ON THE BETTER BEHAVED VERSION OF THE GKZ HYPERGEOMETRIC SYSTEM

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Abstract. We define a version of the generalized hypergeometric system introduced by Gelfand, Kapranov and Zelevinski (GKZ) suited for the case when the underlying lattice is replaced by a finitely generated abelian group. In contrast to the usual GKZ hypergeometric system, the rank of the better behaved GKZ hypergeometric system is always the expected one. We construct explicit solutions as Γ–series and as geometric periods in certain cases.

1. Introduction

Let $\mathcal{A} = \{v_1, \ldots, v_n\}$ be a set of vectors in the lattice $N \cong \mathbb{Z}^d$ such that the elements of $\mathcal{A}$ generate the lattice as an abelian group, and that there exists a group homomorphism $\deg : N \to \mathbb{Z}$ such that $\deg(v) = 1$ for any element $v \in \mathcal{A}$. Let $L \subset \mathbb{Z}^n$ denote the lattice of integral relations among the elements of $\mathcal{A}$ consisting of vectors $l = (l_j) \in \mathbb{Z}^n$ such that $l_1 v_1 + \ldots + l_n v_n = 0$.

For any parameter $\beta \in N \otimes \mathbb{C}$, Gelfand, Kapranov and Zelevinsky [GKZ1] considered a system of differential equations on the function $\Phi(x)$, $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, consisting of the binomial equations

$$\Big(\prod_{i,l_i>0} (\partial_i)^{l_i} - \prod_{i,l_i<0} (\partial_i)^{-l_i}\Big)\Phi = 0,\ l \in L,$$

and the linear equations

$$\Big(\sum_{i=1}^{n} \mu(v_i) x_i \partial_i\Big)\Phi = \mu(\beta)\Phi,\ \text{for all } \mu \in M = \mathrm{Hom}(N, \mathbb{Z}).$$

Gelfand, Kapranov and Zelevinsky showed that this system is holonomic, so the number of solutions at a generic point is finite. Following Batyrev's observation [Bat, Section 14] that the periods of a Calabi–Yau hypersurface in a projective toric variety satisfy a GKZ system, its

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study gained further prominence in connection with mirror symmetry phenomena and algebra geometric applications.

The rank of the GKZ system (the dimension of its solution set at a generic point) and the solution set itself have also been the subject of numerous studies. Its expected dimension is equal to the normalized volume of the convex hull Δ of the elements of the set $\mathcal{A}$. However, if the semigroup generated by the elements of $\mathcal{A}$ is not equal to the integral cone $N \cap K$, where K being the cone spanned in $N \otimes \mathbb{R}$ by the elements of $\mathcal{A}$, then there are non-generic values of β for which the rank jumps. This rank discrepancy has been thoroughly investigated by many authors (see, for example, Adolphson [A], Saito, Sturmfels and Takayama [SST], Cattani, Dickenstein and Sturmfels [CDS]) and a quite definitive explanation for it has been obtained in the work of Matusevich, Miller and Walther [MMW].

In the present work, we propose a *better behaved* version of the GKZ system whose space of solutions always has the expected number of solutions. We frame the definition in a context where the lattice is replaced by a finitely generated abelian group N, and the set $\mathcal{A}$ is replaced by an n-tuple $\mathcal{A} = (v_1, \ldots, v_n)$ of elements of N, with possible repetitions. Given a parameter β in $N \otimes \mathbb{C}$, the better behaved GKZ system consists of the equations

$$\partial_i \Phi_c = \Phi_{c+v_i}, \quad \text{for all } c \in K,\ i \in \{1, \ldots, n\}$$

and the linear equations

$$\sum_{i=1}^{n} \mu(v_i) x_i \partial_i \Phi_c = \mu(\beta - c)\Phi_c, \text{ for all } \mu \in M,\ c \in K.$$

A solution to the better behaved GKZ system is then a sequence of functions of n variables $\big(\Phi_c(x_1, \ldots, x_n)\big)_{c \in K}$, where K is the preimage under the map $N \to N \otimes \mathbb{R}$ of the cone $K_{\mathbb{R}}$ generated by the images of the elements v_i in $N \otimes \mathbb{R}$.

When N is a lattice and $\mathcal{A}$ is a finite subset subject to the the hyperplane condition, the better behaved GKZ equations on Φ_0 imply the usual GKZ equations on that function. The generalization presented in our work fits in the general context of ideas where the usual combinatorial framework of toric geometry is extended from toric varieties and their fans to that of toric Deligne-Mumford stacks and stacky fans provided in the work of Borisov, Chen and Smith [BCS].

We now briefly discuss the content of this paper. In section 2, we give the precise definition to the better behaved GKZ system. In section 3, we give identifications for the spaces of solutions as the logarithmic Jacobian rings (Definition 3.3). As a byproduct, we prove that the spaces of solutions have indeed the expected dimensions, namely the product of the normalized volume of the polytope Δ and the torsion order of the abelian group N.

In section 4, we construct a complete system of linearly independent Γ–series solutions to the better behaved GKZ system. The construction is accomplished in two steps. We first show how to pursue the construction in the presence of torsion in N, and then, for $\beta \in N$, we explicitly give the Γ–series construction for the better behaved GKZ with no torsion. This second part of the construction uses the "shadow modules" K_β (Definition 4.4) associated to the cone K and the parameter $\beta \in N$. For the usual GKZ, Γ–series solutions have been obtained in the book by Saito, Sturmfels and Takayama [SST] in the general case, and by Hosono, Lian and Yau [HLY] and Stienstra [S] in the case of unimodular triangulations.

In the last section, we obtain integral representations for the solutions to the better behaved GKZ in the case $\beta = 0$ as periods of middle dimensional cycles in algebraic tori. The results are implicit in the work of Batyrev [Bat] in the usual GKZ case and offer a potential method for studying the integral structure on the solution space to the better behaved GKZ as an image of the integral structure on the homology of the complement of a hypersurface in the algebraic torus.

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2. The usual and the better-behaved versions of the GKZ hypergeometric system

Throughout this paper, we will use the following notations. We are given a finitely generated abelian group N, and an n-tuple $\mathcal{A} = (v_1, \ldots, v_n)$ of elements of N. We will denote by M the free abelian group $\mathrm{Hom}(N, \mathbb{Z})$. We will assume that there exists an element $\deg \in M$ such that $\deg(v_i) = 1$ for all i. We will denote by Δ the convex hull of the set of v_i in $N \otimes \mathbb{R}$ and by $K_\mathbb{R}$ the cone $\mathbb{R}_{\geq 0}\Delta$. We will denote by K the preimage of $K_\mathbb{R}$ in N under the natural map $\pi : N \to N \otimes \mathbb{R}$ and by $[c]$, for $c \in K$, the corresponding elements in the semigroup ring $\mathbb{C}[K]$ or its variants that will be used below. We will further assume that $\pi(v_i)$ span the lattice $\pi(N)$ as a group. The finite abelian group $\mathrm{tors}(N)$ is the torsion part of N and $|\mathrm{tors}(N)|$ its order. For N torsion-free, we set $|\mathrm{tors}(N)| = 1$.

