangle = \int_{\mathcal{F}} f(x+iy)g(x+iy)y^k \frac{dxdy}{y^2}$$

inner product

Last time:

$$\langle f, P_m\rangle = \widehat{f}(m)\frac{\Gamma(k-1)}{(4\pi m)^{k-1}},\ m > 0$$

This implies

$$\begin{aligned}\langle P_n, P_m\rangle &= \langle P_m, P_n\rangle \\ &= \frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}}\left[\delta_{m=n} + 2\pi i^{-k}\sum_{c=1}^{\infty}\frac{Kl(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right]\end{aligned}$$

Consequences:

<u>Theorem 1</u> $\{P_n : n \geq 1\}$ span the space $S_k(SL(2,\mathbb{Z}))$.

<u>Proof</u> S_k is finite dimensional with an inner product congruent to $\mathbb{C}^n$ with usual inner product.

$S_k' \cong S_k$ via inner product:

Suppose there is $f \in S_k(\Gamma)$ not in the span. May assume f orthogonal to span.

Thus $\langle f, P_n\rangle = 0$ for all n implies

$$\widehat{f}(n)\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} = 0$$

and therefore all coefficients $\widehat{f}(n) = 0$. Thus $f = 0$.

<u>Theorem 2</u> For $m > 0$ and $\mathcal{F}$ is an orthonormal basis of $S_k(SL(2,\mathbb{Z}))$.

$$P_m = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\sum_{f\in\mathcal{F}}\overline{\widehat{f}(m)}f(z)$$

<u>Proof</u>

$$\langle P_m, f\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\overline{\widehat{f}(m)}$$

by earlier formula. Both sides have the same inner product versus a basis element. Hence formula holds.

<u>Pertersson Trace Formula</u>

Let $\mathcal{F}$ be an orthonormal basis of $S_k(\Gamma)$. Then

$$\sum_{f\in\mathcal{F}}\widehat{f}(n)\overline{\widehat{f}(m)} = \frac{(4\pi\sqrt{mn})^{k-1}}{\Gamma(k-1)}\left[\delta_{m=n} + 2\pi i^{-k}\sum_{i=1}^{\infty}\frac{Kl(m,n;c)}{c}J_{k-1}(4\pi\sqrt{mn}/c)\right]$$

<u>Proof</u>

$$\begin{aligned}\langle P_m, P_n\rangle &= \frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}}[\cdots] \\ &= \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\sum_{f\in\mathcal{F}}\overline{\widehat{f}(m)}\langle f, P_n\rangle \\ &= \sum_{f\in\mathcal{F}}\overline{\widehat{f}(m)}f(n)\frac{\Gamma(k-1)^2}{(4\pi m 4\pi n)^{k-1}}\end{aligned}$$

How big are the coefficients of cusp forms?
Rapid Decay
For all N, all cusp forms decay faster than y^{-N} as $y \to \infty$.
Proof

$$f(x+iy) = \int_0^1 [f(x+iy) - f(u+iy)]du$$

because for cusp forms $\int_0^1 f(u+iy)du = 0$.

$$\begin{aligned}f(x+iy) &= \int_0^1 [f(x+iy) - f(u+iy)]du \\ &= \int_0^1\int_u^x f_x(t+iy)dtdu \\ \max_x |f(x+iy)| &\leq \max_x f_x(x+iy)\end{aligned}$$

$SL(2,\mathbb{R})$ acts on i with orbit in $\mathbb{H}$.

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} : i \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} iy \mapsto x+iy$$

$$\max_x \left| f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right)\right| \leq \max_x \left|\frac{\partial}{\partial t}|_{t=0} f\left(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix}\right)\right|$$

Fact: $\partial t|_{t=0} f(g(\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix})) \ll$ bounded

$$
\begin{aligned}
&= \max_x \left| \partial_t|_{t=0} f \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \right) \right| \\
&= \frac{1}{y} \max_x \left| \partial_t|_{t=0} f \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \right) \right| \\
&\le \frac{1}{y} \max(f'(\cdots)) \\
&\le \frac{1}{y^2} \max f''(\cdots) \\
&\cdots \\
&\le \frac{1}{y^n} \max f^{(n)}(\cdots)
\end{aligned}
$$

Trivial Bound

$$\frac{\widehat{f}(n)}{n^{\frac{k-1}{2}}} = O(n^{1/2})$$

i.e. $\widehat{f}(n) \ll n^{k/2}$

Proof

$$
\begin{aligned}
\widehat{f}(n)e^{-2\pi ny} \quad &= \int_0^1 f(x+iy)e(-nx)dx \\
&\text{set } y = 1/n
\end{aligned}
$$

$\widehat{f}(n)e^{-2\pi} = \int_0^1 f(x+i/n)e(-nx)dx$

$\widehat{f}(n) \ll \max_{x\in[0,1]} |f(x+i/n)|$

Observe

$$
\begin{aligned}
g(z) \quad &= \left| y^{k/2} f(x+iy) \right| = |Im(z)^{k/2} f(z)| \\
g(\gamma z) \quad &= |Im(\gamma z)^{k/2} f(\gamma z)| \\
&= \left| \left(\frac{Im(z)}{|cz+d|^2} \right)^{k/2} f(z)(cz+d)^k \right| \\
&= g(z), \text{ invariant}
\end{aligned}
$$

If $y \ge 1$, $|g(x+iy)| \ll y^{k/2} y^{-N}$ for any N because I will show $f(x+iy) \ll y^{-N}$ as $y \to \infty$. $g(z)$ is bounded on fundamental domain so it also is on $\mathbb{H}$.

$f(x+iy) \ll y^{-k/2}$, $f(x+i/n) \ll n^{k/2}$, $\widehat{f}(n) \ll n^{k/2}$.

$$\begin{aligned}
\textstyle\int_0^1 |f(x+iy)|^2 dx &\ll y^{-k} \\
= \textstyle\sum |\widehat{f}(n)|^2 e^{-4\pi n y} & \\
\approx \textstyle\sum_{n=1}^{\infty} |\widehat{f}(n)|^2 &\ll y^{-k} \\
|\widehat{f}(T)|^2 &\leq \sum_{n=1}^{T} |\widehat{f}(n)|^2 \ll T^k \\
\Rightarrow \quad & \widehat{f}(T) \ll T^{k/2} \\
\text{On average} \quad & |\widehat{f}(T)|^2 \ll T^{k-1}
\end{aligned}$$

We expect (Ramanujan conjecture), for all $\epsilon > 0$, there is c_ϵ such that $|\widehat{f}(n)| \leq c_\epsilon n^{\frac{k-1}{2}+\epsilon}$, Deligne showed this is 1972.

Theorem For $\alpha \in \mathbb{R}$

$$\sum_{n\leq T} \widehat{f}(n) e(n\alpha) \ll T^{k/2}$$

independent of α. Remark: Implies the above.

Proof

$$\begin{aligned}
f(x+iy) *_x D_T(x)|_{x=\alpha} &= \sum_{n\leq T} \widehat{f}(n) e^{-2\pi n y} e(n\alpha) \\
D_T(x) &= \sum_{|n|\leq T} e(nx) \\
\|D_t\|_1 &\ll \log T
\end{aligned}$$

Theorem Asymptotics of $J_{k-1}(x)$

$$J_{k-1}(x) \ll \min\left(x^{k-1}, x^{-1/2}\right)$$

$$|Kl(m,n;c)| \ll c^{power}, \text{ power } \leq 1$$

$$\begin{aligned}
& \sum_{c=1}^{\infty} \frac{Kl(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\
& \ll \sum_{c=1}^{\infty} c^{p-1} \min\left\{\left(\frac{\sqrt{mn}}{c}\right)^{k-1}, \frac{c^{1/2}}{(mn)^{1/4}}\right. \\
& \ll \sum_{c<\sqrt{mn}} c^{p-1} c^{1/2} (mn)^{-1/4} \\
& + \sum_{c\geq\sqrt{mn}} c^{p-1} c^{1-k} (mn)^{(k-1)/2} \\
& = (mn)^{-1/4} \sum_{c<\sqrt{mn}} c^{p-1/2} + (mn)^{(k-1)/2} \sum_{c\geq\sqrt{mn}} c^{p-k} \\
& \ll (mn)^{-1/4} (\sqrt{mn})^{p+1/2} + (mn)^{(k-1)/2} (\sqrt{mn})^{1+p-k} \\
& = (mn)^{p/2} + (mn)^{p/2} \\
[\delta_{m=n} + \cdots] \quad & \ll (mn)^{p/2}
\end{aligned}$$

$$\begin{aligned}
P_m \quad & = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \sum_{f\in\mathcal{F}} \overline{\widehat{f}(m)} f \\
\langle P_m, P_m \rangle \quad & = \frac{\Gamma(k-1)^2}{(4\pi m)^{2k-2}} \sum |\widehat{f}(m)|^2 \\
& \text{Petersson with } m = n \Rightarrow \\
\sum |\widehat{f}(n)|^2 \quad & = \left(\frac{4\pi n}{\Gamma(k-1)}\right)^{k-1} [\delta_{m=n} + \cdots] \ll n^{p+k-1}
\end{aligned}$$

For any orthnormal basis,

$$|\widehat{f}(n)|^2 \leq \sum_{f\in\mathcal{F}} |\widehat{f}(n)|^2 \ll n^{p+k-1}$$

$$\widehat{f}(n) \ll n^{p/2+(k-1)/2}$$

$p = 1$ gives trivial bound, $p < 1$ gives improvement, $p = 0$ (false) implies Ramanujan.

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$$S(m,n;c) \quad := \sum_{x\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{mx + nx^{-1}}{c}\right)$$

Basic Properties

$$S(m,n;c) = S(n,m;c) \qquad x \leftrightarrow x^{-1}$$

If $(a,c)=1$, $S(am,n;c) = S(m,an;c) \qquad x \mapsto xa^{-1}$

Twisted Multiplicativity Formula

$(q,r)=1$

$$S(m,n;qr) = S(\overline{q}m,\overline{q}n;r)\cdot S(\overline{r}m,\overline{r}n;q)$$

where $q\overline{q}\equiv 1(r)$, $r\overline{r}\equiv 1(q)$

Consequently, Kloosterman sums are determined by $S(m,n;p^{\alpha})$, p prime.

Proof

$$\begin{aligned} &S(\overline{q}m,\overline{q}n;r)\cdot S(\overline{r}m,\overline{r}n;q)\\ &= \sum_{\substack{ab\equiv 1(r)\\ cd\equiv 1(q)}} e\left(\frac{\overline{q}ma+\overline{q}nb}{r}+\frac{\overline{r}mc+\overline{r}nd}{q}\right)\\ &= \sum_{\substack{ab\equiv 1(r)\\ cd\equiv 1(q)}} e\left(\frac{m(q\overline{q}a+r\overline{r}c)+n(q\overline{q}b+r\overline{r}d)}{qr}\right)\end{aligned}$$

Note: $(q\overline{q}a+r\overline{r}c)(q\overline{q}b+r\overline{r}d)\equiv (r\overline{r})^2cd\equiv 1(q)$

$\equiv (q\overline{q})^2ab\equiv 1(r)$

Since $(r,q)=1$, this is $\equiv 1(qr)$. The map $(a,c)\mapsto q\overline{q}a+r\overline{r}c$ is a bijection from

$$(\mathbb{Z}/r\mathbb{Z})^*\times(\mathbb{Z}/q\mathbb{Z})^*\to(\mathbb{Z}/qr\mathbb{Z})^*$$

by Chinese Remainder Theorem. Thus this is $S(m,n;qr)$. ■

Goal Show: For m,n fixed

$$S(m,n;c)\ll c^{1-\delta} \text{ for some } \delta>0$$

Best possible-Weil: $\delta<1/2$

Trivial bound: $S(m,n;c)\leq\phi(c)\ll c$

Show $a_n\ll \epsilon n^{k/2-1/8+\epsilon}$

Selberg: $a_n\ll n^{k/2-1/4+\epsilon}$

Optimal (Ramanujan's conjecture, 1916, Deligne's Theorem 1972):

$$a_n\ll n^{k/2-1/2+\epsilon}$$

Let's consider $S(a,b;p)$. $S(0,0;p)=p-1$

$$S(0,b;p)=S(b,0;p)=\sum_{x=1}^{p-1}e\left(\frac{bx}{p}\right)=-1$$

if $b \neq 0$

If $b \neq 0$, $S(a,b;p) = S(ab,1;p)$ reduce to $S(a,1;p)$ Weil: $|S(a,b;p)| \leq 2\sqrt{p}$

Theorem (Katz): $S(a,1;p)/2\sqrt{p}$ distributed by Sato-Tate.

Studied Moments: k^{th} moments

$$V_k = \sum_{a(\bmod p), a\neq 0} S(a,1;p)^k$$

$$\begin{aligned}
(p-1)V_k &= \sum_{a,b\in\mathbb{F}_p^*} S(a,b;p)^k \\
V_k &= \sum_{a\in\mathbb{F}_p^*} S(ab,b;p) \\
(p-1)V_k &= \sum_{a,b(\bmod p)} S(a,b;p)^k - S(0,0;p)^k - 2\sum_{b\in\mathbb{F}_p^*} S(b,0;p)^k \\
&= \sum_{a,b(\bmod p)} S(a,b;p)^k - (p-1)^k - 2(p-1)(-1)^k \\
\textstyle\sum_{a,b(\bmod p)} S(a,b;p)^k &= \sum_{a,b(\bmod p)} \sum_{x_i\in\mathbb{F}_p^*} e\left(\frac{ax_1 + b\overline{x}_1 + ax_2 + b\overline{x}_2 + \cdots + ax_k + b\overline{x}_k}{p}\right) \\
&= \sum_{x_i\in\mathbb{F}_p^*} \sum_{a(\bmod p)} e\left(\frac{a(x_1+x_2+\cdots+x_k)}{p}\right) \sum_{b(\bmod p)} e\Bigg(\frac{b(\overline{x}_1+\overline{x}_2+\cdots+\overline{x}_k)}{p} \\
\textstyle\sum_{a(\bmod p)} e\left(\frac{ar}{p}\right) &= \begin{cases} p & \text{if } p|r \\ 0 & \end{cases} \\
\textstyle\sum_{a,b(\bmod p)} S(a,b;p)^k &= p^2\nu_k(p) \\
\nu_k(p) &= \#\{(x_1,\ldots,x_k)\in(\mathbb{F}_p^*)^k : x_1+\cdots+x_k = 0,\ \overline{x}_1+\cdots+\overline{x}_k = 0\} \\
V_k &= \frac{p^2}{p-1}\nu_k(p) - (p-1)^{k-1} - 2(-1)^k
\end{aligned}$$

$$\begin{aligned}
\nu_1(p) &= 0 \\
V_1(p) &= -1 - 2(-1) = 1
\end{aligned}$$

Lemma

$\nu_2(p) = p-1$, $V_2(p) = p^2 - p - 1$

Proof

$x_1 + x_2 = 0$, $x_1 = -x_2$, $\overline{x}_1 + \overline{x}_2 = \overline{x}_1 - \overline{x}_1 = 0$, any x_1 works so $p-1$ solutions.

If I sum squares I get $\sim p$ on average. $|S(a,1;p)| \sim \sqrt{p}$ on average.

Kloosterman sums are real-valued:

$$
\begin{aligned}
\overline{S(m,n;c)} &= \sum_x \overline{e\left(\frac{mx+n\overline{x}}{c}\right)} \\
&= \sum_x e\left(-\frac{mx+n\overline{x}}{c}\right) \\
&= S(m,n;c)
\end{aligned}
$$

Lemma $\nu_3(p) = (p-1)\left[1+(\frac{-3}{p})\right]$

$V_3 = (\frac{-3}{p})p^2 + 2p + 1$

Proof

$x_1 + yx_1 + x_1z = 0$, $x_2 = x_1y$, $x_3 = x_1z$, so $1+y+z = 0$, $z = -(y+1)$. $\overline{x}_1 + \overline{x}_2 + \overline{x}_3 = 0$, $\overline{x}_1 + \overline{x}_1\overline{y} + \overline{x}_1\overline{z} = 0$, so $1 + \overline{y} = \overline{1+y}$. Get a solutions for each x_1 and y such that $1+\overline{y} = \overline{1+y}$. Get $(p-1)$ solutions for each $\overline{y}$ such that $1+\overline{y} = \overline{1+y}$. Equivalently, $(1+\overline{y})(1+y) = 1$, $1+y+\overline{y}+y\overline{y} = 1$, so $1+y+\overline{y} = 0$ iff $y^2+y+y\overline{y} = 0$ or $y^2+y+1 = 0$. Solution is $(-1\pm\sqrt{-3})/2$, so if $(-3/p) = 1$, get 2 solutions. If it is -1, get zero solutions.