The version of the GKZ hypergeometric system associated to a fixed parameter $\beta \in N \otimes \mathbb{C}$ which will be the central object of study of this paper is then defined as follows:

Definition 2.1. Consider the following system of partial differential equations on sequences of functions of n variables $\big(\Phi_c(x_1, \ldots, x_n)\big)_{c \in K}$:

$$\partial_i \Phi_c = \Phi_{c+v_i}, \quad \text{for all } c \in K,\ i \in \{1, \ldots, n\} \tag{1}$$

$$\sum_{i=1}^{n} \mu(v_i) x_i \partial_i \Phi_c = \mu(\beta - c)\Phi_c, \text{ for all } \mu \in M,\ c \in K. \tag{2}$$

We will call this system the *better behaved GKZ* and will denote it by $\mathrm{GKZ}(\mathcal{A}, K; \beta)$.

In order to simplify our notation, we will denote a solution to the better behaved GKZ by $\Phi_K(x_1, \ldots, x_n)$. Alternatively, it can be viewed it as a function in n variables

$$\Phi_K(x_1, \ldots, x_n) = \sum_{c \in K} \Phi_c(x_1, \ldots, x_n)[c]$$

with values in the completion $\mathbb{C}[K]^c$ of the ring $\mathbb{C}[K]$.

Remark 2.2. It is clear that one can reformulate the above system as a system of PDEs on a finite collection of functions of $(x_1, \ldots, x_n)$. Indeed, the set K_{prim} of elements $v \in K$ such that $v - v_i \notin K$ for all i is finite. The functions Φ_c for $c \in K_{\mathrm{prim}}$ then determine the rest of Φ_c. In fact, the number of PDEs can also be made finite, in view of the following. The relations (2) for $c \in K$, together with the relation (1) implies the relations (2) for $c + v_i$. Consequently, one only needs to use (2) for $c \in K_{\mathrm{prim}}$. The relations (1) can then be restated as

$$\Big(\prod_{i=1}^{n} \partial_i^{k_i}\Big)\Phi_{c_1} = \Big(\prod_{i=1}^{n} \partial_i^{l_i}\Big)\Phi_{c_2} \tag{3}$$

for all $k_i, l_i \in \mathbb{Z}_{\geq 0}$ such that

$$c_1 + \sum_i k_i v_i = c_2 + \sum_i l_i v_i$$

and $c_1, c_2 \in K_{\mathrm{prim}}$. To see that (3) follows from a finite number of relations of this type, note that they correspond to the $\mathbb{C}$-basis of the module over the polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_n]$ which is the kernel of the natural map

$$\mathbb{C}[K_{\mathrm{prim}}] \otimes \mathbb{C}[\partial_1, \ldots, \partial_n] \to \mathbb{C}[K]$$

which sends $\partial_i \to [v_i]$. Since K_{prim} is finite, this kernel is a Noetherian module, thus a finite subset of (3) generates the rest.

Remark 2.3. The usual GKZ hypergeometric system coincides with $\mathrm{GKZ}(\mathcal{A}, K; \beta)$ if N has no torsion and v_i generate K *as a semigroup*. Indeed, then $K_{\mathrm{prim}} = \{0\}$, (2) leads to the linear equations of [GKZ1] and (1) leads to

$$\Big(\prod_{i=1}^{n} \partial_i^{k_i}\Big)\Phi_0 = \Big(\prod_{i=1}^{n} \partial_i^{l_i}\Big)\Phi_0$$

whenever $\sum_i (k_i - l_i) v_i = 0$, which are the binomial relations of [GKZ1].

Remark 2.4. The n-tuple $\mathcal{A}$ of elements of N is allowed to contain repeated elements. As one can see from the PDEs defining the better-behaved GKZ system, the effect of having $v_i = v_j$ for some $i \neq j$, is that all functions Φ_c depend on $x_i + x_j$.

Example 2.5. Let $N = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{A} = (v_1, v_2)$, with $v_1 = (1, 0)$, $v_2 = (1, 1)$. Let β be an element in $N \otimes \mathbb{C} \cong \mathbb{C}$. The solution space of the better-behaved GKZ system is isomorphic to the space of pairs of functions $\Phi_{(0,0)}(x_1, x_2), \Phi_{(0,1)}(x_1, x_2)$ satisfying the equations

$$\partial_1 \Phi_{(0,0)} = \partial_2 \Phi_{(0,1)}, \quad \partial_2 \Phi_{(0,0)} = \partial_1 \Phi_{(0,1)},$$

$$(x_1\partial_1 + x_2\partial_2)\Phi_{(0,0)} = \beta\Phi_{(0,0)}, \quad (x_1\partial_1 + x_2\partial_2)\Phi_{(0,1)} = \beta\Phi_{(0,1)}.$$

The first pair of equations implies that both functions $\Phi_{(0,0)}$ and $\Phi_{(0,1)}$ satisfy the wave equation. It follows that

$$\Phi_{(0,0)}(x_1, x_2) = a(x_1 + x_2) + b(x_1 - x_2),$$

$$\Phi_{(0,1)}(x_1, x_2) = a(x_1 + x_2) - b(x_1 - x_2),$$

for some arbitrary functions a, b. The second pair of equations implies then that

$$(x_1 + x_2)a'(x_1 + x_2) = \beta a(x_1 + x_2), \ (x_1 - x_2)b'(x_1 - x_2) = \beta b(x_1 - x_2).$$

It follows that $a(x_1 + x_2) = A(x_1 + x_2)^\beta$ and $b(x_1 - x_2) = B(x_1 - x_2)^\beta$, for some arbitrary complex constants A, B. Hence the better-behaved GKZ system has a two-dimensional solution space. Note that the discriminant locus of the system consists of the reducible curve $x_1^2 - x_2^2 = 0$ in $\mathbb{C}^2$.

Definition 2.6. For any subset S of N which is closed under the additions of v_i we can define the system $\mathrm{GKZ}(\mathcal{A}, S; \beta)$ as in Definition 2.1, but with $c \in S$ rather than $c \in K$.

Remark 2.7. If N has no torsion, then the usual version of GKZ is equivalent to $\mathrm{GKZ}(\mathcal{A}, S; \beta)$ for S the subsemigroup of K generated by v_i. The fact that $\mathrm{GKZ}(\mathcal{A}, K; \beta)$ is better-behaved than the usual GKZ is then related to the fact that the semigroup algebra $\mathbb{C}[K]$ is always Cohen-Macaulay, whereas $\mathbb{C}[S]$ need not be so.

3. Spaces of solutions of the better-behaved GKZ and the logarithmic Jacobian ring

Let $(x_1, \ldots, x_n) \in \mathbb{C}^n$. We introduce a non-degeneracy notion for a degree one element $f = \sum_{i=1}^n x_i[v_i]$ of $\mathbb{C}[K]$ which is closely related to the one used by Batyrev [Bat] in the non-torsion case (see for example theorem 4.8 in [Bat]).

Definition 3.1. The degree one element $f = \sum_{i=1}^n x_i[v_i]$ of $\mathbb{C}[K]$ is said to be *non-degenerate* if the logarithmic derivatives $\sum_i x_i \mu_j(v_i)[v_i]$ form a regular sequence in $\mathbb{C}[K]$ for a basis μ_j, $1 \leq j \leq \mathrm{rk}M$, of M.

Proposition 3.2. *For a generic choice of $f = \sum_{i=1}^n x_i[v_i]$ and any basis (μ_j) of M the log-derivatives $f_j = \sum_i x_i \mu_j(v_i)[v_i]$ of f give a regular sequence in $\mathbb{C}[K]$. Equivalently, the Koszul complex induced by the elements f_j*

$$0 \to \ldots \to \wedge^2 M \otimes \mathbb{C}[K] \to M \otimes \mathbb{C}[K] \to \mathbb{C}[K] \to R(f, K) \to 0 \tag{4}$$

is exact.

Proof. If N has no torsion, the result is [B, Proposition 3.2]. The Koszul complex reformulation is standard. If N has torsion, the result appears to be new, but perhaps not particularly unexpected. In order to prove it, note that the ring $\mathbb{C}[K]$ is the direct sum of $|\mathrm{tors}N|$ copies of $\mathbb{C}[\pi(K)]$, where $\pi : K \to K \otimes \mathbb{R}$ is the natural map. Then the regularity of the sequence needs to be checked at each individual copy of $\mathbb{C}[\pi(K)]$ where it follows again from the non-torsion result. $\square$

Definition 3.3. The ring $R(f, K)$ is called the *logarithmic Jacobian ring* associated to f and K.

Corollary 3.4. *The dimension of the $\mathbb{C}$-vector space $R(f, K)$ is equal to $vol(\Delta) \cdot |\mathrm{tors}(N)|$, where $vol(\Delta)$ is the normalized volume of the polytope Δ in $N \otimes \mathbb{R}$, and $|\mathrm{tors}(N)|$ is the order of the torsion part of N.*

Proof. The dimension of the $\mathbb{C}$-vector space $R(f, K)$ is equal to the product of $(\mathrm{rk}N-1)! \cdot |\mathrm{tors}(N)|$ and the leading coefficient of the Hilbert polynomial of the graded ring $\mathbb{C}[K \otimes \mathbb{Z}]$. But it is well known that this leading coefficient is the quotient of the normalized volume of Δ by $(\mathrm{rk}N - 1)!$. $\square$

The complex (4) is graded with finite-dimensional graded components. We can dualize it component-wise to get another graded exact complex with finite-dimensional graded components

$$0 \to R(f,K)^\vee \to \mathbb{C}[K] \to N \otimes \mathbb{C}[K] \to \wedge^2 N \otimes \mathbb{C}[K] \to \ldots \to 0 \tag{5}$$

We will naturally identify the (graded) dual of $\mathbb{C}[K]$ with itself, since each graded component of $\mathbb{C}[K]$ has a natural basis.

The complex (5) allows us to give the following description of the vector space $R(f,K)^\vee$.

Proposition 3.5. *The space $R(f,K)^\vee$ is the set of elements $\sum_{c\in K} \lambda_c[c]$ in $\mathbb{C}[K]$ such that the linear equations in $N \otimes \mathbb{C}$*

$$\sum_{i=1}^{n} x_i \lambda_{c+v_i} v_i = 0 \tag{6}$$

hold for all $c \in K$.

Proof. The result follows from the observation that the dual of the map $M \otimes \mathbb{C}[K] \to \mathbb{C}[K]$ in the Koszul complex (4) is the map $\mathbb{C}[K] \to N \otimes \mathbb{C}[K]$ in the dual complex (5) given by

$$\sum_{c\in K} \lambda_c[c] \mapsto \sum_{c\in K} \sum_{i=1}^{n} x_i \lambda_{c+v_i} v_i \otimes [c].$$

□

Remark 3.6. Note that equations (6) can be solved degree-by-degree and will have no nontrivial solutions for $\deg(c) > \mathrm{rk}N$. Indeed, the exactness of the complex (5) implies that the Hilbert-Poincaré series of the kernel of the map $\mathbb{C}[K] \to N \otimes \mathbb{C}[K]$ is a polynomial of degree at most $\mathrm{rk}N$.

Let us now consider the solutions to $\mathrm{GKZ}(\mathcal{A}, K; \beta)$.

Theorem 3.7. *The space of analytic solutions to $\mathrm{GKZ}(\mathcal{A}, K; \beta)$ in a neighborhood of a generic f is isomorphic to the space of elements $\sum_{c\in K} \lambda_c[c]$ in $\mathbb{C}[K]^c$ such that the linear equations in $N \otimes \mathbb{C}$*

$$\sum_{i=1}^{n} x_i \lambda_{c+v_i} v_i = \lambda_c(\beta - c) \tag{7}$$

hold for all $c \in K$.

Proof. In one direction, if we have a solution $(\Phi_c), c \in K$, then $\lambda_c = \Phi_c(x_1, \ldots, x_n)$ clearly satisfies (7). In fact, this map from the space of solutions of $\mathrm{GKZ}(\mathcal{A},K;\beta)$ to the space of solutions of (7) is clearly injective in view of Taylor's formula, since knowing all $\Phi_c(x_1,\ldots,x_n)$ implies the knowledge of all the partial derivatives of all Φ_c at $(x_1,\ldots,x_n)$ in view of equation (1).

In the other direction, suppose that we have a solution (λ_c) of (7). Then equation (1) and Taylor formula force us to have

$$\Phi_c(z_1,\ldots,z_n)=\sum_{(l_1,\ldots,l_n)\in\mathbb{Z}^n_{\geq 0}}\lambda_{c+\sum_i l_iv_i}\prod_{i=1}^n\frac{(z_i-x_i)^{l_i}}{l_i!} \tag{8}$$

for all $c\in K$. It remains to show that the above series converges absolutely and uniformly in $c\in K$ and $\mathbf{z}$ in a neighborhood of $(x_1,\ldots,x_n)$. Observe that it suffices to show uniform convergence for a fixed $c\in K_{\mathrm{prim}}$, since the partial derivative of a Taylor series will converge in the same neighborhood and $K_{\mathrm{prim}}\subset K$ is a finite set. From now on we fix $c=c_0$.

We claim that there exists a constant $C_1\in\mathbb{R}$ such that

$$|\lambda_{c_0+\sum_i l_iv_i}|\leq C_1^{(\sum_{i=1}^n l_i)}(\sum_{i=1}^n l_i)! \tag{9}$$

for all nonzero $(l_1,\ldots,l_n)\in\mathbb{Z}^n_{\geq 0}$. This is easily seen to be equivalent to the existence of a constant $C_2\in\mathbb{R}$ such that

$$|\lambda_d|\leq C_2^{\deg d}(\deg d)! \tag{10}$$

for all d with sufficiently high $\deg d$.

Define $\Lambda_k=\max\limits_{d,\deg d=k}|\lambda_d|$. To prove (10) it suffices to show that there exists $C_3\in\mathbb{R}$ such that $\Lambda_{k+1}\leq C_3k\Lambda_k$ for all sufficiently large k.