Proposition $\nu_4(p) = 3(p-1)(p-2)$

$$
\begin{aligned}
V_4 &= \frac{p^2}{p-1}3(p-1)(p-2) - (p-1)^3 - 2 \\
&= 3p^3 - 6p^2 - (p^3 - 3p^2 + 3p - 1) - 2 \\
&= 2p^3 - 3p^2 - 3p - 1 \leq 2p^3 \\
\Rightarrow \quad & |S(m,n;p)| \leq 2^{1/4}p^{3/4}
\end{aligned}
$$

Proof

$x_1 + x_2 + x_3 + x_4 = 0 = \overline{x}_1 + \overline{x_2} + \overline{x_3} + \overline{x}_4$, $x_3 \to -x_3$, $x_4 \to -x_4$ implies $x_1 + x_2 = x_3 + x_4$ and $\overline{x}_1 + \overline{x}_2 = \overline{x}_3 + \overline{x}_4$.

Let $\eta(u,v) = \#\{(x,y) \in \mathbb{F}_p^* : x+y = u, \overline{x}+\overline{y} = v\}$

$\nu_4(p) = \sum_{u,v\in\mathbb{F}_p} \nu(u,v)^2$

$\eta(0,0) = p-1$. If $x+y = 0$, $\overline{x}+\overline{y} = 0$, so $\eta(0,b) = \eta(b,0) = 0$ unless $b = 0$.

If $u, v \neq 0$, $x+y = u$, $\overline{x}+\overline{y} = v$,

$xyv = (\overline{x}+\overline{y})xy = x\overline{x}y + y\overline{y}x = y + x = u$.

$xyv = u$, Solve $z^2 - uz + u\overline{v} \equiv 0(p) = (z-x)(z-y)$. Has $1 + (D/p)$ solutions where $D = u^2 - 4u\overline{v}$.

So $\eta(u,v) = 1+$

$$
\eta(u,v)^2 = 1 + \left(\frac{u^2 - 4u\overline{v}}{p}\right) = 1 + \left(\frac{1 - \frac{u}{uv}}{p}\right)
$$

$$\begin{aligned}
\nu_4(p) &= (p-1)^2 + \sum_{u,v\in\mathbb{F}_p^*} \eta(u,v)^2 \\
&= (p-1)^2 + \sum_{u,v\in\mathbb{F}_P^*} \left(1+\left(\frac{1-\frac{u}{uv}}{p}\right)\right)^2 \\
&= (p-1)^2 + (p-1) \sum_{w\in\mathbb{F}_p^*, z=w+1} \left(1+\left(\frac{z}{p}\right)\right)^2 \\
&= (p-1)^2 + (p-1)\sum_{z\neq 1}\left[1+2(\frac{z}{p})+(\frac{z}{p})^2\right] \\
&= (p-1)^2 + (p-1)[(p-1)-2+p-2] \\
&= (p-1)\left[p-1+p-1+p-4\right] \\
&= (p-1)(3p-6) \\
&= 3(p-1)(p-2)
\end{aligned}$$

Useful facts for prime powers:
For all $p > 2$, there is $g \in \mathbb{Z}$ such that g generates $(\mathbb{Z}/p^k\mathbb{Z})^*$ for any k.
Gauss Sum
$c > 0$ and $(c, 2n) = 1$

$$\begin{aligned}
\textstyle\sum_{t\in\mathbb{Z}/c\mathbb{Z}} e(\frac{nt^2}{c}) &= (\frac{n}{c})\sqrt{c}\epsilon_c \\
\epsilon_c &= \left\{\begin{array}{ll} 1 & c\equiv 1(4) \\ i & c\equiv -1(4)\end{array}\right.
\end{aligned}$$

Salie 1936:
Assume $(n, m, c) = 1$, $c = p^\alpha$, $\alpha > 1$, $p > 2$.
1) $S(lm, ln; c) = S(m, n; c) = 0$ unless $n = l^2 m(\mod c)$ for some $(l, c) = 1$.
2) $S(n, n; c) = 2(\frac{n}{c})\sqrt{c}Re[\epsilon_c e(\frac{2n}{c})]$

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$$\sum_{a=1}^{p-1} S(a, 1; p)^k$$

for $k \leq 4$, polynomials except $k = 3$ where $(-3/p)$ comes up. For large k the formulas involve other arithmetic info, e.g. Fourier coefficients of modular forms, e.g. on $GL(3)$.

In 1916, Ramanujan conjectured

$$c(z)\prod_{n>0}(1-e(nz))^{24} = \Delta = \sum_{n>0}\tau(n)e(nz)$$

(Deligne, 1972)

$$|\tau(p)| \leq 2p^{11/2}$$

(Mordell, 1917)

$$\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11}\tau(mn/d^2)$$

In particular, if $(m,n) = 1$,

$$\tau(m)\tau(n) = \tau(mn) \text{ (multiplicative)}$$

Mordell's Theorem implies $\tau(n)$ is determined by n prime power

$$\tau(p^k)\tau(p) = \tau(p^{k+1}) + p^{11}\tau(p^{k-1}) \text{ if } k \geq 1$$

$$\tau(p^{k+1}) = \tau(p^k)\tau(p) - p^{11}\tau(p^{k-1})$$

If $|\tau(p)| \leq 2p^{11/2}$, then $\tau(p^k) \ll_\epsilon p^{k(11/2+\epsilon)}$ for any $\epsilon > 0$ which implies $\tau(n) \ll_\epsilon n^{11/2+\epsilon}$ since both sides are mutliplicative in n. In the other direction, assume $\tau(n) \ll_\epsilon n^{11/2+\epsilon}$. In particular, $\tau(p^k) \ll_\epsilon p^{k(11/2+\epsilon)}$.

Write $\tau(p) = p^{11/2}(\alpha_p + \alpha_p^{-1})$

Claim:

$$\tau(p^k) = p^{k\cdot 11/2}\frac{\alpha_p^{k+1} - \alpha_p^{-k-1}}{\alpha_p - \alpha_p^{-1}}$$

Proof: agrees for $k = 1$.

$$\begin{aligned}
p^{(k+1)11/2}\frac{\alpha^{k+2} - \alpha^{-k-2}}{\alpha - \alpha^{-1}} &= p^{k11/2}\frac{\alpha^{k+1} - \alpha^{-k-1}}{\alpha - \alpha^{-1}}p^{11/2}(\alpha + \alpha^{-1}) - p^{11}p^{(k-1)11/2}\frac{\alpha^k - \alpha^{-k}}{\alpha - \alpha^{-1}} \\
\Leftrightarrow \qquad & \alpha^{k+2} - \alpha^{-k-2} = (\alpha^{k+1} - \alpha^{-k-1})(\alpha + \alpha^{-1}) - \alpha^k + \alpha^{-k} \\
&= \alpha^{k+2} + \alpha^k - \alpha^{-k} - \alpha^{-k-2} - \alpha^k + \alpha^{-k} \\
&= \alpha^{k+2} - \alpha^{-k-2} \blacksquare
\end{aligned}$$

So

$$\begin{aligned}
\tau(p^k) &= p^{k11/2}\frac{\alpha^{k+1} - \alpha^{-k-1}}{\alpha - \alpha^{-1}} \ll_\epsilon p^{k(11/2+\epsilon)} \\
& \frac{\alpha^{k+1} - \alpha^{-k-1}}{\alpha - \alpha^{-1}} \ll_\epsilon p^{k\epsilon} \\
|\tau(p)| \leq 2p^{11/2} &\Leftrightarrow |\alpha + \alpha^{-1}| \leq 2 \\
\alpha^k \ll_\epsilon p^{k\epsilon} &\Rightarrow |\alpha| \leq p^\epsilon \text{ for all } \epsilon > 0 \Rightarrow |\alpha| \leq 1
\end{aligned}$$

Ramanujan's

$$|\tau(p)| \leq 2p^{11/2} \Leftrightarrow \tau(n) \ll_\epsilon n^{11/2+\epsilon} \text{ for all } \epsilon > 0$$

$$\tfrac{\tau(m)}{m^{11/2}}\tfrac{\tau(n)}{n^{11/2}} \quad = \sum \frac{\tau(mn/d^2)}{(mn/d^2)^{11/2}}$$

Hecke operators:
Group-theoretic background:
Let

$$G_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = n \right\}$$

e.g. $\Gamma = SL(2, \mathbb{Z}) = G_1$
$G_n = \Gamma G_n = G_n \Gamma$
<u>Main Lemma</u>
Let

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad = n, 0 \leq b < d \right\}$$

Then Δ_n is a complete set of right coset representatives for G_n mod Γ

$$G_n = \coprod_{\rho \in \Delta_n} \Gamma \rho$$

<u>Proof</u>

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + nc & b + nd \\ c & d \end{pmatrix}$$

If $c = 0$:

$$\Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \Gamma \begin{pmatrix} a & b + nd \\ 0 & d \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \text{ and } \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}$$

are inequivalent if $b \neq b'$ and $0 \leq b, b' < d$.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$$

implies $\gamma = 0$ so $\delta = \pm 1$.

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b' \\ 0 & d \end{pmatrix}$$

$a' = a$, $b = b'$, $d' = d$ so are all inequivalent

The rest of the proof involves demonstrating that each element $g \in G_n$ can be multiplied by some $\gamma \in \Gamma$ so that γg is upper triangular.

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$ra + sc = 0$. Try $r = c/(a,c)$, $s = -a/(a,c)$. Then there exist p, q such that $ps - qr = 1$.

$$\gamma g = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Fact If $g \in GL(n,\mathbb{Q})$, $g = \gamma b$, where $\gamma \in GL(n,\mathbb{Z})$, $b \in GL(n,\mathbb{Q})$ and upper triangular. (p-adic version of Gram-Schmidt, also called the Iwasawa Decomposition)

Hecke Operators T_n, $n \geq 1$

$$\begin{aligned} (T_n f)(z) &= \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right) \\ &= n^{k/2-1} \sum_{\rho \in \Delta_n} \overline{\chi(\rho)} f|_\rho \end{aligned}$$

where the slash operator

$$f|_A(z) = (detA)^{k/2}(cz+d)^{-k} f(Az)$$

$f|_A|_B = f|_{AB}$
Look at $SL(2,\mathbb{Z})$

$$\begin{aligned} (T_n f)(\gamma z) &= n^{k/2-1} \sum_{\rho \in \Delta_n} f|_\rho|_\gamma \\ &= n^{k/2-1} \sum_{\rho \in \Delta_n} f|_{\rho\gamma} \end{aligned}$$

$\Gamma \backslash G_n$, $\Delta_n \gamma$ is still a complete set of coset representatives.
Thus

$$T_n : \mathcal{M}_k(SL(2,\mathbb{Z})) \to \mathcal{M}_k(SL(2,\mathbb{Z}))$$

If f is a cusp form $f(z) \to 0$ rapidly as $Imz \to \infty$. So $f((az+b)/d) \to 0$ since $a, d > 0$. Therefore, $T_n f(z) \to 0$ iff $T_n f$ is cuspidal if f is.

Hence,

$$T_n : \mathcal{S}_k(SL(2,\mathbb{Z})) \to \mathcal{S}_k(SL(2,\mathbb{Z}))$$

e.g. $k = 12$ this one-dimensional, hence $T_n \Delta = \lambda_n \Delta$ for some λ_n, Hecke eigenfunction.

Action on Fourier Coefficients

$$\begin{aligned}
T_n f(z) &= \frac{1}{n} \sum_{ad=n,b(\mod d)} a^k f\left(\frac{az+b}{d}\right) \\
&= \frac{1}{n} \sum_{m} \sum_{ad=n,b|d} a^k \tau(m) e\left(\frac{maz+bm}{d}\right) \\
\textstyle\sum_{b|d} e\left(\frac{bm}{d}\right) &= \begin{cases} 0 & d \nmid m \\ d & d|m \end{cases} \\
T_n f(z) &= \sum_{m} \sum_{d|(m,n),a=n/d} \left(\frac{n}{d}\right)^{k-1} \tau(m) e\left(\frac{mn}{d^2} z\right) \\
N^{th} \text{ Fourier Coefficient} &= \sum_{mn=d^2N,d|mn} \left(\frac{n}{d}\right)^{k-1} \tau(m) \\
&= \sum_{d|n,n|dN} \left(\frac{n}{d}\right)^{k-1} \tau\left(\frac{d^2N}{n}\right)
\end{aligned}$$

$N = 1$: $d|n$, $n|d$, so $d = n$

$$= \tau(d^2/n) = \tau(d) = \tau(n)$$

$T_n\Delta$'s first coefficient is $\tau(n) = \lambda_n \Delta$. First coefficient is $\lambda_n \tau(1) = 1$ implies $\lambda_n = \tau(n)$.

The n^{th} Hecke operator T_n has $T_n f = \lambda_n f$ then $\lambda_n =$ nth Fourier coefficient of f.

m^{th} coefficient on $T_n f = \lambda_n \tau(m) = \tau(n)\tau(m)$.

$$= \sum_{d|n,n|dm} \left(\frac{n}{d}\right)^{k-1} \tau\left(\frac{d^2m}{n}\right)$$

$d \mapsto n/d$ is a bijection of divisors of n

$$= \sum_{d|n,n|nm/d} d^{k-1} \tau(mn^2/d^2n) = \sum_{d|n,n|nm/d} d^{k-1} \tau(mn/d^2)$$

$d|n$ and $n|nm/d$ iff $dn|nm$ iff $d|m$ iff $d|(n,m)$

Lemma

$$T_n T_m = \sum_{d|(m,n)} d^{k-1} T_{nm/d^2}$$

implies T_n's commute.

Proof

$$
\begin{aligned}
&= (nm)^{k/2-1} \sum_{ad=n, a'd'=m, b(d), b'(d')} f\Big|\begin{pmatrix} a' & b \\ & d' \end{pmatrix}\begin{pmatrix} a & b \\ & d \end{pmatrix} \\
&= \sum_{ad=n, a'd'=m, b(d), b'(d')} f\Big|\begin{pmatrix} aa' & a'b+db' \\ & dd' \end{pmatrix}
\end{aligned}
$$

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Hecke Operators

$$
\begin{aligned}
T_n f &= n^{k/2-1} \sum_{\substack{\rho\in\Gamma\backslash G_n \\ \rho\in\Delta_n}} f|_\rho \\
(T_n f)(z) &= \frac{1}{n} \sum_{\substack{ad=n \\ b \mod d}} a^k f\left(\frac{az+b}{d}\right)
\end{aligned}
$$

Today: $\Gamma = SL(2,\mathbb{Z})$

Theorem

$$
T_n T_m = \sum_{d|(n,m)} d^{k-1} T_{nm/d^2}
$$

In particular, $\{T_n\}$ commute, $T_n T_m = T_m T_n$ if $(n,m)=1$ and

$$
T_{p^{n+1}} = T_p T_{p^n} - p^{k-1} T_{p^{n-1}}
$$

Proof

$$
\begin{aligned}
mn(T_n \circ T_m f)(z) &= \sum_{\substack{a_1d_1=m\\ a_2d_2=n\\ b_1 \bmod d_1\\ b_2 \bmod d_2}} (a_1a_2)^k f\left(\frac{a_1\frac{a_2z+b_2}{d_2}+b_1}{d_1}\right)\\
&= \sum_{\substack{a_1d_1=m\\ a_2d_2=n\\ b_1 \bmod d_1\\ b_2 \bmod d_2}} (a_1a_2)^k f\left(\frac{(a_1a_2)z+(b_2a_1+b_1d_2)}{d_1d_2}\right)\\
&= \sum_{\delta|(m,n)} \sum_{\substack{(a_1,d_2)=\delta\\ a_1d_1=m\\ a_2d_2=n\\ b_i \bmod d_i}} (a_1a_2)^k f\left(\frac{(a_1a_2)z+(b_2a_1+b_1d_2)}{d_1d_2}\right)
\end{aligned}
$$

$a_1 \mapsto a_1\delta$, $d_2 \mapsto d_2\delta$, $(a_1,d_2)=1$

$$
\begin{aligned}
&= \sum_{\delta|(m,n)} \sum_{\substack{(a_1,d_2)=1\\ a_1d_1\delta=m\\ a_2d_2\delta=n\\ b_1 \bmod d_1\\ b_2 \bmod \delta d_2}} (a_1a_2\delta)^k f\left(\frac{(a_1a_2)z+(b_2a_1+b_1d_2)}{d_1d_2}\right)\\
&= \sum_{\delta|(m,n)} \sum_{\substack{(a_1,d_2)=1\\ a_1d_1=m/\delta\\ a_2d_2=n/\delta\\ b_1 \bmod d_1\\ b_2 \bmod \delta d_2}} (a_1a_2)^k f\left(\frac{a_1a_2z+a_1b_2+b_1d_2}{d_1d_2}\right)
\end{aligned}
$$

Since a_1, d_2 are relatively prime, $(b_1,b_2) \mapsto a_1b_2+b_1d_2$ is onto. Since the image contains $\{1\}$, the map is onto. Kernel is $\{(b_1,b_2)|a_1b_2+b_1d_2 \equiv 0(d_1d_2)\}$ order δ.

$$
= \sum_{\delta|(m,n)} \delta^{k+1} \sum_{\substack{(a_1,d_2)=1\\ a_1d_1=m/\delta\\ a_2d_2=n/\delta\\ b \bmod d_1d_2}} (a_1a_2)^k f\left(\frac{a_1a_2z+b}{d_1d_2}\right)
$$

Let's consider the factorization $(a_1a_2)(d_1d_2)=mn/\delta^2=ad$?