The ideal I of $\mathbb{C}[K]$ generated by logarithmic derivatives of f contains $[d]$ for all d of $\deg d=\mathrm{rk}N+1$. It is easy to see that every d_1 of sufficiently high degree can be written as $d_1=d+d_2$ with $d,d_2\in K$ and $\deg d=\mathrm{rk}N+1$. We can write each $[d]$ of degree $\mathrm{rk}N+1$ as

$$[d]=\sum_{i=1}^n\sum_{j=1}^{\mathrm{rk}N}x_i\mu_j(v_i)[v_i]t_{d,j}$$

for some $t_{d,j}\in\mathbb{C}[K]_{\deg=\mathrm{rk}N}$ and some basis $(\mu_1,\ldots,\mu_{\mathrm{rk}N})$ of M. Consequently, for each d_1 of sufficiently high degree we have that

$$[d_1]=\sum_{i=1}^n\sum_{j=1}^{\mathrm{rk}N}x_i\mu_j(v_i)[v_i]t_{d,j}[d_2]$$

for some d_2. By considering the maximum size of the coefficients of $t_{d,j}$ we observe that, for $\deg d_1 = k+1$,

$$[d_1] = \sum_{i=1}^{n} \sum_{j=1}^{\mathrm{rk} N} \sum_{d_3, \deg d_3 = k} \beta_{d_3,j} x_i \mu_j(v_i)[d_3 + v_i]$$

with $\sum_{d_3,j} |\beta_{d_3,j}|$ bounded by a constant independent of d_1 and k. Equation (7) implies then that

$$\sum_{d_3,j} \beta_{d_3,j} \lambda_{d_3} \mu_j(\beta - d_3) = \sum_{i,d_3,j} \beta_{d_3,j} x_i \lambda_{d_3+v_i} \mu_j(v_i) = \lambda_{d_1}.$$

Since $\sum_{d_3,j} |\beta_{d_3,j}|$ is bounded by a constant, $|\lambda_{d_3}|$ is bounded by Λ_k and $\mu(\beta - d_3)$ is bounded by a constant times k, we get $|\lambda_{d_1}| \leq C_3 k \Lambda_k$ as required.

This allows us to establish estimates (10) and (9). Since the multinomial coefficients

$$\frac{(\sum_{i=1}^{n} l_i)!}{\prod_{i=1}^{n} l_i!}$$

are bounded by $n^{\sum_{i=1}^{n} l_i}$, the terms of the series (8) are bounded by $\prod_{i=1}^{n} (nC_1)^{l_i} |z_i - x_i|^{l_i}$. By making $|z_i - x_i|$ sufficiently small, the absolute convergence is obtained by comparing to a product of convergent geometric series. $\square$

Having identified the space of solutions of $\mathrm{GKZ}(\mathcal{A}, K; \beta)$ in a neighborhood of $(x_1, \ldots, x_n)$ with the space of solutions of the equations (7), we can now consider a natural filtration on it. We define the subspaces F_k of the space of solutions of $\mathrm{GKZ}(\mathcal{A}, K; \beta)$ in a neighborhood of a generic $(x_1, \ldots, x_n)$ to be characterized by the fact that $\Phi_c(x_1, \ldots, x_n) = 0$ for all c with $\deg c < k$. We have that $F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots$.

Theorem 3.8. *The quotient F_k/F_{k+1} is naturally isomorphic to the dual of the degree k component of $R(f, K)$.*

Proof. The essential observation is that the equations (7) satisfied by the elements λ_c can be solved recursively in the degree of c. Indeed, suppose that we have found $\lambda_d, \deg d \leq k$, which satisfy (7) for all c, $\deg c \leq k-1$. In order to check that a solution exists for all d of degree $k+1$, we need to check that $\sum_{c, \deg c = k} \lambda_c(\beta - c)[c]$ sits in the degree k component of the image of the map $C[K]^c \to N \otimes \mathbb{C}[K]^c$ of the complex (5). Since this is an exact complex, it suffices to check that it is in the kernel of the map

$$N \otimes \mathbb{C}[K]^c \to \wedge^2 N \otimes \mathbb{C}[K]^c.$$

of (5). The coefficient of its image at $[c_1]$ is given by the element of $\wedge^2 N \otimes \mathbb{C}[K]^c$

$$\sum_{i=1}^{n} x_i \lambda_{c_1+v_i} v_i \wedge (\beta - c_1) = \lambda_{c_1}(\beta - c_1) \wedge (\beta - c_1) = 0.$$

Observe that as we are solving recursively the equations (7), the ambiguity at each step is precisely an element of the corresponding component of $R(f,K)^\vee$, which leads to the result. $\square$

Corollary 3.9. *The space of solutions to the true GKZ system is of the same dimension as* $R(f,K)$.

Remark 3.10. The same argument applies with obvious modifications when one replaces K by its interior.

Remark 3.11. The argument of this section is likely philosophically the same as the general arguments used in the theory of holonomic D-modules, but it has an advantage of being self-contained.

4. Gamma series solutions to the better behaved GKZ system

A standard way of obtaining solutions for the GKZ hypergeometric system is given by a Γ-series. We will adopt the same approach to our current situation.

We first analyze the role of played by the torsion part of the finitely generated abelian group N and by the possible repetitions that may appear in the n-tuple $\mathcal{A}$. Let $\{w_1, \ldots, w_m\} \subset N \otimes \mathbb{R}$ be the set consisting of the elements $\pi(v_i), 1 \leq i \leq n$, in $N \otimes \mathbb{R}$ where $\pi : N \to N \otimes \mathbb{R}$ is the natural map. For each $j, 1 \leq j \leq m$, let I_j be the set of indices i with $\pi(v_i) = w_j$.

Let $\rho : N \to \mathbb{C}^\times$ be a multiplicative group character. Define the map $p_\rho : \mathbb{C}^n \to \mathbb{C}^m$ by

$$p_\rho(x_1, \ldots, x_n) := (\sum_{i \in I_1} \rho(v_i) x_i, \ldots, \sum_{i \in I_m} \rho(v_i) x_i). \tag{11}$$

To a sequence of functions $(\Psi_w(z_1, \ldots, z_m))_{w \in \pi(K)}$, we associate a sequence of functions $(\Phi_c(x_1, \ldots, x_n))_{c \in K}$ such that, for any $c \in K$,

$$\Phi_c(x_1, \ldots, x_n) := \rho(c) \Psi_{\pi(c)}(p_\rho(x_1, \ldots, x_n)). \tag{12}$$

What makes this definition useful is the following result.

Proposition 4.1. *For any character* $\rho \in \mathrm{Hom}(N, \mathbb{C}^\times)$, *if the function*

$$\Psi_{\pi(K)}(z_1, \ldots, z_m) = \sum_{w \in p(K)} \Psi_w(z_1, \ldots, z_m)[w],$$

is a solution on an open set $U \subset \mathbb{C}^m$ to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by $\{w_1, \ldots, w_m\}$ in $\pi(N)$, then the associated function

$$\Phi_K(x_1, \ldots, x_n) = \sum_{c \in K} \Phi_c(x_1, \ldots, x_n)[c],$$

is a solution on the open set $p_\rho^{-1}(U) \subset \mathbb{C}^n$ to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by $(v_1, \ldots, v_n)$ in N.