Claim: Given any factorization $ad=mn/\delta^2$, there is a unique factorization $a_1a_2=a$, $d_1d_2=d$, $(a_1,d_2)=1$, given by $a_1=m/(m,\delta d)$

$$
a_1=\frac{m}{(m,\delta d)},\ d_2=\frac{\delta d}{(m,\delta d)},\ d_1=\frac{(m,\delta d)}{\delta},\ a_2=\frac{(m,\delta d)}{\delta^2 d}n
$$

$$
= \sum_{\delta|(m,n)} \delta^{k+1} \sum_{\substack{ad=mn/\delta^2\\ b \bmod d}} a^k f\left(\frac{az+b}{d}\right)
$$

In this setup, $(a_1, d_2) = 1$, $a_1 d_1 = m/\delta$, $a_2 d_2 = n/\delta$, $\delta | m$ and $\delta | \delta d$ implies $d_1 \in \mathbb{Z}$. $ad = mn/\delta^2$, $d | mn/\delta^2$, $\delta^2 d | mn$, $a_2 = (mn, \delta dn)/\delta^2 d$, $\delta^2 d | mn$ iff $\delta^2 (mn/a\delta^2) | mn$ iff $mn/a | mn$, check. $\delta^2 d | \delta dn$ iff $\delta | n$, check.

Conversely, if $ad = nm/\delta^2$ and $a = a_1 a_2$, $d = d_1 d_2$, $(a_1, d_2) = 1$. $a_1 d_1 = m/\delta$, $a_2 d_2 = n/\delta$

Again, reduces to prime powers:

Situation:

$$\begin{aligned} p^a p^d &= \frac{p^n p^m}{p^{2\delta}} \\ p^{a_1} p^{d_1} &= p^m / p^\delta \\ p^{a_2} p^{d_2} &= p^n / p^\delta \\ p^a &= p^{a_1} p^{a_2} \\ p^d &= p^{d_1} p^{d_2} \end{aligned}$$

So

$$\begin{aligned} a + d &= n + m - 2\delta \\ a_1 + d_1 &= m - \delta \\ a_2 + d_2 &= n - \delta \\ a &= a_1 + a_2 \\ d &= d_1 + d_2 \end{aligned}$$

Without loss of generality coprimality means $a_1 = 0$.

$$\begin{aligned} d_1 &= m - \delta \\ a_2 + d_2 &= n - \delta \\ a &= a_2 \\ d &= d_1 + d_2 \end{aligned}$$

Translate back to multiplication:

$$\begin{aligned} d_1 &= m/\delta \\ d_2 &= n/\delta \\ a &= a_2 \\ d &= mn/\delta^2 a \end{aligned}$$

a_1 claim iff $(m, \delta d)$ and m have same p-part.

$\delta + d_1 + d_2 = \delta d$'s p-part $\geq \delta + d_1 = m$'s part.

Once you have a_1 you have d_1, which then forces the product of a_2, d_2. Only one way to do that, so d_2 follows.

Lemma :et X be a formal variable.

$$(1 - T_p X + p^{k-1}X^2)\sum_{n\geq 0} T_{p^n}X^n = 1$$

Proof

$$\begin{aligned}
&\sum_{n\geq 0} T_{p^n}X^n - \sum_{n\geq 0} T_p T_{p^n} X^{n+1} + p^{k-1}\sum_{n\geq 0} T_{p^n}X^{n+2} \\
&= \sum_{n\geq 0} T_{p^n}X^n - \sum_{n\geq 0} T_{p^{n+1}}X^{n+1} + p^{k-1}\sum_{n\geq 1} T_{p^{n-1}}X^{n+1} + \sum_{n\geq 0} T_{p^n}X^{n+2}p^{k-1} \\
&= 1
\end{aligned}$$

Consequence: f cusp form
If $T_p f = \lambda_p f$ for all p, then $T_n f = \lambda_n f$ for all n.

$$f(z) = \sum \lambda_n e(nz)$$

$$\sum \lambda_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + p^{k-1-2s})^{-1}$$

Proof $\lambda_{nm} = \lambda_n \lambda_m$ if $(n,m) = 1$. so reduces to

$$\sum_{n\geq 0} \lambda_{p^n} p^{-ns} = \frac{1}{1 - \lambda_p p^{-s} + p^{k-1-2s}}$$

follows from lemma if $X = p^{-s}$.

Lemma

$$T_n P_m = \sum_{d|(m,n)} \left(\frac{n}{d}\right)^{k-1} P_{mn/d^2}$$

by

$$T_n T_m = \sum_{d|(n,m)} d^{k-1} T_{nm/d^2}$$

lemma iff $m = 1$.

Warm Up

$$\begin{aligned}
\tfrac{1}{n}\textstyle\sum_{\substack{ad=n \\ b \bmod d}} a^k e\left(m\frac{az+b}{d}\right) &= \frac{1}{n}\sum_{\substack{ad=n \\ b \bmod d}} a^k e\left(\frac{maz}{d}\right) e\left(\frac{mb}{d}\right) \\
&\sum_{b \bmod d} e\left(\frac{mb}{d}\right) = \left\{ \begin{array}{ll} d & d|m \\ 0 & o.w. \end{array} \right. \\
&= \frac{1}{n}\sum_{\substack{ad=n \\ d|m}} a^k d e\left(\frac{maz}{d}\right) \\
T_n e(mz) &= \sum_{d|(n,m)} \left(\frac{n}{d}\right)^{k-1} e\left(\frac{mn}{d^2}z\right)
\end{aligned}$$

Sum both sides over $\Gamma_\infty\backslash\Gamma$, we get P_m identity.

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Let

$$P_m = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e\left(m\frac{az+b}{cz+d}\right)(cz+d)^{-k}$$

Then

$$T_n P_m = \sum_{d|(m,n)} \left(\frac{n}{d}\right)^{k-1} P_{mn/d^2}$$

In particular, when $m = 1$,

$$T_m P_1 = m^{k-1} P_m$$

We now check that

$$T_n T_m P_1 = T_n m^{k-1} P_m$$

On the one hand,

$$\begin{aligned} T_n m^{k-1} P_m &= m^{k-1} \sum_{d|(m,n)} \left(\frac{n}{d}\right)^{k-1} P_{mn/d^2} \\ &= \sum_{d|(m,n)} \left(\frac{mn}{d}\right)^{k-1} P_{mn/d^2} \end{aligned}$$

On the other hand,

$$\begin{aligned} T_n T_m P_1 &= \left(\sum_{d|(m,n)} d^{k-1} T_{nm/d^2}\right) P_1 \\ &= \sum_{d|(m,n)} d^{k-1} \left(\frac{nm}{d^2}\right)^{k-1} P_{nm/d^2} \end{aligned}$$

As a consequence, we have

$$\begin{aligned} m^{k-1} T_n P_m &= T_n T_m P_1 \\ &= T_m T_n P_1 \\ &= n^{k-1} T_m P_n \end{aligned}$$

symmetric in $(n,m) \leftrightarrow (m,n)$.

Oet f be a modular from of weight k

$$f(z) = \sum_{n\geq 0} \widehat{f}(n) e(nz)$$

Consider

$$T_n f = \sum a_n(m) e(nz)$$

We have

$$\langle m^{k-1} T_n f, P_m \rangle = m^{k-1} \langle T_n f, P_m \rangle = m^{k-1} \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_n(m)$$

where $a_n(m)$ is the m^{th} coefficient of $T_n(f)$.

Then

$$\langle m^{k-1} T_n f, P_m \rangle = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} a_n(m)$$

which is symmetric in n and m. We showed before $a_n(m) = a_m(n)$. Therefore, we can conclude that

$$\langle m^{k-1} T_n f, P_m \rangle = \langle n^{k-1} T_m f, P_n \rangle$$

Now let's let $f = P_l$:

$$\langle m^{k-1} T_n P_l, P_m \rangle = \langle n^{k-1} T_m P_l, P_n \rangle$$

On the one hand,

$$\begin{aligned} \langle m^{k-1} T_n P_l, P_m \rangle &= \left(\frac{m}{l}\right)^{k-1} \langle l^{k-1} T_n P_l, P_m \rangle \\ &= \left(\frac{m}{l}\right)^{k-1} \langle n^{k-1} T_l P_n, P_m \rangle \\ &= \left(\frac{mn}{l}\right)^{k-1} \langle P_m, T_l P_n \rangle \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle n^{k-1} T_m P_l, P_n \rangle &= \left(\frac{n}{l}\right)^{k-1} \langle l^{k-1} T_m P_l, P_n \rangle \\ &= \left(\frac{n}{l}\right)^{k-1} \langle m^{k-1} T_l P_m, P_n \rangle \end{aligned}$$

These being equal gives us that

$$\langle T_l P_m, P_n \rangle = \langle P_m, T_l P_n \rangle$$

Since $\{P_m\}$ span the space of cusp forms

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle$$

for all cusp forms. Therefore, the T_n are self-adjoint operators on a finite dimensional Hilbert space.

Hence T_n can be diagonalized by a basis of Hecke eigenforms. Note the P_0 Eisenstein series is a Hecke eigenform and $\langle P_o, \text{cuspform} \rangle = 0^{th}$ coefficent $= 0$. This gives an orthonormal basis of Hecke eigenforms fro all modular forms.

Last Time:

$$\left(1 - T_p X + p^{k-1}X^2\right)\sum T_{p^n}X^n = 1$$

for $|X| < 1$. If f is a Hecke eigenform, cuspidal $a_0 = 0$,

$$f = \sum a_n e(nz)$$

Normalize with $a_1 = 1$, $T_n f = a_n$. Therefore,

$$\left(1 - a_p X + p^{k-1}X^2\right)\sum a_{p^n}X^n = 1$$

Let $X = p^{-s}$ with $Re(s) > 1$.

$$\frac{1}{1 - a_p p^{-s} + p^{k-1-2s}} = \sum a_{p^n}p^{-ns}$$

Euler product:

$$\begin{aligned}\textstyle\prod_p \frac{1}{1-a_p p^{-s}+p^{k-1-2s}} &= \prod_p\left(\sum a_{p^n}p^{-ns}\right)\\ &= \sum_m m^{-s}\left(a_{p_1^{m_1}}\cdots a_{p_r^{m_r}}\right) \text{ where } m = p_1^{m_1}\cdots p_r^{m_r}\\ &= \sum_{m=1}^{\infty} a_m m^{-s}\text{L function of f}\end{aligned}$$

$a_n a_m = a_{nm}$ if $(n,m) = 1$ and $T_n T_m = T_{nm}$
Riemann ζ-function:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

Simple pole at $s = 1$ but holomorphically continues to $\mathbb{C}\backslash\{1\}$.
How did Riemann analytically continue this?

$$\begin{aligned}\Theta(z) &= \sum_{n\in\mathbb{Z}} e(n^2 z) \qquad\qquad z \mapsto -1/4z\\ \omega(t) &= \sum_{n=1}^{\infty} e^{-\pi n^2 t} = \frac{1}{2}\left[\Theta\left(\frac{ti}{z}\right) - 1\right]\\ &= \frac{1}{2}\left[\frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} - 1\right]\\ &= \frac{1}{2\sqrt{t}}\left[\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t} - \sqrt{t}\right]\end{aligned}$$

$$\omega(t) \quad = \frac{1}{\sqrt{t}}\omega(1/t) + \frac{1}{2t^{1/2}} - \frac{1}{2}$$

$$I(s) = \int_0^\infty \omega(t)t^{s-1}dt$$

converge fine when $Re(s) \gg 1$. For t small, $\omega(t) \ll t^{-1/2}$. For t small the integrand is about $t^{s-3/2}$. This is fine if $s - 3/2 > -1$, or $s > 1/2$.

$$\begin{aligned}
I(s) \quad &= \int_0^\infty \sum_{n=1}^\infty e^{-\pi n^2 t} t^{s-1}dt \\
&= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{s-1}dt \\
&= \sum_{n=1}^\infty (\pi n^2)^{-s} \int_0^\infty e^{-t}t^{s-1}dt \\
&= \Gamma(s)\pi^{-s}\zeta(2s) = I(s)
\end{aligned}$$

$$\begin{aligned}
I(s) \quad &= \int_0^\infty \left[\frac{1}{\sqrt{t}}\omega(1/t) + \frac{1}{2t^{1/2}} - \frac{1}{2}\right] t^{s-1}dt \\
&= \int_0^\infty \left[\sqrt{t}\omega(t) + \frac{1}{2}t^{1/2} - \frac{1}{2}\right] t^{-s-1}dt \\
&= \int_0^\infty \left[\omega(t) + \frac{1}{2} - \frac{1}{2\sqrt{t}}\right] t^{-s-1/2}dt \\
&= I(1/2 - s)
\end{aligned}$$

still absolutely convergent!

This argument gives the analytic continuation of $\zeta(s)$ and shows

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$$

(Functional equation)

Hecke applied this idea of Riemann to modular forms other than Θ.

Assume f is a cusp form for $SL(2,\mathbb{Z})$

$$f(z) = z^{-k}f(-1/z)$$
$$f(iy) = (iy)^{-k}f(i/y)$$

Rapidly decays as $y \to 0$ or ∞.

Hence

$$I(s) = \int_0^\infty f(iy)y^{s-1}dy$$

is entire. Compute it:

$$\begin{aligned}
I(s) &= \int_0^\infty f(iy)y^{s-1}dy \\
&= \int_0^\infty \sum_{n>0} a_n e^{-2\pi ny} y^{s-1}dy \\
&= \sum_{n>0} a_n \int_0^\infty e^{-2\pi ny} y^{s-1}dy
\end{aligned}$$

absolutely convergent if $Re(s) \gg 1$.

$$\begin{aligned}
&= \sum_{n=1}^\infty a_n (2\pi n)^{-s}\Gamma(s) \\
&= (2\pi)^{-s}\Gamma(s)\sum \frac{a_n}{n^s}
\end{aligned}$$

$(2\pi)^{-s}$ is entire, nonzero. $\Gamma(s)$ is nonzero so $\sum a_n/n^s$ is entire and vanshes at $s = 0, -1, -2, \ldots$.

Note: You have to analytically continue, before you can talk about the functional equation.