Proof. For any $c \in N$, equation (12) implies that given some $v_i \in N$ and $w_j \in \pi(N)$ such that $\pi(v_i) = w_j$ we have that

$$\partial_i \Phi_c(x_, \ldots, x_n) = \rho(c + v_i)\partial_j \Psi_{\pi(c)}(\sum_{i \in I_1} \rho(v_i)x_i, \ldots, \sum_{i \in I_m} \rho(v_i)x_i).$$

Since the functions $\Psi_w, w \in \pi(K)$, are solutions to the better behaved GKZ in $\pi(N)$, and $\pi(c)+w_j = \pi(c)+\pi(v_i) = \pi(c+v_i)$, we obtain indeed that $\partial_i \Phi_c(x_, \ldots, x_n) = \Phi_{c+v_i}$. Similarly, for any $\mu \in M = \mathrm{Hom}(N, \mathbb{Z}) = \mathrm{Hom}(\pi(N), \mathbb{Z})$, we have that

$$\begin{aligned} &\sum_{i=1}^n \mu(v_i) x_i \partial_i \Phi_c(x_1, \ldots, x_n) \\ &= \rho(c) \sum_{j=1}^m \mu(w_j) \big(\sum_{i \in I_j} \rho(v_i) x_i \big) \partial_j \Psi_{\pi(c)}(\sum_{i \in I_1} \rho(v_i)x_i, \ldots, \sum_{i \in I_m} \rho(v_i)x_i) \\ &= \mu(\beta - c)\Phi_c(x_1, \ldots, x_n). \end{aligned}$$

□

Let G denote the torsion part of N. Since G is finite abelian group, we have that $\mathrm{Hom}(G, \mathbb{C}^x) \simeq G$. Assume that G has order k, and let $(\rho_g)_{g \in G}$ be the corresponding set of independent characters in $\mathrm{Hom}(G, \mathbb{C}^x) \simeq G$. When there is no torsion, we set $|G| = 1$ and $\rho_1 = 1$. The characters ρ_g can be easily extended to become multiplicative characters of N by imposing that they take the value 1 on the free part of N, after a choice of splitting. Under this convention, we will view the characters ρ_g as elements in $\mathrm{Hom}(N, \mathbb{C}^x)$. As in formula (11), for each $g \in G$, we define the linear surjective maps $p_g : \mathbb{C}^n \to \mathbb{C}^m$ by

$$p_g(x_1, \ldots, x_n) := (\sum_{i \in I_1} \rho_g(v_i)x_i, \ldots, \sum_{i \in I_m} \rho_g(v_i)x_i).$$

Let $U \subset \mathbb{C}^m$ a nonempty open set in $\mathbb{C}^m$ with the property that there exists an open set V in $\mathbb{R}^m$ such that

$$\begin{aligned} U = \{(z_1, \ldots, z_m) \,:\, &(\log|z_1|, \ldots, \log|z_m|) \in V, \\ &(\arg z_1, \ldots, \arg z_m) \in (-\pi, \pi) \times \ldots \times (-\pi, \pi)\} \end{aligned} \tag{13}$$

for a choice of the argument functions $(\arg z_1, \ldots, \arg z_m) \in \mathbb{R}^m$. For such a set $U \subset \mathbb{C}^m$, we have that:

Lemma 4.2. $\cap_{g\in G}\ p_g^{-1}(U) \neq \emptyset$.

Proof. For each set of indices I_j, choose exactly one $i_j \in I_j$ and a complex number x_{i_j} such that

$$(\log|x_{i_1}|, \ldots, \log|x_{i_m}|) \in V \setminus \{0\}$$

and, for all $j, 1 \leq j \leq m$,

$$\arg x_{i_j} + \pi < 2\pi/|G|.$$

From (13), we see that $\arg(\rho_g(v_{i_j})x_{i_j}) \in (-\pi, \pi)$, for any $g \in G$ and $1 \leq j \leq m$, hence

$$(\rho_g(v_{i_1})x_{i_1}, \ldots, \rho_g(v_{i_m})x_{i_m}) \in U,$$

for any $g \in G$. By continuity, it is now possible to choose all the other complex numbers $x_i, 1 \leq i \leq n$, for those indices i different from any of the i_j's, in a small enough neighborhood of the origin in the complex plane such that $p_g(x_1, \ldots, x_n) \in U$, for any $g \in G$. The lemma follows. $\square$

Theorem 4.3. *Let $\Psi^{\lambda}_{\pi(K)}, \lambda \in \Lambda$, be a set of linearly independent analytic solutions on an open set $U \subset \mathbb{C}^m$ satisfying condition (13) to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by $(w_1, \ldots, w_m)$ in $\pi(N)$. The associated set of $|\Lambda| \cdot |G|$ functions $\Phi^{\lambda,g}_K$, $\lambda \in \Lambda, g \in G$, is a set of linearly independent analytic solutions on the non-empty open set $p_1^{-1}(U) \cap \ldots \cap p_m^{-1}(U) \subset \mathbb{C}^n$ to the better behaved GKZ associated to $\beta \in N \otimes \mathbb{C}$ defined by $(v_1, \ldots, v_n)$ in N.*

Proof. Assume that there exists constants $\alpha_{\lambda,g}$ such that

$$\sum_{\lambda\in\Lambda, g\in G} \alpha_{\lambda,g}\Phi^{\lambda,g}_c(x) = 0,$$

for any $c \in K, x \in \cap_{g\in G}\ p_g^{-1}(U)$. It follows that

$$\sum_{\lambda\in\Lambda, g\in G} \alpha_{\lambda,g}\rho_g(c + c_h)\Psi^{\lambda}_{\pi(c)}(p_g(x)) = 0,$$

for any $c \in K, x \in \cap_{g\in G}\ p_g^{-1}(U)$, and $c_h \in K$ such that $\pi(c_h) = 0$.

For each fixed $c \in K$, we have $|G|$ linear relations indexed by $h \in G$. The orthogonality relations for the characters of the representations of the finite group G imply that

$$\sum_{\lambda\in\Lambda} \alpha_{\lambda,g}\Psi^{\lambda}_{\pi(c)}(p_g(x)) = 0,$$

for any $g \in G, c \in K$ and $x \in \cap_{g \in G}\ p_g^{-1}(U)$. The linear independence of the analytic functions $\Psi^{\lambda}_{\pi(K)}, \lambda \in \Lambda$, in U shows that $\alpha_{\lambda,g}$ are all zero. $\square$

The previous discussion clarifies the construction of solutions in the presence of torsion and repetitions, so for the rest of this section we will assume that N is a lattice isomorphic to $\mathbb{Z}^d$ and that the elements in the n-tuple $\mathcal{A} = (v_1, \dots, v_n)$ form a set.

We are assuming that $\beta \in N$. Consider a regular triangulation of the polytope $\Delta = \mathrm{Conv}(\mathcal{A})$ with all the vertices among the elements of $\mathcal{A}$ and let Σ be the induced simplicial fan supported on the cone $K = \mathbb{R}_{\geq 0}$. If σ_1 and σ_2 are cones of Σ, we will use the notation $\sigma_1 \prec \sigma_2$ to indicate that the cone σ_1 is a subcone of σ_2. For any $c \in K$, we denote by $\sigma(c)$ the minimal cone of the fan containing c.

We define the partial semigroup ring $\mathbb{C}[K, \Sigma]$ to be the complex vector space with a basis given by the symbols $[c]$ for all $c \in K$ and the multiplication defined such that $[c_1] \cdot [c_2] = [c_1 + c_2]$, whenever the images of c_1 and c_2 under the map $N \to N \otimes \mathbb{R}$ belong to a cone of the fan Σ, and $[c_1] \cdot [c_2] = 0$, otherwise.

Definition 4.4. For any $\beta \in N$, we define *the shadow K_β of K with respect to β* to be the subset of lattice points $c \in K$ such that $c + \epsilon\beta \in K_{\mathbb{R}}$ for all sufficiently small $\epsilon > 0$.

If $\beta \in K$, then $K_\beta = K$, and if $\beta \in -K^\circ$, then $K_\beta = K^\circ$. Note that $K_{\beta - c} \subset K_\beta$ for any $c \in K$.

Let $\mathbb{C}[K_\beta, \Sigma]$ be the ideal in $\mathbb{C}[K, \Sigma]$ generated by $[c]$, for all c in K_β. The arguments used by Borisov [B] essentially show that the following theorem holds:

Proposition 4.5. *The ring $\mathbb{C}[K, \Sigma]$ and the module $\mathbb{C}[K_\beta, \Sigma]$ (over $\mathbb{C}[K, \Sigma]$) are Cohen-Macaulay of dimension d. Moreover, for any basis $(\mu_1, \dots, \mu_d)$ of $M = \mathrm{Hom}(N, \mathbb{Z})$, the elements*

$$Z_j = \sum_{i, \mathbb{R}_{\geq 0} v_i \in \Sigma} \mu_j(v_i)[v_i]$$

form a regular sequence in $\mathbb{C}[K, \Sigma]$ (and hence in $\mathbb{C}[K_\beta, \Sigma]$).

Corollary 4.6. *The quotients*

$$\mathbb{C}[K, \Sigma]/Z\mathbb{C}[K, \Sigma] := \mathbb{C}[K, \Sigma]/(Z_1, \dots, Z_d)\mathbb{C}[K, \Sigma]$$

and

$$\mathbb{C}[K_\beta, \Sigma]/Z\mathbb{C}[K_\beta, \Sigma] := \mathbb{C}[K_\beta, \Sigma]/(Z_1, \dots, Z_d)\mathbb{C}[K_\beta, \Sigma]$$

have dimension equal to the normalized volume of Δ.

For any cone $\sigma \in \Sigma$, we define the finite set $\mathrm{Box}(\sigma)$ of elements in N as the set

$$\{c_0 \,:\, c_0 = \sum_{i=1}^{n} q_i v_i,\ 0 \le q_i < 1,\ q_i = 0, \text{ if } \mathbb{R}_{\ge 0} v_i \text{ is not a ray of } \sigma\}.$$

Let $\mathrm{Box}(\Sigma) \subset N$ be the union of the sets $\mathrm{Box}(\sigma)$ for all the maximal dimensional cones $\sigma \in \Sigma$. If $c_0 \in \mathrm{Box}(\Sigma)$, we denote by $\sigma(c_0)$ the smallest cone of Σ that contains c_0, and let $\gamma_i(c_0), 1 \le i \le n$, be the rational numbers such that

$$\sum_{i=1}^{n} \gamma_i(c_0) v_i = c_0,$$

with $\gamma_i(c_0) = 0$ for those j such that v_i does not generate a ray of $\sigma(c_0)$.

Definition 4.7. For each $c \in K$, we define the set $L(c_0, c) \subset \mathbb{Q}^n$ of collections $l = (l_i)_{1 \le i \le n}$ such that

$$\sum_{i=1}^{n} l_i v_i = \beta - c$$

and such that $l_i - \gamma_i(c_0)$ are integers.

The assumption that the elements of $\mathcal{A}$ generate the lattice N implies that $\beta - c_0 - c$ can be written as a linear combination with integer coefficients of the elements of $\mathcal{A}$, so the set $L(c_0, c)$ is a translate in $\mathbb{C}^n$ of the lattice of integral relations L among the elements of $\mathcal{A}$, where

$$L = \{(l_i)_{1 \le i \le n} \ :\ \sum_{i=1} l_i v_i = 0,\ l_i \in \mathbb{Z} \text{ for } 1 \le j \le n\}.$$

It is also useful to note that for any distinct elements c_0 and c_1 in $\mathrm{Box}(\Sigma)$, the sets $L(c_0, c)$ and $L(c_1, c)$ are disjoint.

For a given $x = (x_1, \ldots, x_n)$ in $(\mathbb{C}^\star)^n$, and $c \in K$, we introduce the formal $\mathbb{C}[K, \Sigma]^c$–valued Γ–series

$$\Phi_c(x) := \sum_{c_0 \in \mathrm{Box}(\Sigma)} [c_0] \sum_{l \in L(c_0,c)} \prod_{i=1}^{n} \frac{x_i^{l_i + D_i}}{\Gamma(l_i + D_i + 1)}, \tag{14}$$

where

$$D_i := [v_i] \text{ if } \mathbb{R}_{\ge 0} v_i \in \Sigma, \text{ and } D_i := 0, \text{ otherwise},$$

and

$$x_i^{l_i + D_i} := e^{(l_i + D_i)(\log|x_i| + \sqrt{-1}\arg x_i)},$$

for a choice of $(\arg x_1, \ldots, \arg x_n) \in \mathbb{R}^n$.

According to [BH, Proposition 2.12], for each $c \in K$, the formal series $\Phi_c(x)$ induces a well defined map $\Psi_c(x_1, \ldots, x_n)$ from a non-empty open set U_Σ in $\mathbb{C}^n$ to the completion $\mathbb{C}[K,\Sigma]^c$ of the graded ring $\mathbb{C}[K,\Sigma]$. The aforementioned proposition also states that the set U_Σ satisfies condition (13). This is an important fact required in theorem 4.3 and insures the consistency of our construction for the torsion and non-torsion cases.

Lemma 4.8. *For any $c_0 \in \mathrm{Box}(\Sigma)$, $c \in K$, $l \in L(c_0, c)$ and $x \in \mathbb{C}^n$, the product*

$$[c_0] \prod_{i=1}^{n} \frac{x_i^{l_i + D_i}}{\Gamma(l_i + D_i + 1)}$$

belongs to the shadow module $\mathbb{C}[K_\beta, \Sigma]$.

Proof. Note first that the product is zero, unless there exists a maximal cone in Σ containing $\sigma(c_0]$ whose rays contain all the vectors v_i with $l_i \in \mathbb{Z}_{<0}$. Assume that the product is not zero. It lies in the ideal of $\mathbb{C}[K,\Sigma]$ generated by $[w]$ where

$$w = c_0 + \sum_{i, l_i \in \mathbb{Z}_{<0}} v_i,$$

so it is enough to show that $w \in K_\beta$. For any $\epsilon > 0$ we can write that

$$w + \epsilon(\beta - c) = w + \epsilon \sum_{i=1}^{n} l_i v_i$$

$$= \sum_{i, \mathbb{R}_{\geq 0} v_i \prec \sigma(c_0)} (\gamma_i(c_0) + \epsilon l_i) v_i + \sum_{i, l_i \in \mathbb{Z}_{<0}} (1 + \epsilon l_i) v_i + \sum_{i, l_i \in \mathbb{Z}_{\geq 0}} \epsilon l_i v_i.$$

Since $\gamma_i(c_0) > 0$ in the first summation, for any sufficiently small $\epsilon > 0$, we have that $w + \epsilon(\beta - c) \in K_\mathbb{R}$. Since $c \in K$, it follows that $w + \epsilon\beta \in K_\mathbb{R}$. $\square$

Theorem 4.9. *For any linear map*

$$h : \mathbb{C}[K_\beta, \Sigma]/(Z_1, \ldots, Z_d)\mathbb{C}[K_\beta, \Sigma] \to \mathbb{C},$$

the sequence of functions $(h \cdot \Psi_c(x_1, \ldots, x_n))_{c \in K}$ satisfies the better behaved GKZ hypergeometric equations on U_Σ corresponding to the set $\mathcal{A}$ and parameter $\beta \in N$.