Investigate symmetry:

$$\begin{aligned}
I(s) &= \int_0^\infty f(iy)y^{s-1}dy \\
&= \int_0^\infty (iy)^{-k} f(i/y) y^{s-1}dy \\
&= i^k \int_0^\infty f(iy)y^{k-s-1}dy \\
I(s) &= i^k I(k-s)
\end{aligned}$$

Consider

$$L(s,f) := \sum \frac{a_n}{n^{\frac{k-1}{2}}} n^{-s} = \sum a_n n^{-s-\frac{k-1}{2}}$$

(normalization)

$$I(s) = \Gamma(s)(2\pi)^{-s} L\left(s - \frac{k-1}{2}\right)$$

Completed L-function

$$\begin{aligned}
\Lambda(s,f) &= I\left(s + \frac{k-1}{2}\right) = L(s)(2\pi)^{-s-\frac{k-1}{2}}\Gamma\left(s + \frac{k-1}{2}\right) \\
&= i^k I\left(k - \left(s + \frac{k-1}{2}\right)\right) = I\left(1 - s + \frac{k-1}{2}\right) \\
\Lambda(s,f) &= i^k \Lambda(1-s,f)
\end{aligned}$$

To Summarize: (Analytic Properties of Modular Form L-functions)
Let $f \in \mathcal{S}_k(SL(2,\mathbb{Z}))$, Hecke eigenfunction

$$f = \sum a_n e(nz)$$

Then

$$\begin{aligned}
\Lambda(s,f) &:= \pi^{-s-\frac{k-1}{2}} \Gamma\left(s + \frac{k-1}{2}\right) L(s,f) \\
L(s,f) &= \sum a_n n^{-s-\frac{k-1}{2}} = \prod_p \left(1 - a_p p^{-s-\frac{k-1}{2}} + p^{-2s}\right)^{-1}
\end{aligned}$$

is entire and satisfies

$$\Lambda(s,f) = i^k \Lambda(1-s,f)$$

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L-functions of modular forms

Recall

$$\zeta(s) = \sum_{n>0} n^{-s} = \prod_p (1-p^{-s})^{-1}$$

For the coefficients $\{a_n\}$ of a cusp form for $SL(2,\mathbb{Z})$ of weight k,

$$L(s) = \sum a_n n^{-s-\frac{k-1}{2}} = \prod \left(1 - \frac{a_p}{p^{\frac{k-1}{2}}} p^{-s} + p^{-2s}\right)^{-1}$$

entire.

Functional Equations:

$$\begin{aligned}
\Gamma_{\mathbb{R}}(s) &= \pi^{-s/2}\Gamma(s/2) \\
\Gamma_{\mathbb{C}}(s) &= 2(2\pi)^{-s}\Gamma(s) \\
&= \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1/2) \\
\Gamma_{\mathbb{R}}(s)\zeta(s) &= \Gamma_{\mathbb{R}}(1-s)\zeta(1-s) \\
\Gamma_{\mathbb{C}}\left(s + \tfrac{k-1}{2}\right) L(s) = i^k \Gamma_{\mathbb{C}}\left(1 - s + \tfrac{k-1}{2}\right) L(1-s)
\end{aligned}$$

HW1 due in class March 26:

Look at the L-series for E_k, the Eisenstein series of weight k and show it factors as a product of $\zeta(s+a)\zeta(s+b)$ for some a,b which you should identify.

Hecke's Converse Theorem

Let $\{a_n : n > 0\}$ be complex numbers bounded by some polynomial in n. Let

$$f(z) = \sum_{n>0} a_n e(nz)$$

$$\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)\sum a_n n^{-s}$$

Then f is modular for $SL(2,\mathbb{Z})$ if and only if

$$\Lambda_f(s) + \frac{a_0}{s} + \frac{i^k a_0}{k-s}$$

is entire and

$$\Lambda_f(s) = i^k \Lambda_f(k-s)$$

<u>Proof</u> (somewhat repeats what we did last time) Let's look at

$$\int_0^\infty [f(iy) - a_0] y^{s-1} dy$$

This is holomorphic for $Re(s) \gg 1$. $f(iy) - a_0$ decays rapidly for y large. Last time, we showed this was $\Lambda_f(s)$. Hence,

$$\Lambda_f(s) = \int_0^1 [f(iy) - a_0] y^{s-1} dy + \int_1^\infty [f(iy) - a_0] y^{s-1} dy$$

Modularity gives

$$f(z) = z^{-k} f(-1/z)$$

so

$$f(iy) = i^k y^{-k} f(i/y)$$

Then

$$\Lambda_f(s) = \int_1^\infty [i^k y^k f(iy) - a_0] y^{-s-1} dy + \int_1^\infty [f(iy) - a_0] y^{s-1} dy$$

The second integral is entire.

$$\begin{aligned}\Lambda_f(s) &= \int_1^\infty i^k y^k [f(iy) - a_0] y^{-s-1} dy + \int_1^\infty i^k y^k a_0 y^{-s-1} - a_0 y^{-s-1} dy \\ &+ \int_1^\infty [f(iy) - a_0] y^{s-1} dy\end{aligned}$$

$$\begin{aligned}\textstyle\int_1^\infty i^k y^k a_0 y^{-s-1} - a_0 y^{-s-1} dy &= i^k a_0 \int_1^\infty y^{k-s-1} dy - a_0 \int_1^\infty y^{-s-1} dy \\ &= \frac{i^k a_0}{s-k} - \frac{a_0}{s}\end{aligned}$$

So,

$$\begin{aligned}\Lambda_f(s) + \tfrac{i^k a_0}{k-s} + \tfrac{a_0}{s} &= \int_1^\infty [f(iy) - a_0] y^{s-1} dy (entire) \\ &+ i^k \int_1^\infty [f(iy) - a_0] y^{k-s-1} dy (entire)\end{aligned}$$

and satisfies functional equation.

This shows the only if.

Mellin Inversion

$$f(x) = \frac{1}{2\pi i}\int_{Res=c\gg 1} H(s)x^{-s}ds$$

$H(s)$ holomorphic, $H(\sigma+it)$ is rapidly decaying integrand, bounded in vertical strips. Shift c large gives $f(x) \ll x^{-c}$, so $f(x) \to 0$ rapidly as $x \to \infty$. Then

$$\int_0^\infty f(x)x^{s-1}dx = H(s)$$

for $Re(s)$ large.

Let $H(s) = \Lambda_f(s)$.

$$\begin{aligned} M^{-1}H \quad &= f(iy) - a_0 \\ &= \frac{1}{2\pi i}\int_{Re(s)=c\gg 1} \Lambda_f(s)y^{-s}dy \end{aligned}$$

$\Lambda_f(s)$ has poles at $s = k$ and $s = 0$. At $s = k$, residue if $i^k a_0 y^{-k}$. At $s = 0$, residue is $-a_0$. Shift $c \ll -1$.

$$\begin{aligned} f(iy) - a_0 \quad &= \frac{1}{2\pi i}\int_{Res=c\ll -1} \Lambda_f(s)y^{-s} + i^k y^{-k}a_0 - a_0 \\ f(iy) \quad &= i^k y^{-k}a_0 + \frac{1}{2\pi i}\int_{Res=c\ll -1} \Lambda_f(k-s)y^{s-k}ds \end{aligned}$$

Apply functional equation:

$$\Lambda_f(s) = i^k\Lambda_f(k-s)$$

$$\begin{aligned} f(iy) \quad &= i^k a_0 y^{-k} + \frac{1}{2\pi i}\int_{Res=c\gg 1} i^k\Lambda_f(s)y^{s-k}ds \\ &= i^k a_0 y^{-k} + \frac{1}{2\pi i}y^{-k}i^k\int_{Res=c\gg 1} \Lambda_f(s)y^{s}ds \\ &= y^{-k}i^k a_0 + \left[i^k y^{-k}\left(f(i/y) - a_0\right)\right] \\ f(iy) \quad &= i^k y^{-k} f(i/y) \end{aligned}$$

Thus,

$$f(z) = z^{-k}f(-1/z)$$

holds for $Rez = 0$. Hence on all of $\mathbb{H}$.

Defn χ, a Dirichlet character of conductor N.

$$\begin{aligned} \chi(n+N) &= \chi(n) (\text{periodic mod N}) \\ \chi(n) &= 0 \text{ if } (n,N) > 1 \\ \chi|_{(\mathbb{Z}/n\mathbb{Z})^*} & \quad \text{is a character} \end{aligned}$$

Imprimitive:

$$\chi(n) = \begin{cases} \chi_1(n) & (n,N) = 1 \\ 0 & (n,N) > 1 \end{cases}$$

for χ_1 a Dirichlet character of conductor $N'|N$.

Primitive is opposite of imprimitive.

Theorem Suppose $f \in M_k(\Gamma_0(q), \chi)$. Let χ be a primitive Dirichlet character of conductor r. Write

$$f = \sum_{n \geq 0} a_n e(nz)$$

Then

$$f_\chi(z) = \sum a_n \chi(n) e(nz) \in M_k(\Gamma_0(N), \chi\psi^2)$$

χ is a Dirichlet character of conductor $q^*|q$

Twist: Multiply coefficients by Dirichlet character. Example:

$$L(s,\chi) = \sum \chi(n) n^{-s}$$

are twists of ζ.

HW2 Relate primitive twists of E_k (and their L-function) to products of Dirichlet L-functions.

Proof: Uses Gauss sums

$$\tau(\psi) = \sum_{u \mod r} \psi(u) e(u/r)$$

Note

$$\begin{aligned} \overline{\tau(\psi)} &= \sum \overline{\psi(u)} e(-u/r) \\ &= \sum \psi(u^{-1}) e(-u/r) \end{aligned}$$

Lemma

$$|\tau(\psi)| = \sqrt{r}$$

Proof We'll show

$$\begin{aligned}
\tau(\psi)\overline{\tau(\psi)} &= r \\
&= \sum_{u_1,u_2\in(\mathbb{Z}/r\mathbb{Z})^*} \psi(u_1u_2^{-1})e\left(\frac{u_1-u_2}{r}\right) \\
u_1 \mapsto u_1u_2 & \\
&= \sum_{u_1,u_2\in(\mathbb{Z}/r\mathbb{Z})^*} \psi(u_1)e\left(\frac{u_1u_2-u_2}{r}\right) \\
&= \sum_{u_1}\psi(u_1)\sum_{u_2} e\left(\frac{u_2}{r}(1-u_1)\right) \\
u_1 \mapsto 1-u_1 & \\
&= \sum_{u_1}\psi(1-u_1)\sum_{u_2} e\left(\frac{u_2}{r}u_1\right)
\end{aligned}$$

Continue next time

$$\begin{aligned}
\tau(\overline{\psi})\psi(n) &= \sum_{u \mod r} \overline{\psi}(u)e\left(\frac{un}{r}\right) \text{ for } (n,r)=1 \\
\tau(\overline{\psi})f_\psi(z) &= \sum_n a_n e(nz)\psi(n)\tau(\overline{\psi}) \\
&= \sum_n a_n e(nz) \sum_{u \mod r} \overline{\psi}(u)e(un/r) \\
&= \sum_n \sum_{u \mod r} a_n e\left(n(z+\frac{u}{r})\right)\overline{\psi}(u) \\
&= \sum_{u \mod r} \overline{\psi}(u)f(z+u/r) \\
&= f\left(\begin{pmatrix} 1 & u/r \\ & 1 \end{pmatrix}\right)
\end{aligned}$$

How to show modularity:

$$\begin{aligned}
\begin{pmatrix} 1 & u/r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a+uc/r & b+ud/r \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} a+uc/r & b-bcdu/r \\ c & d-cd^2u/r \end{pmatrix}\begin{pmatrix} 1 & d^2u/r \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

integral if $r^2|c$. Hence in $\Gamma_0(q)$ if $qr^2|c$, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

$$\begin{aligned}
\tau(\overline{\psi})f_\psi(z)|_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} &= \sum_{u \mod r} \overline{\psi}(u) f|_{\begin{pmatrix} 1 & u/r \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \\
&= \sum \overline{\psi}(u) f|_{\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}} \begin{pmatrix} 1 & d^2u/r \\ & 1 \end{pmatrix} \\
&= \sum_{u \mod r} \overline{\psi}(u)\chi(d) f|_{\begin{pmatrix} 1 & d^2u/r \\ & 1 \end{pmatrix}} \\
u \mapsto ud^{-2} \qquad & \\
&= \sum \overline{\psi}(u)\psi(d^2)\chi(d) f|_{\begin{pmatrix} 1 & u/r \\ & 1 \end{pmatrix}} \\
&= \chi(d)\psi(d)^2\tau(\overline{\psi})f_\psi
\end{aligned}$$

Hence dividing by $\tau(\overline{\psi})$ get modularity.

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Theorem

Let $f \in S_k(\Gamma_0(q), \xi)$, $f = \sum a(n)e(nz)$, ψ a primitive character of conductor r such that $(r, q) = 1$. Let $w(\psi) = \xi(r)\psi(q)\tau(\psi)^2 r^{-1}$ and

$$\Lambda(s, f \bigotimes \psi) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) \sum_{n>0} a(n)\psi(n)n^{-s}$$

Then $\Lambda(s, f \bigotimes \psi)$ is eniter and

$$\Lambda(s, f \bigotimes \psi) = i^k w(\psi)\Lambda(k-s, g \bigotimes \overline{\psi})$$

where $g = f|_{\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}}$.

Definition A Dirichlet character to the modulus r is a function $\xi : \mathbb{Z} \to \mathbb{C}$ with $\xi(n) = 0$ for $(n, r) > 1$, periodic mod r and satisfying $\xi(nm) = \xi(n)\xi(m)$ completely multiplicative.

Dirichlet characters of conductor r are in one-to-one correspondence with characters of $(\mathbb{Z}/r\mathbb{Z})^*$.

Imprimitive: character of $(\mathbb{Z}/r_1\mathbb{Z})^*$, $r_1|r$ extended to $(\mathbb{Z}/r\mathbb{Z})^*$ via $(\mathbb{Z}/r\mathbb{Z})^* \to (\mathbb{Z}/r_1\mathbb{Z})^*$. Primitive: not imprivitive.

Weil's Umkehrsatz "Converse Theorem" Let $k \in 2\mathbb{Z}_{>0}$, ξ character of modulus q. Let $f = \sum a(n)e(nz)$, $g = \sum b(n)e(nz)$. Assume $a(n), b(n) \ll n^A$ for some A. Suppose $\Lambda(s, f)$ and $\Lambda(s, g)$ are entire and of finite order.

Assume

$$\Lambda(s,f) = i^k \Lambda(k-s,g)$$

Also assume for all primes $r \nmid q$ and all primitive $\psi \mod r$, the functional equation

$$\Lambda(s, f \bigotimes \psi) = i^k w(\psi)\Lambda(k-s, g \bigotimes \overline{\psi})$$

holds. Then $f \in S_k(\Gamma_0(g), \xi)$, $g \in S_k(\Gamma_0(q), \overline{\xi})$, $g = f|_{\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}}$.

Group Ring

Let G be a group. $\mathbb{C}[G]$ is the the set of finite formal sums $\sum a_n g_n$, $a_n \in \mathbb{C}$, $g_n \in G$, forms a ring.

Slash in defined on $GL(2,\mathbb{Q})^+$ extends to $\mathbb{C}[GL(2,\mathbb{Q})^+]$ by $f|_{\sum a_n g_n} = \sum a_n f|_{g_n}$. Let $\alpha(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. For $(v,r)=1$, $(r,q)=1$, we have $(r,vq)=1$, so there exist a,b such that $ar + bqv = 1$. $\begin{pmatrix} r & -v \\ bq & q \end{pmatrix}$ has determinant 1. Let $b = -u$ (notation).

Define

$$\gamma\left(\frac{v}{r}\right) := \begin{pmatrix} r & -v \\ -uq & * \end{pmatrix} \in \Gamma_0(q)$$

Let

$$\beta\left(\frac{v}{r}\right) = [\xi(r) - \gamma(v/r)]\alpha(v/r) \in \mathbb{C}[GL(2,\mathbb{Q})^+]$$

Lemma 1 With our hypotheses and $g = f|_{\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}}$, $g|_{\beta(v/r)}$ is independent of $(v,r)=1$.