Proof. For the set of equations (1), note that taking the j-th partial derivative of the summand over $(l_i)_{1 \leq i \leq n} \in L(c_0, c)$ in the series Φ_c, replaces it by a summand over $(l_i - \delta_j^i)_{1 \leq i \leq n}$. It is then enough to note that

$$L(c_0, c) - (\delta_j^i)_{1 \leq i \leq n} = L(c_0, c + v_j).$$

For the set of equations (2), our convention that $D_j = 0$ if $\mathbb{R}_{\geq 0} v_j \notin \Sigma$, implies that

$$\sum_{i=1}^{n} \mu(v_i) D_i = \sum_{i, \mathbb{R}_{\geq 0} v_i \in \Sigma} \mu(v_i) D_i,$$

for any $\mu \in M = \mathrm{Hom}(N, \mathbb{Z})$. Hence

$$\big(- \mu(\beta - c) + \sum_{i=1}^{n} \mu(v_i) x_i \partial_i \big) \Psi_c = \big(\sum_{i=1}^{n} \mu(v_i) D_i \big) \Psi_c.$$

The previous lemma shows that the series Ψ_c takes values in $\mathbb{C}[K_\beta, \Sigma]$, so the result follows after we observe that $\sum_{i=1}^{n} \mu(v_i) D_i$ is a linear combination of the Z_j's. $\square$

Accordingly, it is convenient to view each Γ–series $\Psi_c(x), c \in K$, as a map from the non-empty open set U_Σ in $\mathbb{C}^n$ to the finite dimensional vector space $\mathbb{C}[K_\beta, \Sigma]/(Z_1, \ldots, Z_d)\mathbb{C}[K_\beta, \Sigma]$.

Lemma 4.10. *Any $w \in K_\beta$ determines a unique $c_0 \in \mathrm{Box}(\Sigma)$ such that*

$$w = \sum_{i=1}^{n} p_i v_i$$

with $p_i - \gamma_i(c_0) \in \mathbb{Z}_{\geq 0}$ for all i, and $p_i = 0$ unless v_i generates a one dimensional subcone of $\sigma(w)$. Moreover, there exists $c_1 \in K$ and $l \in L(c_0, c_1)$ such that $p_i \in \mathbb{Z}_{>0}$ for all i such that $l_i \in \mathbb{Z}_{<0}$.

Proof. Note first that, since $\sigma(w)$ is the minimal cone containing w, we can write in a unique way

$$w = c_0 + \sum_{i=1}^{n} m_i v_i = \sum_{i=1}^{n} (\gamma_i(c_0) + m_i) v_i, \ m_i \in \mathbb{Z}_{\geq 0},$$

for some $c_0 \in \mathrm{Box}(\sigma)$, with $\gamma_i(c_0) = m_i = 0$ for those i such that v_i does not generate a one dimensional subcone of $\sigma(w)$. Set $p_i := \gamma_i(c_0) + m_i$.

The definition of K_β implies that for any $w \in K_\beta$, there exists a cone σ in the fan such that $w + \epsilon\beta \in \sigma$ for any small enough $\epsilon > 0$. In particular, we have that $w \in \sigma(w) \subset \sigma$. Let $P > 0$ be a large enough positive integer such that, for any $c \in K$, the element Pc belongs to the semigroup generated by the elements v_i generating the cone σ, and such that $c + \beta/P \in K_{\mathbb{R}}$. It follows that

$$Pc = \sum_{i=1}^{n} q_i' v_i, \ q_i' \in \mathbb{Z}_{\geq 0},$$

and

$$Pc + \beta = \sum_{i=1}^{n} q_i'' v_i, \ q_i'' \in \mathbb{Z}_{\geq 0},$$

where $q_i' = q_i'' = 0$, for those i such that v_i does not generate a one dimensional subcone of σ. Hence

$$\beta = \sum_{i=1}^{n} q_i v_i, \ q_i \in \mathbb{Z},$$

with $q_i = q_i'' - q_i'$. Since $w + \epsilon\beta \in K_{\mathbb{R}}$ for sufficiently small $\epsilon > 0$, we conclude that $\gamma_i(c_0) + m_i > 0$ for all i with $q_i \in \mathbb{Z}_{<0}$.

Now we choose c_1 to be the element in $\mathrm{Box}(\Sigma)$ such that $\sigma(c_0) = \sigma(c_1)$ and, if $c_0 \neq 0$,

$$c_0 + c_1 = \sum_{i, \mathbb{R}_{\geq 0} v_i \prec \sigma(c_0)} v_i.$$

We can write that

$$\beta - c_1 = \sum_{i=1}^{n} l_i v_i,$$

where $l_i = q_i + \gamma_i(c_0) - 1$, if v_i generates a one dimensional subcone of $\sigma(c_0)$, and $l_i = q_i$, otherwise. We see that $l = (l_i)_{1 \leq i \leq n}$ belongs to $L(c_0, c_1)$ and has the required properties, so the result follows. □

We now prove the required linear independence result.

Proposition 4.11. *If*

$$h : \mathbb{C}[K_\beta, \Sigma]/(Z_1, \ldots, Z_d)\mathbb{C}[K_\beta, \Sigma] \to \mathbb{C},$$

is a linear map such that $h \cdot \Psi_c(x) = 0$, *for all* $c \in K$ *and any* $x \in U_\Sigma$, *then* $h = 0$.

Proof. The proof is very similar to the proof of [BH, Proposition 2.19]. We include it here since the context and the notation are slightly changed. Assume that there exists some $x \in \mathbb{C}[K_\beta, \Sigma]/Z\mathbb{C}[K_\beta, \Sigma]$ such that $h(x) \neq 0$. Let R be the largest degree of such an element. Furthermore, we can assume that x has representative $[w]$ mod $Z\mathbb{C}[K_\beta, \Sigma]$, with $w \in K_\beta$. According to the previous lemma w determines a unique $c_0 \in \mathrm{Box}(\Sigma)$, an element $c_1 \in K$, and a relation $l \in L(c_0, c_1)$ such that

$$w = c_0 + \sum_{i, l_i \in \mathbb{Z}_{<0}} v_i + \sum_{i=1}^{n} n_i v_i \tag{15}$$

with $n_i \in \mathbb{Z}_{\geq 0}$ for those i such that v_i generates a one dimensional subcone in $\sigma(w)$, and $n_i = 0$ otherwise.

For each i such that v_i generates a cone in Σ, consider the loop of the form $x_i(t) = \epsilon\exp(2\pi\sqrt{-1}\,t), x_j(t) = \epsilon$, $j \neq i$, $0 \leq t \leq 1$, with $\epsilon > 0$ a small real positive number. The action of the induced monodromy operator T_i on the Γ-series Ψ_{c_1} is given by $\exp(D_i)$. Since D_i is nilpotent in $\mathbb{C}[K_\beta, \Sigma]/Z\mathbb{C}[K_\beta, \Sigma]$, there is a polynomial $g(T_i)$ such that $g(T_i)\Psi_{c_1} = D_i\Psi_{c_1}$, for every i such that v_i generates a cone in Σ. Hence

$$\prod_i g(T_i)^{n_i}\Psi_{c_1}(x) = \prod_i D_i^{n_i}\Psi_{c_1}(x).$$

Since $h \cdot \Psi_c(x) = 0$ and we have analytically continued Ψ_{c_1}, we also have that

$$h(\prod_i D_i^{n_i}\Psi_{c_1}(x)) = 0.$$

The definition of the Γ–series $\Psi_{c_1}(x)$ and the fact that $D_i = [v_i]$ are nilpotent in $\mathbb{C}[K_\beta, \Sigma]/Z\mathbb{C}[K_\beta, \Sigma]$ shows that any induced GKZ solution can be written as the product of a monomial in the variables x_i and an element of $\mathbb{C}[u_k^{-1}, \log u_k][[u_k]]$ where $u_k, 1 \leq k \leq n - \mathrm{rank} N$, invariant variables under the action of the character torus $\mathrm{Hom}(L, \mathbb{C}^\times)$.