Proof $N = qr^2$. Recall

$$\tau(\overline{\psi}) f_\psi = \sum_{u \mod r} \overline{\psi}(u) f|_{\alpha(u/r)}$$

implies

$$\tau(\overline{\psi}) f_\psi|_{\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}} = \sum_{u \mod r} \overline{\psi}(u) f|_{\alpha(u/r)\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}}$$

Likewise

$$\tau(\psi) g_{\overline{\psi}} = \sum_{u \mod r} \psi(u) g|_{\alpha(u/r)}$$

implies

$$\xi(r)\psi(-q) \sum_{u \mod r} \psi(u) g|_{\alpha(u/r)} = \sum_{u \mod r} \overline{\psi}(u) f|_{\alpha(u/v)\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}}$$

Note $uvq \equiv 1 \mod r$ so $v \equiv (uq)^{-1} \mod r$. Calculation:

$$\alpha(u/v)\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} 1 & u/v \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} Nu/v & -1 \\ N & 0 \end{pmatrix}$$

$$r\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix}\begin{pmatrix} 1 & v/r \\ 0 & 1 \end{pmatrix} = r\begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}\begin{pmatrix} r & 0 \\ -uq & 1/r \end{pmatrix} = \begin{pmatrix} uqr & -1 \\ N & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} Nu/v & -1 \\ N & 0 \end{pmatrix} = \begin{pmatrix} N(u/v+l) & -1 \\ N & 0 \end{pmatrix}$$

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Lemma 1

:et r be prime, $r \nmid q$. Let f, $g = f|_{\begin{pmatrix} & -1 \\ q & \end{pmatrix}}$ be such that

$$\Lambda(s, f \bigotimes \psi) = i^k w(\psi)\Lambda(k-s, g \bigotimes \overline{\psi})$$

$$w(\psi) = \frac{\xi(r)\psi(q)\tau(\psi)^2}{r}$$

be entire and finite order for any primitive character ψ of conductor r. Then $g|_{\beta(v/r)}$ is independent of the choice of v so long as $(v,r) = 1$. $N = qr^2$,

$$\beta(v/r) = \left[\xi(r) - \begin{pmatrix} r & -v \\ -uq & a \end{pmatrix}\right]\begin{pmatrix} 1 & v/r \\ & 1 \end{pmatrix}$$

Proof (continued) We got to

$$\begin{aligned} &\xi(r)\psi(-q)\sum_{u \mod r}\psi(u)g|_{\begin{pmatrix} 1 & u/r \\ & 1 \end{pmatrix}} \\ &= \sum_{u \mod r}\overline{\psi}(u)f|_{\alpha(u/r)w_N},\ w_N = \begin{pmatrix} & -1 \\ N & \end{pmatrix} \end{aligned}$$

Note:

$$\alpha(u/r)\begin{pmatrix} & -1 \\ N & \end{pmatrix} = \begin{pmatrix} 1 & u/r \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ qr^2 & \end{pmatrix} = \begin{pmatrix} qur & -1 \\ qr^2 & \end{pmatrix}$$

$$\begin{pmatrix} & -1 \\ q & \end{pmatrix}\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix}\begin{pmatrix} r & v \\ & r \end{pmatrix} = \begin{pmatrix} & -1 \\ q & \end{pmatrix}\begin{pmatrix} r^2 & 0 \\ -uqr & 1 \end{pmatrix} = \begin{pmatrix} uqr & -1 \\ q^2 & 0 \end{pmatrix}$$

Hence

$$\alpha(u/r)w_N = r\begin{pmatrix} & -1 \\ q & \end{pmatrix}\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix}\alpha(v/r)$$

$$\begin{aligned}
&\sum_{u \mod r} \overline{\psi}(u) f|_{\alpha(u/r) w_N} \\
&= \sum_{u \mod r} \overline{\psi}(u) f|_{w_q}|_{\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix} \alpha(v/r)} \\
&= \sum_{u \mod r} \overline{\psi(u)} g|_{\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix} \alpha(v/r)} \\
&= \xi(r)\psi(-q) \sum_{u \mod r} \psi(u) g|_{\alpha(u/r)}
\end{aligned}$$

$uvq \equiv -1 \mod r$, $u, v \neq 0 \mod r$

$$\begin{aligned}
&\sum_{u \mod r} \overline{\psi(u)} g|_{\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix} \alpha(v/r)} \\
&= \sum_{v \mod r} \overline{\psi}(-(vq)^{-1}) g|_{\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix} \alpha(v/r)} \\
&= \psi(-q) \sum_{v \mod r} \psi(v) g|_{\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix} \alpha(v/r)} \\
RHS \quad &= \xi(r)\psi(-q) \sum_{v \mod r} \psi(v) g|_{\alpha(v/r)}
\end{aligned}$$

Conclusion

$$\sum_{v \mod r} \psi(v) g|_{\begin{pmatrix} r & -v \\ -uq & a \end{pmatrix} \alpha(v/r)} = \xi(r) \sum_{v \mod r} \psi(v) g|_{\alpha(v/r)}$$

$$\Rightarrow$$

$$\sum_{v \in \mathbb{F}_r^*} \psi(v) g|_{\beta(v/r)} = 0$$

for all primitive $\psi \mod r$. Since r is prime, this means all nontrivial characters. $\langle g|_{\beta(v/r)}, \overline{\psi} \rangle = 0$

Let's think of $f(v) = \psi(v) g|_{\beta(v/r)}$, $v \in \mathbb{F}_r^* \cong \mathbb{Z}/(r-1)\mathbb{Z}$ as a function on this group. There are $r-1$ characters of $\mathbb{F}_r^*$, $\xi_1, \ldots \xi_{r-1}$, ξ_1 trivial.

$$\begin{aligned}
&f(v) = \sum c_j \xi_j(v) \\
&\langle f, \psi \rangle = c_k, \text{ where } \psi = \xi_k \\
&\Rightarrow c_j = 0 \text{ for } j > 1 \\
&f(v) = c_1 \xi_1 = c_1 = \text{ independent of } v
\end{aligned}$$

Lemma 2

Let m, n be primes not dividing q with the hypothesis of the theorem,

$$g|_{\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}} = \overline{\xi(\gamma)}g$$

for all $u, v \in \mathbb{Z}$ such that the determinant is 1.

Proof

$$\begin{aligned} & g|_{\left[_{\xi(m)-}\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}\right]_{\alpha(v/m)}} \\ & = g|_{\left[_{\xi(m)-}\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\right]_{\alpha(-v/m)}} \\ & g|_{\left[_{\xi(m)-}\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}\right]} \\ & = g|_{\left[_{\xi(m)-}\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\right]_{\alpha(-2v/m)}} \end{aligned}$$

n, m play the same role, $mn \equiv 1 \mod r$, $\xi(m) = \overline{\xi(n)}$

$$g|_{\left[\xi(n)-\begin{pmatrix} n & -v \\ -uq & m \end{pmatrix}\right]}$$

$$= g|_{\left[\xi(n)-\begin{pmatrix} n & v \\ uq & m \end{pmatrix}\right]\alpha(-2v/n)}$$

Use

$$\xi(n)-\begin{pmatrix} n & -v \\ -uq & m \end{pmatrix} = -\xi(n)\left[\xi(m)-\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\right]\begin{pmatrix} n & -v \\ -uq & m \end{pmatrix}$$

$$\xi(n)-\begin{pmatrix} n & v \\ uq & m \end{pmatrix} = -\xi(n)\left[\xi(m)-\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}\right]\begin{pmatrix} n & v \\ uq & m \end{pmatrix}$$

Consider

$$g|_{\left[\xi(m)-\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\right]\begin{pmatrix} n & -v \\ -uq & m \end{pmatrix}}$$

$$= -\xi(n)^{-1}g|_{\left[\xi(n)-\begin{pmatrix} n & -v \\ -uq & m \end{pmatrix}\right]}$$

$$= -\xi(n)^{-1}g|_{\left[\xi(n)-\begin{pmatrix} n & v \\ uq & m \end{pmatrix}\right]\alpha(-2v/n)}$$

$$= g|_{\left[\xi(m)-\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}\right]\begin{pmatrix} n & v \\ uq & m \end{pmatrix}\alpha(-2v/n)}$$

$$= g|_{\left[\xi(n)-\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\right]\begin{pmatrix} n & -v \\ -uq & m \end{pmatrix}}$$

Thus,

$$g|_{\left[\xi(m)-\begin{pmatrix} m & -v \\ uq & n \end{pmatrix}\right]}$$

$$= g|_{\left[\xi(m)-\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\right]\alpha(-2v/m)}$$

$$= g|_{\left[\xi(m)-\begin{pmatrix} m & -v \\ uq & n \end{pmatrix}\right]\begin{pmatrix} n & v \\ uq & m \end{pmatrix}\alpha(-2v/n)\begin{pmatrix} m & v \\ uq & n \end{pmatrix}\alpha(-2v/m)}$$

Let

$$\begin{aligned}
\beta &= \begin{pmatrix} n & v \\ uq & m \end{pmatrix} \alpha(-2v/n) \begin{pmatrix} m & v \\ uq & n \end{pmatrix} \alpha(-2v/m) \\
&= \begin{pmatrix} n & -v \\ uq & -2uvq/n + m \end{pmatrix} \begin{pmatrix} m & -v \\ uq & -2uvq/m + n \end{pmatrix} \\
&= \begin{pmatrix} nm - uvq & -2vn + 2v^2uq/m \\ uqm - 2v/n(uq)^2 + mqu & * \end{pmatrix} \\
&= \begin{pmatrix} 1 & -2vn + 2v^2uq/m \\ uqm - 2(v/n)u^2q^2 + mqu & * \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& uqm - \frac{2v}{n}u^2q^2 + mq \\
&= \frac{nmuq - vu^2q^2 + mnq - vu^2q^2}{n} \\
&= \frac{2}{n}\left[uq(nm - vuq)\right] = \frac{2uq}{n}
\end{aligned}$$

$$\begin{aligned}
& -2vn + \frac{2v^2uq}{m} \\
&= \frac{2v}{m}(-nm + uvq) = -\frac{2v}{m}
\end{aligned}$$

$$\beta = \begin{pmatrix} 1 & -2v/m \\ 2uq/n & 4/mn - 3 \end{pmatrix}$$

Thus $h = h|_\beta$ and holomorphic.

Lemma β fixes a point in $\mathbb{H} = \{Imz > 0\}$ and rotates by angle $2\pi\alpha$, $\alpha \notin \mathbb{Q}$. Hence if I take a power series around that point

$$\begin{aligned}
\sim h(z) &= \sum c_n z^n \\
= h(ze^{2\pi i\alpha}) \qquad &= \sum c_n e^{2\pi i\alpha n} z^n \\
\Rightarrow \qquad & c_n = e^{2\pi i\alpha n} c_n \\
\Rightarrow \qquad & c_n = 0 \text{ for } n > 0 \\
\Rightarrow \qquad & h = \text{ constant}
\end{aligned}$$

Lemma 4 $c_0 = 0$, i.e. h at fixed point of β is zero.

How to get Converse Theorem from this:

Invariant under $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}$, m, n prime.

Hence

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & -t \\ & 1 \end{pmatrix}\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix}\begin{pmatrix} 1 & -s \\ & 1 \end{pmatrix} \\
\begin{pmatrix} m & -v \\ -uq & n \end{pmatrix} &= \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a+tc & * \\ c & cs+d \end{pmatrix}
\end{aligned}$$

Proof of Lemma 3

$$\beta = \begin{pmatrix} 1 & -2v/m \\ 2uq/n & 4/nm-3 \end{pmatrix}$$

$$\begin{aligned}
\beta z \quad &= z \Rightarrow \\
&z - \frac{2v}{m} = z\left(\frac{2uq}{n}z + \frac{4}{mn} - 3\right) \\
&4mnz - 2vn = m2uqz^2 + 4z \\
&2muqz^2 + (4-4mn)z + 2vn = 0 \\
&\text{Hence} \\
&z = \frac{4mn - 4 \pm \sqrt{16(mn-1)^2 - 16muqvn}}{4muq} \\
&= \frac{mn - 1 \pm \sqrt{m^2n^2 - 2mn + 1 - muqvn}}{muq} \\
&= \frac{mn - 1 \pm \sqrt{1 - 2mn + mn(mn - uvq)}}{muq} \\
&= \frac{mn - 1 \pm \sqrt{1 - mn}}{muq} \\
&= \frac{mn - 1 + i\sqrt{mn-1}}{muq} \in \mathbb{H}
\end{aligned}$$

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fix $x+iy$

$\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ fixes iy

$\begin{pmatrix} \sqrt{y}^{-1} & 0 \\ 0 & \sqrt{y} \end{pmatrix}\begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$ fixes i

Matrices with a fixed point in $\mathbb{H}$ are precisely those conjugate to an element of $SO(2,\mathbb{R})$. Write

$$\begin{pmatrix} \sqrt{y}^{-1} & 0 \\ 0 & \sqrt{y} \end{pmatrix} \begin{pmatrix} 1 & -x \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Trace is $2\cos\theta = a+d = 4/mn-2$. β has finite order iff $\cos^{-1}(2/mn-1)/2\pi \in \mathbb{Q}$. $\cos\theta$ is rational for $\theta = 0, \pi/2, \pi, 3\pi/2, \ldots, \pi/3, 2\pi/3, \ldots$.

15 23 March 2010

Nonholomorphic Eisenstein Series

Idea: holomorphic ones are

$$\zeta(k) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-k} \sum_{(m,n) \in \mathbb{Z}^2_{\neq (0,0)}} (mz+n)^{-k}$$

Defn

$$E_s(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Im(\gamma z)^s$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$Im(\gamma z) = \frac{Im(z)}{|cz+d|^2}$$

$$E_s(x+iy) = \frac{1}{2} y^s \sum_{(c,d)=1} |cz+d|^{-2s}$$

$$E_s(z) = E_s\left(\frac{az+b}{cz+d}\right)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. This converges absolutely for $Re(s) > 1$. Holomorphic there in s.

Completed

$$E_s^*(z) = \zeta(2s) E_s(z) = \pi^{-s} y^s \Gamma(s) \sum_{(c,d) \neq (0,0)} |cz+d|^{-2s}$$

Fourier expansion, using Poisson summation.

$$E_s(x+iy) = \sum_{n \in \mathbb{Z}} c_n(y) e(nx)$$

$$\begin{aligned} c_n(y) &= \int_0^1 E(x+iy)e(-nx)dx \\ &= \int_0^1 y^s 1 e(-nx)dx + \frac{1}{\zeta(2s)}\int_0^1 y^s \sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}} |c(x+iy)+d|^{-2s} e(-nx)dx \end{aligned}$$

The first part contributes

$$\begin{cases} y^s & n=0 \\ 0 & n\neq 0 \end{cases}$$

The second part is the beef:

$$= \frac{1}{\zeta(2s)} y^s \int_0^1 \sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z}} [(cz+d)^2 + c^2y^2]^{-s} e(-nx)dx$$

$x \mapsto x - d/c$

$$= \frac{1}{\zeta(2s)} y^s \sum_{\substack{c=1 \\ d\in\mathbb{Z}}}^{\infty} \int_{d/c}^{1+d/c} [(cx)^2 + (cy)^2]^{-s} e(-nx)e(nd/c)dx$$

$$= \frac{1}{\zeta(2s)} y^s \sum_{\substack{c\geq 1 \\ d\in\mathbb{Z}}} c^{-2s} e(nd/c) \int_{d/c}^{1+d/c} (x^2+y^2)^{-s} e(-nx)dx$$

$d \mapsto d + kc$, $d \in \mathbb{Z}/c\mathbb{Z}$, $k \in \mathbb{Z}$

$$= \frac{y^s}{\zeta(2s)} \sum_{c=1}^{\infty} \sum_{d\in\mathbb{Z}/c\mathbb{Z}} e(nd/c)c^{-2s} \int_{\mathbb{R}} (x^2+y^2)^{-s} e(-nx)dx$$

$x \mapsto xy$, $dx \mapsto ydx$

$$= \frac{y^{1-s}}{\zeta(2s)} \sum_{c=1}^{\infty} \sum_{d\in\mathbb{Z}/c\mathbb{Z}} \frac{e(nd/c)}{c^{2s}} \int_{\mathbb{R}} (x^2+1)^{-s} e(-nxy)dx$$

$$= \frac{y^{1-s}}{\zeta(2s)} \sum_{c=1}^{\infty} \sum_{d\in\mathbb{Z}/c\mathbb{Z}} \frac{e(nd/c)}{c^{2s}} \widehat{\phi}(ny)$$

$$= \frac{y^{1-s}}{\zeta(2s)} \sum_{c=1}^{\infty} c^{1-2s} \widehat{\phi}(ny) \begin{cases} 1 & c|n \\ 0 & c\nmid n \end{cases}$$

Thus

$$\begin{aligned} E_s(x+iy) &= y^x + \frac{y^{1-s}}{\zeta(2s)}\zeta(2s-1)\widehat{\phi}(0) \\ &+ \sum_{n\neq 0} \frac{y^{1-s}}{\zeta(2s)} \sigma_{1-2s}(n)\widehat{\phi}(ny)e(nx) \end{aligned}$$

$$\sigma_w(n) = \sum_{d|n} d^w$$

It remains to compute the special functions $\widehat{\phi}(0)$ and $\widehat{\phi}(t)$, $t \neq 0$.
First look at

$$\begin{aligned}
\widehat{\phi}(0) &= \int_{\mathbb{R}} (x^2+1)^{-s} dx \\
& x \mapsto x/y \\
&= \int_{\mathbb{R}} \left(\frac{x^2}{y^2}+1\right)^{-s} \frac{dx}{y} \\
&= y^{2s-1} \int_{\mathbb{R}} (x^2+y^2)^{-s} dx
\end{aligned}$$

$$\int (x^2+y^2)^{-s} dx = y^{1-2s} \widehat{\phi}(0)$$

$$\begin{aligned}
\Gamma_{\mathbb{R}}(s) &= \pi^{-s/2}\Gamma(s/2) \\
\widehat{\phi}(0) &= \frac{\Gamma_{\mathbb{R}}(2s-1)}{\Gamma_{\mathbb{R}}(2s)} = \frac{\pi^{1/2-s}\Gamma(s-1/2)}{\pi^{-s}\Gamma(s)} \\
&= \pi^{1/2}\frac{\Gamma(s-1/2)}{\Gamma(s)}
\end{aligned}$$

We need to check this. We need to check whether

$$\begin{aligned}
\textstyle\int (x^2+y^2)^{-s} dx \quad &? = y^{1-2s}\frac{\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} \\
&= y^{1-2s}\frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)}
\end{aligned}$$

This is the β-integral.

$$\widehat{\phi}(t) \quad = \int_{\mathbb{R}} (x^2+1)^{-s} e(-xt) dx = \frac{K_{s-1/2}(t)t^{power}}{\Gamma(s)\pi^{-s}}$$

Derive series expansion in t

$$\begin{aligned}
E_s(x+iy) &= y^s + y^{1-s}\frac{\psi(2s-1)}{\psi(2s)} \\
&+ \frac{1}{\psi(2s)} \sum_{n\neq 0} e(nx) K_{s-1/2}(|n|y)(|n|y)^{s-1/2} y^{1-s} \sigma_{1-2s}(|n|) \\
&= y^s + y^{1-s}\frac{\psi(2s-1)}{\psi(2s)} \\
&+ \frac{1}{\psi(2s)} \sum_{n\neq 0} e(nx) K_{s-1/2}(|n|y)\sqrt{y}|n|^{s-1/2} \sigma_{1-2s}(|n|)
\end{aligned}$$

$$\begin{aligned} E_s^*(z) &= \psi(2s)E_s(z) \\ &= \psi(2s)y^s + \psi(2s-1)y^{1-s} \\ &+ \sum_{n\neq 0} e(nx)\sqrt{y}K_{s-1/2}(|n|y)|n|^{s-1/2}\sigma_{1-2s}(|n|) \\ |n|^w\sigma_{-2w}(n) &= \sum d^{-2w}n^w \\ &= \sum \left(\frac{n}{d}\right)^{-2w} n^w \\ &= n^{-w}\sigma_{2w}(n) \end{aligned}$$

Theorem

$$E_s^*(z) = E_{1-s}^*(z)$$

has simple poles at $s = 0, 1$ homomorphic everywhere else.

$$Res_{s=1}E_s^*(z) = \text{constant}$$

Rankin-Selberg Integral

Let $f, g \in M_k(SL(2,\mathbb{Z}))$. Assume one is cuspidal. $y^k f(z)\overline{g(z)}$ invariant under $SL(2,\mathbb{Z})$, decay rapidly on cusp.

$$I(s) = \int_{\Gamma\backslash\mathbb{H}} y^k f(x+iy)\overline{g(x+iy)}E_s^*(x+iy)\frac{dxdy}{y^2}$$

analytically continue to $\mathbb{C}\backslash\{0,1\}$.

Compute

$$I(s) = \pi^{-s}\Gamma(s)\zeta(2s)\int_{\Gamma\backslash\mathbb{H}} y^{k-2}dxdyf(x+iy)\overline{g(x+iy)}E_s(x+iy)$$

The integrand not including E_s is γ-invariant.

Unfolding

$$\begin{aligned} &= \pi^{-s}\Gamma(s)\zeta(2s)\int_{\Gamma_\infty\backslash\mathbb{H}} y^{k-2}dxdyf(x+iy)\overline{g(x+iy)}y^s \\ &= \psi(2s)\int_{y=0}^{\infty} dy \int_{x=-1/2}^{1/2} dxy^{s+k-2}\sum c_n e(nx)e(iny)d_m e(-mx)e(imy) \\ &= \psi(2s)\sum_n \int_{y=0}^{\infty} y^{s+k-2}dyc_n\overline{d_n}e^{-4\pi ny} \\ &= \psi(2s)(4\pi)^{s+k-1}\Gamma(s+k-1)\sum \frac{c_n\overline{d_n}}{n^{s+k-1}} \end{aligned}$$

where c_n comes from f and d_m come from g.

Euler Product: Assume f, g are Hecken eigenfunctions. $c_n = a_n n^{(k-1)/2}$, $d_n = b_n n^{(k-1)/2}$

$$L(s, f \times g) = \sum a_n b_n n^{-s}$$

$$\begin{aligned} a_{p^k} &= \frac{(\alpha_p^{k+1} - \alpha_p^{-k-1})}{\alpha_p - \alpha_p^{-1}} \\ b_{p^k} &= \frac{(\beta_p^{k+1} - \beta_p^{-k-1})}{\beta_p - \beta_p^{-1}} \end{aligned}$$

Claim:

$$\zeta(2s)L(s, f \times g) = \prod_P (1 - \alpha_p\beta_p p^{-s})^{-1}(1 - \alpha_p^{-1}\beta_p p^{-s})^{-1}(1 - \alpha_p\beta_p^{-1}p^{-s})^{-1}(1 - \alpha_p^{-1}\beta_p^{-1}p^{-s})^{-1}$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

then

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s + (k-1)/2)\prod_p (--) =$$

itself with $s \to 1 - s$.

Claim reduces to Lemma:

$$\begin{aligned} &(1 - \alpha\beta X)^{-1}(1 - \alpha^{-1}\beta X)^{-1}(1 - \alpha\beta^{-1}X)^{-1}(1 - \alpha^{-1}\beta^{-1}X)^{-1}(1 - X^2) \\ &= \sum \frac{(\alpha^{k+1} - \alpha^{-k-1})(\beta^{k+1} - \beta^{-k-1})}{(\alpha - \alpha^{-1})(\beta - \beta^{-1})} X^k \\ &(1 - X^2)(\alpha - \alpha^{-1})(\beta - \beta^{-1}) \\ &= \sum_k X^k(\alpha^{k+1} - \alpha^{-k-1})(\beta^{k+1} - \beta^{-k-1})(1 - \alpha\beta X)(1 - \alpha^{-1}\beta X)(1 - \alpha\beta^{-1}X)(1 - \alpha^{-1}\beta^{-1}X) \end{aligned}$$

X^0 term on right hand side. X^1: $(\alpha^2 - \alpha^{-2})(\beta^2 - \beta^{-2})$

16 26 March 2010

Last time: Calculation

$$\sum_{n \geq 1} a_n b_n n^{-s}$$

where a_n, b_n are normalized coefficients of cusp forms of weight k for $SL(2, \mathbb{Z})$ We proved the analytic continuation and functional equation of this series last time. Assume furthermore that the cusp forms are Hecke eigenforms. In particular

$$a_{p^k} = \frac{\alpha_p^{k+1} - \alpha_p^{-k-1}}{\alpha_p - \alpha_p^{-1}}$$

$$b_{p^k} = \frac{\beta_p^{k+1} - \beta_p^{-k-1}}{\beta_p - \beta_p^{-1}}$$

Claim:

$$\begin{aligned}
&\zeta(2s) \sum_{n \geq 1} a_n b_n n^{-s} \\
&= \prod_p (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \alpha_p \beta_p^{-1} p^{-s})^{-1} (1 - \alpha_p^{-1} \beta_p p^{-s})^{-1} (1 - \alpha_p^{-1} \beta_p^{-1} p^{-s})^{-1}
\end{aligned}$$

This reduces to

$$\begin{aligned}
&(1 - p^{-2s})^{-1} \sum_{k \geq 0} \frac{(\alpha_p^{k+1} - \alpha_p^{-k-1})(\beta_p^{k+1} - \beta_p^{-k-1})}{(\alpha_p - \alpha_p^{-1})(\beta_p - \beta_p^{-1})} X^k \\
&= (1 - \alpha_p \beta_p X)^{-1} (1 - \alpha_p \beta_p^{-1} X)^{-1} (1 - \alpha_p^{-1} \beta_p X)^{-1} (1 - \alpha_p^{-1} \beta_p^{-1} X)^{-1}
\end{aligned}$$

iff

$$\begin{aligned}
&(1 - \alpha\beta X)(1 - \alpha\beta^{-1} X)(1 - \alpha^{-1}\beta X)(1 - \alpha^{-1}\beta^{-1} X) \\
&\times \sum \left(\alpha^{k+1}\beta^{k+1} + \alpha^{-k-1}\beta^{-k-1} - \alpha^{k+1}\beta^{-k-1} - \alpha^{-k-1}\beta^{k+1} \right) X^k \\
&= \left(\alpha\beta + \alpha^{-1}\beta^{-1} - \alpha^{-1}\beta - \alpha\beta^{-1} \right) (1 - X^2)
\end{aligned}$$

Expand LHS:

$$\begin{aligned}
&(1 - \alpha\beta X - \alpha^{-1}\beta^{-1} X + X^2)(1 - \alpha\beta^{-1} X - \alpha^{-1}\beta X + X^2) \\
&\times \left[\frac{\alpha\beta}{1 - \alpha\beta X} + \frac{\alpha^{-1}\beta^{-1}}{1 - \alpha^{-1}\beta^{-1} X} - \frac{\alpha\beta^{-1}}{1 - \alpha\beta^{-1} X} - \frac{\alpha^{-1}\beta}{1 - \alpha^{-1}\beta X} \right] \\
&= \alpha\beta (1 - \alpha^{-1}\beta X)(1 - \alpha\beta^{-1} X)(1 - \alpha^{-1}\beta^{-1} X) \\
&+ \alpha^{-1}\beta^{-1} (1 - \alpha\beta X)(1 - \alpha^{-1}\beta X)(1 - \alpha\beta^{-1} X) \\
&- \alpha\beta^{-1} (1 - \alpha\beta X)(1 - \alpha^{-1}\beta X)(1 - \alpha^{-1}\beta^{-1} X) \\
&- \alpha^{-1}\beta (1 - \alpha\beta X)(1 - \alpha\beta^{-1} X)(1 - \alpha^{-1}\beta^{-1} X)
\end{aligned}$$

$$\begin{aligned}
&= \alpha\beta \left(1 - [\alpha^{-1}\beta + \alpha\beta^{-1} + \alpha^{-1}\beta^{-1}] X + [1 + \alpha^{-2} + \beta^{-2}] X^2 + \alpha^{-1}\beta^{-1} X^3 \right) \\
&+ \alpha^{-1}\beta^{-1} \left(1 - [\alpha\beta^{-1} + \alpha^{-1}\beta + \alpha\beta] X + [1 + \alpha^2 + \beta^2] X^2 - \alpha\beta X^3 \right) \\
&- \alpha\beta^{-1} \left(1 - [\alpha\beta + \alpha^{-1}\beta + \alpha^{-1}\beta^{-1}] X + [\beta^2 + 1 + \alpha^{-2}] X^2 - \alpha^{-1}\beta X^3 \right) \\
&- \alpha^{-1}\beta \left(1 - [\alpha^{-1}\beta^{-1} + \alpha\beta^{-1} + \alpha\beta] X + [\beta^{-2} + 1 + \alpha^2] X^2 - \alpha\beta^{-1} X^3 \right)
\end{aligned}$$

Coefficient of X^3 is 0.
Coefficient of X^2 is

$$\begin{aligned}
&= \alpha\beta[1 + \alpha^{-2} + \beta^{-2}] + \alpha^{-1}\beta^{-1}[1 + \alpha^2 + \beta^2] \\
&- \alpha\beta^{-1}[\beta^2 + 1 + \alpha^{-2}] - \alpha^{-1}\beta[\beta^{-2} + 1 + \alpha^2] \\
&= \alpha\beta + \alpha^{-1} + \alpha\beta^{-2} + \alpha^{-1}\beta^{-1} + \alpha\beta^{-1} + \alpha^{-1}\beta \\
&- \alpha\beta - \alpha\beta^{-1} - \alpha^{-1}\beta^{-1} - \alpha^{-1}\beta^{-1} - \alpha^{-1}\beta - \alpha\beta \\
&= +\alpha^{-1}\beta + \alpha\beta^{-1} - \alpha\beta - \alpha^{-1}\beta^{-1}
\end{aligned}$$

Coefficient of X is 0.
Coefficient of 1:

$$\alpha\beta + \alpha^{-1}\beta^{-1} - \alpha^{-1}\beta - \alpha\beta^{-1}$$

Symmetric Power L-functions:

$$\prod_p (1-\alpha_p^k p^{-s})^{-1}(1-\alpha_p^{k-1}\alpha_p^{-1}p^{-s})^{-1}\cdots(1-\alpha_p^{-k}p^{-s})^{-1}$$

$$= \prod_p \prod_{l=0}^{k} (1-\alpha_p^{k-l}\alpha_p^{-l}p^{-s})^{-1}$$

$$\begin{aligned}
&\zeta(2s)\sum a_{n^2}n^{-s} \\
&= \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p^{-2}p^{-s})^{1}(1-p^{-s})^{-1} \\
&= \frac{\text{above product}}{\zeta(s)} \text{ if } \alpha_p = \beta_p
\end{aligned}$$

$$\begin{aligned}
&(1-X^2)^{-1}\sum_{k\geq 0}\frac{(\alpha^{2k+1}-\alpha^{-2k-1})}{\alpha-\alpha^{-1}}X^k \\
&= \frac{1}{1-X^2}\frac{1}{\alpha-\alpha^{-1}}\left(\alpha\sum X^k\alpha^{2k} - \alpha^{-1}\sum X^k\alpha^{-2k}\right) \\
&= \frac{1}{1-X^2}\frac{1}{\alpha-\alpha^{-1}}\left[\frac{\alpha}{1-\alpha^2X} - \frac{\alpha^{-1}}{1-\alpha^{-2}X}\right] \\
&= \frac{1}{1-X^2}\frac{1}{\alpha-\alpha^{-1}}\left[\frac{\alpha-\alpha^{-1}X-\alpha^{-1}+\alpha X}{(1-\alpha^2X)(1-\alpha^{-2}X)}\right] \\
&= \frac{1}{(1-\alpha^2X)(1-X)(1-\alpha^{-2}X)}
\end{aligned}$$

Example:
Ramanujan Formula

$$\sum_{n\geq 1}\frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-1)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

$a_n = \sigma_a(n)$, $b_n = \sigma_b(n)$.
Barnes' Integral

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\substack{Re(s)=\sigma \\ \sigma>Re(-\alpha),Re(-\beta) \\ \sigma<Re(\gamma),Re(\delta)}} \Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s) \\
&= \frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\delta+\gamma)}
\end{aligned}$$

Another Proof

Let

$$\begin{aligned}\phi(z) &= \sum_{k\geq 0} a_{p^k} z^k = \frac{1}{(1-\alpha z)(1-\alpha^{-1}z)}\\ \psi(z) &= \sum_{k\geq 0} b_{p^k} z^k = \frac{1}{(1-\beta z)(1-\beta^{-1}z)}\end{aligned}$$

Calculate

$$\frac{1}{2\pi i}\int_{\text{circle of radius } r} \phi(wz)\psi(z^{-1})\frac{dz}{z}$$

w small and $r \geq |\beta| \neq 1$, $r \leq |\alpha|/|w|$

$$\begin{aligned}&\text{Integral}\\ &= \frac{1}{2\pi i}\int \sum_{k,l} a_{p^k} b_{p^l} w^k z^{k-l-1} dz\\ &= \sum_{k=0}^{\infty} a_{p^k} b_{p^k} w^k\end{aligned}$$

The Residue Theorem gives that this is equal to

$$\begin{aligned}&Res_{z=\beta}\phi(wz)\psi(z^{-1}) + Res_{z=0}\phi(wz)\psi(z^{-1})\frac{1}{z}\\ &+Res_{z=\beta^{-1}}\phi(wz)\psi(z^{-1})\\ &= \frac{1}{(1-\alpha w\beta)(1-\alpha^{-1}w\beta)(1-\beta^{-2})}\\ &+0+\frac{1}{(1-\alpha w\beta^{-1})(1-\alpha^{-1}w\beta^{-1})(1-\beta^{2})}\\ &= (1-\alpha w\beta)^{-1}(1-\alpha^{-1}w\beta)^{-1}(1-\beta^{-2})^{-1}(1-\alpha w\beta^{-1})^{-1}(1-\alpha^{-1}w\beta^{-1})^{-1}(1-\beta^{2})^{-1}\\ &\times\left[(1-\alpha w\beta)(1-\alpha^{-1}w\beta)(1-\beta^{-2}) + (1-\alpha w\beta^{-1})(1-\alpha^{-1}w\beta^{-1})(1-\beta^{2})\right]\\ &= (1-\alpha w\beta)^{-1}(1-\alpha^{-1}w\beta)^{-1}(1-\beta^{-2})^{-1}(1-\alpha w\beta^{-1})^{-1}(1-\alpha^{-1}w\beta^{-1})^{-1}(1-\beta^{2})^{-1}\\ &\times\left[\begin{array}{l} 1-\alpha w\beta-\alpha^{-1}w\beta-\beta^{-2}+w^2\beta^2+\alpha w\beta^{-1}+\alpha^{-1}w\beta^{-1}-w^2\\ +1-\alpha w\beta^{-1}-\alpha^{-1}w\beta^{-1}-\beta^{2}+w^2\beta^{-2}+\alpha w\beta^{2}+\alpha^{-1}\beta w^2\end{array}\right]\\ &= (1-\alpha w\beta)^{-1}(1-\alpha^{-1}w\beta)^{-1}(1-\beta^{-2})^{-1}(1-\alpha w\beta^{-1})^{-1}(1-\alpha^{-1}w\beta^{-1})^{-1}(1-\beta^{2})^{-1}\\ &\times[(w^2-1)(\beta^2+\beta^{-2}-2)]\\ &= (1-\alpha w\beta)^{-1}(1-\alpha^{-1}w\beta)^{-1}(1-\beta^{-2})^{-1}(1-\alpha w\beta^{-1})^{-1}(1-\alpha^{-1}w\beta^{-1})^{-1}(1-\beta^{2})^{-1}\\ &\times(1-w^2)(1-\beta^2-\beta^{-2}+\beta^2\beta^{-2})\\ &= (1-w^2)(1-\alpha w\beta)^{-1}(1-\alpha^{-1}w\beta)^{-1}(1-\alpha w\beta^{-1})^{-1}(1-\alpha^{-1}w\beta^{-1})^{-1}\end{aligned}$$

Now take $\psi(z) = \sum_{k\geq 0} z^{2k} = 1/(1-z^2)$.

Integral is

$$\frac{1}{2\pi i}\int\sum_{k,l} a_{p^k} w^k z^{k-2l-1} dz$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} a_{p^{2k}} w^k \\
&= Res_{z=1} + Res_{z=-1} \\
&= \frac{1}{2(1-\alpha w)(1-\alpha^{-1}w)} + \frac{1}{2(1+\alpha w)(1+\alpha^{-1}w)} \\
&= \frac{1}{2(1-\alpha^2 w^2)(1-\alpha^{-2}w^2)}\left[1+\alpha w+\alpha^2 w+w^1+1-\alpha w-\alpha^{-1}w+w^2\right] \\
&= \frac{4w^2}{(1-\alpha^2 w^2)(1-\alpha^{-2}w^2)}
\end{aligned}$$

Possible mistake above.

$$\begin{aligned}
\zeta(s+w)\zeta(s) &= \sum n^{-s-w} m^{-s} \\
&= \sum n^{-w}(mn)^{-s} \\
&= \sum \sigma_{-w}(n) n^{-s} \\
\zeta(s)\zeta(s-a) &= \sum \sigma_a(n) n^{-s} \\
\zeta(s+a/2)\zeta(s-a/2) &= \sum \frac{\sigma_a(n)}{n^{a/2}} n^{-s}
\end{aligned}$$

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Goal Chapters 9-12

Recall If A is a real symmetric matrix, it can be orthogonally diagonalized. $A = U^T D U$, where U is orthogonal and D is diagonal.

If A is positive definite, then the diagonal entries of D are positive.

Then $\sqrt{D}$ exists and $B = U^T \sqrt{D} U$ has

$$BB^T = B^2 = U^T \sqrt{D}^2 U = A.$$

Then B is positive definite and symmetric. Call it $\sqrt{A}$.

A is positive definite iff A^{-1} is positive definite, symmetric.

A-Laplacian Δ_A on $\mathbb{R}^n$:

$$\Delta_A = \sum_{1\le i,j\le n} a^*_{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j},$$

where $(a^*_{ij}) = A^{-1}$.

$$\Delta_A = [\partial_1 \partial_2 \cdots \partial_n] A^{-1} \begin{bmatrix} \partial_1 \\ \vdots \\ \partial_n \end{bmatrix}$$

where $\partial_i = \frac{\partial}{\partial x_i} = \partial_{x_i}$

A smooth function f on $\mathbb{R}^n$ is called A-Harmonic if $\Delta_A f = 0$.

Example: $A = I$, $\Delta_A = \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$

For $n = 2$, A-harmonic is $f_{11} + f_{22} = 0$, usual notion. Holomorphic on $\mathbb{C} \cong \mathbb{R}^2$.

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\Delta_A = f_{11} - f_{22}$, A-harmonic: $f_{11} = f_{22}$, e.g. $f(x_1, x_2) = (x_1 + x_2)^2$, $\partial_1 f = 2(x_1 + x_2) = \partial_2 f$, $\partial_1^2 f = 2 = \partial_2^2 f$.

Definition Harmonic Polynomials

a.k.a. Spherical functions

are homogeneous polynomials that are also A-harmonic, i.e. annihilated by Δ_A.

Theorem Let f be a homogeneous polynomial and $A =$ symmetric matrix. The following are equivalent.

1. f is spherical, i.e. $\Delta_A f = 0$
2. $\int_{\{x \in \mathbb{R}^n | x^t A x \leq 1\}} f(x) g(x) dx = 0$ for any homogeneous polynomial g of lower degree.
3. If $\nu = deg(f)$ then $\nu \leq 1$ implies f is a linear function or constant. If $\nu \geq 2$, then f is a linear combination of $(u^T A x)^\nu$ where $u^T A u = 0$.

If positive definite, then $\nu \leq 1$.

Explanation of case where A is positive definite.

First step: Reduce to $A = I$. $A = B^T B$. Let $y = Bx$, $v = Bu$. $u^T A x = u^T B^T B x = v^T y$. $u^T A u = v^T v$.

Lemma 1 This change of variables converts Δ_A to Δ. I.e., if $f \in C^\infty(\mathbb{R}^n)$, $(\Delta_A f)(Bx) = \Delta g$, where $g(x) = f(Bx) = f(\sum b_{1j} x_j, \sum b_{2j} x_j, \ldots)$

Proof

$$\begin{aligned}
\tfrac{\partial}{\partial x_l} g(x) \quad &= \sum_{i=1}^{n} f_i(Bx) b_{il} \\
\tfrac{\partial}{\partial x_l} \tfrac{\partial}{\partial x_l} g(x) \quad &= \sum_{i=1}^{n} b_{il} \frac{\partial}{\partial x_l} f(Bx) \\
\partial_l^2 g \quad &= \sum_{i=1}^{n} b_{il} \sum_{j=1}^{n} f_{ij}(Bx) b_{jl} \\
\Delta g = \textstyle\sum_{l=1}^{n} \partial_l^2 g \quad &= \sum_{i,j,l \le n} b_{il} b_{lj} f_{ij}(Bx) \\
\text{Use} \quad & a_{ij} = \sum_{l=1}^{n} b_{il} b_{lj} \\
\Delta g \quad &= \left(\sum_{i,j} a_{ij} \partial_{ij} f \right) (Bx)
\end{aligned}$$

■

For now, we can take $A = I$ and prove there.

$$\partial_i^2 x_1^{d_1} \cdots x_n^{d_n} = \frac{d_i(d_i - 1)}{x_i^2} x_1^{d_1} \cdots x_n^{d_n}$$

If $f = x_1^{d_1} \cdots x_n^{d_n}$, then

$$\Delta f = \sum \frac{d_i(d_i - 1)}{x_i^2} f$$

If f is homogeneous of degree $\nu \ge 2$, then Δf is homogeneous of degree $\nu - 2$.

Therefore, $\Delta f = 0$ only if $deg(f) \le 1$.

Conclusion:

If A is positive definite, symmetric and f is an A-harmonic polynomial, either f is constant or $f(x) = cx$ for some $c \in \mathbb{R}^n$.

These are used to construct general Θ-functions.

Suppose A is positive definite, symmetric and P is an A-harmonic polynomial, homogeneous.

$$\Theta(z) = \sum_{m \in \mathbb{Z}^n} P(m) e\left(\frac{z}{2} A[m] \right)$$

where $A[m] = m^t A m$.

Also for $N \ge 1$

$$\Theta(z, h) = \sum_{\substack{m \in \mathbb{Z}^n \\ m \equiv h(N)}} P(m) e\left(\frac{z}{2} \frac{A[m]}{N^2} \right)$$

Congruent Θ-function.

e.g.

$$\Theta(z,0) = N^\nu \Theta(z)$$

where $\nu = deg(P)$.

Assume A and NA^{-1} are integral. These will be shown to be weight $\nu + n/2$ modular forms.

Example $n = 1$, $\nu = 0$, gives usual $\Theta(z/2)$ has weight $1/2$.

$n = 1$, $\nu = 1$

$$\sum_{m\in\mathbb{Z}} m e^{\pi i m^2 z} = \frac{1}{2\pi i}\frac{\partial}{\partial z}\Theta(z/2)$$

has weight $3/2$.

Preliminary Facts

Let $arg(z) \in (-\pi, \pi]$ for a branch of $\log z$. $z^s = e^{s \log z}$. If z is not in $(-\infty, 0)$. $z^{u+v} = z^u z^v$ for any u, v, $(z^{1/2})^s = z^{s/2}$, $z^{-s} = (1/z)^s$, $(z_1 z_2)^m = z_1^m z_2^m$ if $m \in \mathbb{Z}$. $(cz)^s = c^s z^s$ if $c > 0$. $w^s (z/w)^s = z^s$ if both $Imz, w > 0$.

Proposition Let P be homogeneous, A-harmonic and $\nu = deg(P)$. Let A be positive definite, symmetric, $n \times n$, $z \in \mathbb{H}$, $x \in \mathbb{C}^n$. Then

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^n} P(m+x) e\left(\frac{1}{2}A[m+x]z\right)\\ &= \frac{i^{-\nu}}{|A|^{1/2}}\left(\frac{i}{z}\right)^k \sum_{m\in\mathbb{Z}^n} P^*(m) e\left(\frac{-A^{-1}[m]}{2z} + m^T x\right)\end{aligned}$$

where $P^*(x) = P(A^{-1}x)$ harmonic for A^{-1}.

For example If $x = 0$,

$$\begin{aligned}\sum_{P(m)e\left(\frac{1}{2}A[m]z\right)} &= \Theta_{A,P}(z)\\ &= i^{-\nu}|A|^{-1/2}(i/z)^k \Theta_{A^{-1},P^*}\left(\frac{-1}{z}\right)\end{aligned}$$

Modularity

Another Example $n = 1$, $P = 1$, $A = [1]$.

$$\sum e\left(\frac{m^2 z}{2}\right) = (i/z)^{1/2} \sum_m e\left(-\frac{m^2}{2z}\right)$$

Let $z = \pi i y$

$$\sum e^{-\pi n^2 y} = y^{-1/2} \sum e^{-\pi n^2/y}$$

classical Θ-identity.

Computation Let $f(u) = e\left(\frac{z}{2}A[u+x]\right)$

$$\begin{aligned}
\widehat{f}(v) &= \int_{\mathbb{R}^n} f(u)e(-u\cdot v)du \\
&= \int_{\mathbb{R}^n} e\left(\frac{z}{2}A[u+x]\right)e(-u\cdot v)du \\
& u \to u-x \\
&= e(x\cdot v)\int_{\mathbb{R}^n} e\left(\frac{z}{2}u^tAu\right)e(-u\cdot v)du \\
& A = B^TB,\ u = B^{-1}w,\ du = \frac{1}{|B|}dw, \\
& u^TAu = w^Tw,\ u\cdot v = B^{-1}w\cdot v = w\cdot (B^{-1})^Tv \\
&= \frac{1}{|B|}e(x\cdot v)\int_{\mathbb{R}^n} e\left(\frac{z}{2}w^Tw\right)e(-w\cdot B^{-T}v)dw \\
&= \frac{1}{|B|}e(x\cdot v)\prod_{j=1}^n\int_{\mathbb{R}} e\left(\frac{z}{2}w_j^2\right)e(-w_j\cdot\psi_j)dw_j
\end{aligned}$$

if $z = iy$, $e^{-\pi y w_j^2}$ has Fourier transform $e^{-i\pi\psi_j^2/iy} = e^{-i\pi\psi_j^2/z}y^{-1/2}$ so

$$\begin{aligned}
\widehat{f}(v) &= \frac{e(x\cdot v)}{|A|^{1/2}}\prod e^{-i\pi/z\psi_j^2}y^{-1/2} \\
&= \frac{e(x\cdot v)}{|A|^{1/2}}e\left(-\frac{1}{2z}\|B^{-T}v\|^2\right)\left(\frac{i}{z}\right)^{n/2} \\
&= \frac{i^{n/2}}{z^{n/2}}\frac{e(x\cdot v)}{|A|^{1/2}}e\left(-\frac{1}{2z}A^{-1}[v]\right)
\end{aligned}$$

Poisson summation has

$$\sum_{m\in\mathbb{Z}^n} f(m) = \sum_{m\in\mathbb{Z}^n}\widehat{f}(m)$$

so

$$\sum e\left(\tfrac{z}{2}A[m+x]\right) = \tfrac{i^{n/2}}{z^{n/2}|A|^{1/2}}\sum\nolimits_m e(x\cdot m)e\left(-\tfrac{1}{2z}A^{-1}[m]\right)$$

Therefore, we have proven the proposition for $\nu = 0$.

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Last time we showed for A, a $n\times n$ positive definite, symmetric matrix, P, an A-harmonic homogeneous polynomial, $z\in\mathbb{H}$, $x\in\mathbb{C}^\nu$, ν is the degree of P. This formula

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^n} P(m+x)e\left(\frac{1}{2}A[m+x]z\right) \\
&= i^{-\nu}|A|^{-1/2}(i/z)^{\nu+n/2}\sum_{m\in\mathbb{Z}^n} P(A^{-1}m)e\left(-\frac{A^{-1}[m]}{2z}+m\cdot x\right)
\end{aligned}$$

was shown to be valid if P is constant. We will derive the formula for general P by differentiation.

Study $\partial/\partial x_l$ on both sides. Compute

$$\begin{aligned}
\tfrac{\partial}{\partial x_l} A[x] &= \frac{\partial}{\partial x_l} \sum_{i,j} a_{ij} x_i x_j \\
&= \sum_{i,j} a_{ij} \frac{\partial}{\partial x_l}(x_i x_j) \\
&= \sum_{i,j} a_{ij} \left[x_j \frac{\partial x_i}{\partial x_l} + X_i \frac{\partial x_j}{\partial x_l} \right] \\
&= \sum_{i,j} a_{ij} \left[x_j \delta_{i=l} + x_i \delta_{j=l} \right] \\
&= \sum_j a_{lj} x_j + \sum_i a_{il} x_i \\
&= 2 \sum_j a_{lj} x_j
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{\partial}{\partial x_l} e\left(\frac{1}{2} A[m+x] z\right) \\
&= \frac{2\pi i z}{2} e\left(\frac{1}{2} A[m+x] z\right) 2 \sum_j a_{lj}(m_j + x_j)
\end{aligned}$$

Get $z \sum a_{lj} x_j$. Left hand side is now:

$$2\pi i z P(m+x), \text{ where } P(y_1, \ldots, y_n) = \sum a_{lj} y_j$$

On right hand side, $\partial/\partial x_l$ multiplies the summand by $2\pi i m_l = 2\pi i P(A^{-1} m)$ Recall:

$$\begin{aligned}
\Theta(z:l) &= \sum_{\substack{m \in \mathbb{Z}^n \\ m \equiv l \mod N}} P(m) e\left(\frac{z}{2N^2} A[m]\right) \\
&= (-1)^\nu \Theta(z:-l)
\end{aligned}$$

depends on $l \mod N$. Initially, N^n choices but most are irrelevant.

Relevant classes:

$$\mathcal{H} = \{h \mod N : Ah \equiv 0 \mod N\}$$

Lemma $\mathcal{H}$ is a finite abelian group of order $|\det A|$.

Proof Let $\mathcal{B} = [0,1)^n$. We take h from $N\mathcal{B} \cap \mathbb{Z}^n$ such that $Ah \in N\mathbb{Z}^n$,

$$N^{-1} A : N\mathcal{B} \to \mathcal{P} = A\mathcal{B} = A[0,1)^n$$

$\mathcal{H} \to$ integral points in $\mathcal{P}$, onto. $|\mathcal{H}| = |\mathcal{P} \cap \mathbb{Z}^n|$.

How many lattice points of $\mathbb{Z}^n$ lie in P?

$\mathcal{P}$ is a fundamental domain for the lattice $\Lambda = \langle \lambda_1, \dots \lambda_n \rangle \in \mathbb{Z}^n$, $\lambda_i = Ae_i$, $e_i = [0 \dots 0, 1, 0, \dots, 0]$. For $\lambda \in \Lambda$, $\mathcal{P} + \lambda$, corners are still integral and $|(P + \lambda) \cap \mathbb{Z}^n| = |\mathcal{P} \cap \mathbb{Z}^n|$.

Hence $|\mathcal{P} \cap \mathbb{Z}^n| = vol(\mathcal{P}) = |\det(A)|$.

$$A = \gamma_1 \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \gamma_2$$

$\gamma_1, \gamma_2 \in GL(n, \mathbb{Z})$, $d_1 \geq d_2 \geq \dots \geq d_n$, $d_1 \cdots d_n = |\det(A)|$.

$$\mathcal{H} \cong \mathbb{Z}/d_1\mathbb{Z} \bigoplus \mathbb{Z}/d_2\mathbb{Z} \bigoplus \cdots \bigoplus \mathbb{Z}/d_n\mathbb{Z}$$

Calculation:

$$\begin{aligned} A[h + Nm] &= A[h] + A[Nm] + +2h^T ANm \\ &= A[h] + N(2h^T Am) + N^2 A[m] \end{aligned}$$

If $h \in \mathcal{H}$, then $A[h + Nm] \equiv A[h] \mod N^2$ and $A[h + Nm] \equiv A[h] \mod 2N^2$ if $diagA$ is even.

Proposition For $h \in \mathcal{H}$,

$$\Theta(z + 2 : h) = e\left(\frac{A[h]}{N^2}\right)\Theta(z; h)$$

If furthermore, $diag(A)$ is even,

$$\Theta(z + 1; h) = e\left(\frac{A[h]}{2N^2}\right)\Theta(z : h)$$

Proposition For all $h \in \mathcal{H}$,

$$\Theta(\frac{-1}{z} : h) = i^{-\nu}|A|^{-1/2}(-iz)^k \sum_{l \in \mathcal{H}} \psi(h, l)\Theta(z : l)$$

$\psi(h, l) = e\left(\frac{h^T Al}{N^2}\right)$, bilinear pairing on $\mathcal{H}$.

Proof

$$\begin{aligned} &N^{-\nu} \sum_{m \in \mathbb{Z}^n} P(Nm + h)e\left(\frac{-z^{-1}}{2N^2} A[Nm + h]\right) \\ &= i^{-\nu}|A|^{-1/2}(-iz)^k \sum P(A^{-1}m)e\left(\frac{zA^{-1}[m]}{2} + m\frac{h}{N}\right) \end{aligned}$$

We need to show that

$$\sum_{m\in\mathbb{Z}^n} P(A^{-1}m)e\left(\frac{zA^{-1}[m]}{2}+\frac{mh}{N}\right)$$

equals

$$\sum_{l\in\mathcal{H}} e\left(\frac{h^TAl}{N^2}\right)\sum_{m\equiv l(N)} P(m)e\left(\frac{z}{2N^2}A[m]\right)$$

Let $p=NA^{-1}m$ on left-hand side. $m\in\mathbb{Z}^n \leftrightarrow \{p\in\mathbb{Z}^n : Ap\equiv 0(N)\}$, $m=N^{-1}Ap\in\mathbb{Z}^n$. $A^{-1}m=Np$, $A^{-1}[m]=m^TA^{-1}m=N^{-2}p^TA^TA^{-1}Ap=\frac{A[p]}{N^2}$.

Left hand side is

$$\begin{aligned}&sum_{\substack{p\in\mathbb{Z}^n\\ Ap\equiv 0(N)}} P(p/N)e\left(\frac{z}{2N^2}A[p]+\frac{p^TAh}{N^2}\right)\\ &=\sum_{l\in\mathcal{H}}\sum_{m\equiv l(N)} N^{-\nu}P(p)e\left(\frac{zA[m]}{2N^2}+\frac{m^TAh}{N^2}\right)\end{aligned}$$

Done if we can check

$$m^TAh\equiv h^TAl \mod N^2$$

if $m\equiv l(N)$, done.

<u>Transformation Law</u> Assume $diagA$ is even. Let $\tau=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma(4N)$, modular form of weight $\nu+n/2$, $h\in\mathcal{H}$,

$$\Theta\left(\frac{az+b}{cz+d}:h\right)=\left(\frac{2c}{d}\right)^n(cz+d)^{\nu+n/2}\Theta(z:h)$$

<u>Lemma</u> Let $\tau=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix} b & -a\\ d & -c\end{pmatrix}$

Then

$$\Theta\left(\frac{az+b}{cz+d}:h\right)=i^{-2\nu}d^{-n/2}|A|^{-1}(cz+d)^{\nu+n/2}\sum_{l\in\mathcal{H}}\Phi(h,l)\Theta(z:l)$$

$$\Phi(h,l)=\sum_{h'\in\mathcal{H}}\phi(h,h')\psi(h',l)$$

$$\phi(h,h')=\sum_{g(dN)g\equiv h(N)} e\left(\frac{bA[g]+2(h')^TAg-cA[h^{-1}]}{2dN^2}\right)$$

<u>Proof</u>

$$\frac{bz-a}{dz-c}=\gamma z=\frac{b}{d}-\frac{1}{d(dz-c)}$$

$$\begin{aligned}\Theta(\gamma z : h) &= \sum_{m\equiv h(N)} P(m) e\left(\frac{A[m]}{2N^2}\left(\frac{b}{d} - \frac{1}{d(dz-c)}\right)\right)\\ &= \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{A[g]b}{2N^2 d}\right)\sum_{m\equiv g(N)} P(m)e\left(\frac{A[m]}{2dN^2}\frac{1}{c-dz}\right)\end{aligned}$$

$z' = dz - c$, $A \to dA$, $N \to dN$.

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Let $\mathcal{H} = \{h \mod N : Ah \equiv 0 \mod N\}$, A is symmetric, positive definite, integral, $diag(A)$ has even entries. Integer $N > 0$ such that NA^{-1} is integral.

$$\Theta_N(z,h) = \sum_{\substack{m\in\mathbb{Z}^n\\ m\equiv h(N)}} P(m)e\left(\frac{z}{2N^2}A[m]\right)$$

Proposition

$$\Theta(-\frac{1}{z}, h) = i^{-\nu}|A|^{-1/2}(-iz)^k \sum_{l\in\mathcal{H}} \psi(h,l)\Theta(z,l)$$

$$\psi(h,l) = e\left(\frac{h^T A l}{N^2}\right)$$

Theorem For $\tau \in \Gamma(2N)$, $\tau = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $a \equiv d \equiv 1(2N)$, $2N|b,c$.

$$\Theta(\tau z, h) = \left(\frac{2c}{d}\right)^n (cz+d)^{\nu+n/2}\Theta(z,h)$$

modular form of weight $\nu + n/2$, where $\left(\frac{2c}{d}\right)$ is the Kronecker symbol.

We must have $d \neq 0$. Without loss of generality, we can assume $d > 0$. $\gamma = \tau\begin{pmatrix} & -1\\ 1 & \end{pmatrix} = \begin{pmatrix} b & -a\\ d & -c\end{pmatrix}$. $\gamma z = \frac{b/d}{-}\frac{1}{d(dz-c)}$ Then

$$\begin{aligned}\Theta_{NA}(\gamma z, h) &= \sum_{m\equiv h(N)} P(m)e\left(\frac{A[m]}{2N^2}\left(\frac{b}{d} - \frac{1}{d(dz-c)}\right)\right)\\ &= \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g]}{2N^2 d}\right)\sum_{m\equiv g(dN)} P(m)e\left(\frac{dA[m]}{2(dN)^2}\frac{-1}{dz-c}\right)\end{aligned}$$

because $A[m] = A[m + dN\mathbb{Z}^n] \mod 2dN^2$.

$$
\begin{aligned}
\Theta_{NA}(\gamma z, h) &= \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g]}{2N^2 d}\right) \Theta_{dN,dA}\left(-\frac{1}{dz-c}, g\right)\\
&= \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g]}{2dN^2}\right) i^{-\nu} |det(dA)|^{-1/2} (i(dz-c))^{\nu+n/2} \sum_{\substack{l \ \mathrm{mod}\ N\\ dAl\equiv 0 \ \mathrm{mod}\ dN}} e\left(\frac{l^T dAg}{(dN)^2}\right) \sum_{m\equiv l(dN)} e \Bigg(\\
&= \frac{(i(dz-c))^{\nu+n/2}}{i^{\nu} d^{n/2} |A|^{1/2}} \sum_{\substack{l \ \mathrm{mod}\ dN\\ Al\equiv 0(N)}} \phi(h,l) \sum_{m\equiv l(dN)} P(m) e\left(\frac{A[m]z}{2N^2}\right)
\end{aligned}
$$

where

$$
\phi(h,l) = \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g] + 2l^T Ag - cA[l]}{2dN^2}\right)
$$

We used the fact that $A[m] \equiv A[l] \mod 2dN^2$.

Lemma

$$
\begin{aligned}
\phi(h,l) &= e\left(\frac{2al^t Ah - acA[l]}{2N^2}\right) \phi(h-cl, 0)\\
&= \psi(ah, l) e\left(\frac{-acA[l]}{2N^2}\right) \phi(h-cl, 0)
\end{aligned}
$$

Proof

$g \mapsto g + cl$

$$
\begin{aligned}
\phi(h,l) &= \sum_{\substack{g(dN)\\ g\equiv h-cl}} e\left(\frac{1}{2dN^2}\left(bA[g+cl] + 2l^T A(g+cl) - cA[l]\right)\right)\\
\text{Numerator} &= bA[g] + 2cbg^T Al + bc^2 A[l] + 2l^T Ag + 2cA[l] - cA[l]\\
&= bA[g] + (2bc+2) g^T Al + c(bc+1)A[l]\\
&= bA[g] + 2adg^T Al + adcA[l]\\
\phi(h,l) &= \sum_{\substack{g(dN)\\ g\equiv h-cl}} e\left(\frac{bA[g]}{2dN^2} + \frac{2ag^T Al + acA[l]}{2N^2}\right)
\end{aligned}
$$

We need to show $ag^t Al + acA[l] \equiv 2ah^t Al - acA[l] \mod 2N^2$

This is if and only if $ag^T Al + acA[l] \equiv ah^t Al \mod N^2$. We have $N|Al$, so we have this identity and we have the Theorem.

From the theorem, we see in particular, $\phi(h,l)$ depends only on $l \mod N$.

We now glue from $\mod dN$ to $\mod N$ and find

$$\Theta_{NA}(\gamma z, h) \quad = \frac{(i(c-dz))^{\nu+n/2}}{i^\nu d^{n/2}|A|^{1/2}} \sum_{h' \in \mathcal{H}} \phi(h,h')\Theta(z,h')$$

We have $\tau z = \gamma(-1/z)$. Send $z \mapsto -1/z$:

$$\begin{aligned} \Theta(\tau z, h) \quad &= \frac{(i(c+d/z))^k}{i^\nu d^{n/2}|A|^{1/2}} \sum_{h' \in \mathcal{H}} \phi(h,h')\Theta(-1/z,h') \\ &= \frac{(i(c+d/z))^{\nu+n/2}(-iz)^{\nu+n/2}}{i^{2\nu} d^{n/2}|A|} \sum_{h' \in \mathcal{H}} \phi(h,h') \sum_{l \in \mathcal{H}} \psi(h,l)\Theta(z,l) \\ &= i^{-2\nu} d^{-n/2}|A|^{-1}(cz+d)^k \sum_{l \in \mathcal{H}} \Phi(h,l)\Theta(z,l) \end{aligned}$$

where

$$\Phi(h,l) = \sum_{h' \in \mathcal{H}} \phi(h,h')\psi(h,l)$$

Lemma

$$\Phi(h,l) = \begin{cases} \phi(h,0)|A| & l \equiv -ah(N) \\ 0 & \text{otherwise.} \end{cases}$$

Proof Let $l \in \mathbb{Z}^n$ such that $Al \equiv 0(N)$ [that is, $l \in \mathcal{H}$]. $A[l] = l^t Al$, divisible by N.

$$\phi(h,l) = e\left(\frac{-acA[l]}{2N^2}\right)\psi(ah,l)\phi(h-cl,0)$$

We have ac divisible by $2N$ and $A[l]$ divisible by N means that $\phi(h,l) = \psi(ah,l)\phi(h,0)$.

$$\phi(h-cl,0) = \sum_{\substack{g(dN) \\ g \equiv h-cl(N)}} e\left(\frac{bA[g]}{2dN^2}\right) = \phi(h,0)$$

$$\begin{aligned} \Phi(h,l) \quad &= \sum_{h' \in \mathcal{H}} \phi(h,0)\psi(ah,h')\psi(h',l) \\ &= \phi(h,0) \sum_{h' \in \mathcal{H}} \psi(h',ah+l) \end{aligned}$$

Treating $\psi(h',ah+l)$ as a character of h' is nonzero only when $ah+l \equiv 0(N)$. Then

$$\Phi(h,l) = \begin{cases} |A|\phi(h,0) & l \equiv -ah \\ 0 & \text{otherwise} \end{cases}$$

$$\Theta(\tau,z) = \frac{(cz+d)^k}{d^{n/2}}\phi(h,0)\Theta(z,h)$$

Question: What is $\phi(h,0)$?

Answer: a Gauss sum

$$\phi(h,0) \quad = \sum_{\substack{g(dN)\\ g\equiv h(N)}} e\left(\frac{bA[g]}{2dN^2}\right)$$

$g \equiv (ad-bc)h(N) \equiv adh$, $a \equiv 1(2N)$, $g \equiv adh + xN$, $x \mod d$.
$bA[g] = bA[adh+xN] = ba^2d^2A[h] + bN^2A[x] + 2badNh^tAx$
$b \equiv 0(N)$, so $2badNh^tAx \equiv 0 \mod 2dN^2$.

$$\phi(h,0) = e\left(\frac{abA[h]}{2N^2}\right)\sum_{x(d)} e\left(\frac{-2cA[x]}{d}\right)$$

The sum is the Gauss sum.