Hence, in order to obtain the contradiction it is enough to show that, for $l \in L(c_0, c_1)$ used in formula (15), the Fourier coefficient of $x^l = \prod x_i^{l_i}$ in the expansion of $h(\prod_i D_i^{n_i}\Psi_{c_1}(x))$ is non-zero. Indeed, this coefficient is given by

$$h(\prod D_i^{n_i} \cdot [c_0] \cdot \prod \frac{1}{\Gamma(l_i + D_i + 1)}).$$

Notice that the terms that occur in the expansion of the expression in the argument of h have degree at least R, while $[w]$ is the only element of that degree that occurs and its coefficient is nonzero. Since $h([w]) \neq 0$, the maximal property of R implies that the coefficient of x^l in the expansion of $h(\prod_i D_i^{n_i}\Psi_{c_1}(x))$ is indeed non-zero. This ends the proof of the linear independence result. $\square$

Since the dimension of the space of solutions to the better behaved GKZ system is exactly $\mathrm{vol}(\Delta)$, we conclude that the formal Γ–series produces the expected number of linearly independent analytic solutions to the better behaved GKZ system. More precisely, we have proved the following theorem.

Theorem 4.12. *The map*

$$(\mathbb{C}[K_\beta, \Sigma]/(Z_1, \ldots, Z_d)\mathbb{C}[K_\beta, \Sigma])^\vee \to \mathcal{S}ol(U_\Sigma)$$
$$f \to (f \cdot \Psi_c)_{c\in K}$$

produces a complete system of $\operatorname{vol}(\Delta)$ *linearly independent solutions to the better behaved GKZ system which are analytic in* U_Σ.

Remark 4.13. The result of this theorem should be compared with the results of [BH, Corollary 2.21] which employ the "leading term module" $M(\beta)$ associated to the usual GKZ system. The modules $M(\beta)$ are useful algebraic tools defined in loc. cit., but the dimensions of $M(\beta)/ZM(\beta)$ are quite hard to control for general values of β. In contrast, the shadow modules $\mathbb{C}[K_\beta, \Sigma]$ are maximal Cohen-Macaulay. It is possible to show that $\mathbb{C}[K_\beta, \Sigma] = \sum_{c\in K} M(\beta - c)$.

We summarize the results of this section in the following remark.

Remark 4.14. Let N be a finitely generated abelian group and let $\operatorname{tors}(N)$ denote its torsion part. Assume that $\mathcal{A} = (v_1, \ldots, v_n)$ is an n-tuple of elements in N subject to the conditions detailed at the beginning of section 2, and $\beta \in N$. Let Σ be a fan supported on $K_\mathbb{R}$ induced by a regular triangulation of the convex hull of the elements $\pi(v_i)$ with all the vertices among the elements $\pi(v_i)$, where $\pi : N \to N \otimes \mathbb{R}$ is the natural map.

Theorems 4.3 and 4.12 provide the explicit construction of a complete system of $\operatorname{vol}(\Delta) \cdot |\operatorname{tors}(N)|$ linearly independent analytic solutions to the better behaved GKZ system on an non-empty open set $U_\Sigma \subset \mathbb{C}^n$.

5. Periods and the better behaved GKZ with $\beta = 0$

In this section, we assume that N is a lattice, and, as before, we denote by Δ the convex hull of the vectors v_i. We will further assume that v_i are all of the lattice points of Δ. We denote by T the algebraic torus given by $\operatorname{Spec}(N_1)$ where N_1 is the sublattice of N of degree zero elements. For any degree one element $f = \sum_i x_i t^{v_i}$ one can consider the hypersurface $Z_f \subseteq T$ given by $f/t^v = 0$ for some (any) choice of degree one lattice point v. Batyrev [Bat] calculated the homology of Z_f and of the complement $T \setminus Z_f$. In this section we will show how these are related to the true GKZ with $\beta = 0$.

The key idea is provided by [Bat, Theorem 14.2]. Let d be the dimension of T, so that $N = \mathbb{Z}^{d+1}$.

Theorem 5.1. *For any* $\gamma \in H_d(T \setminus Z_f, \mathbb{Z})$ *and any* $c \in K$ *consider*

$$\Phi(x_1, \ldots, x_n) = \int_\gamma \frac{t^c}{f^{\deg c}} w$$

where w *is the standard* T*-invariant* d*-form on* T*. Then* $\Phi(x_1, \ldots, x_n)$ *satisfies the usual GKZ hypergeometric equation with* $\beta = -c$.

Proof. See the proof of [Bat, Theorem 14.2]. $\square$

The above theorem allows us to construct solutions to the true GKZ hypergeometric system from the following data. Let us assume that we operate in a neighborhood of some nondegenerate x. Consider cycles $\gamma \in H_d(T \setminus Z_f, \mathbb{Z})$ such that one can pick a branch of $\ln(f/t^v)$ on it for some choice of v of degree one, and such that $\int_\gamma w = 0$.

Proposition 5.2. *For all such γ and all $c \in K, c \neq 0$ define*

$$\Phi_c(x_1, \ldots, x_n) = \int_\gamma (-1)^{\deg c - 1} (\deg c - 1)! \frac{t^c}{f^{\deg c}} w.$$

Define $\Phi_0(x_1, \ldots, x_n) = \int_\gamma (\ln f/t^v) w$. Then $(\Phi_c)_{c \in K}$ satisfy the better behaved GKZ equations for $\beta = 0$.

Proof. To calculate $\partial_i \Phi_c$, observe that we can differentiate under the integral. If $c \neq 0$, we get

$$\begin{aligned}\partial_i (-1)^{\deg c - 1} (\deg c - 1)! \frac{t^c}{f^{\deg c}} w &= (-1)^{\deg c} (\deg c)! \frac{t^c}{f^{\deg c} + 1} \partial_i f w \\ &= (-1)^{\deg c} (\deg c)! \frac{t^{c+v_i}}{f^{\deg c + 1}} w\end{aligned}$$

so $\partial_i \Phi_c = \Phi_{c+v_i}$. The calculation for $c = 0$ is similar. The linear relations (2) in the $c \neq 0$ case follow from [Bat, Theorem 14.2] and its proof. The linear relations in $c = 0$ case can be obtained directly as follows. Observe that

$$\sum_i \mu(v_i) x_i \partial_i \Phi_0 = \int_\gamma \frac{\sum_i \mu(v_i) x_i t^{v_i}}{f} w.$$

If $\mu = \deg$, then the integral reduces to $\int_\gamma w = 0$, by our assumption on the cycle γ. Otherwise, we can assume that $w = \wedge_{j=1}^d \frac{dt^{l_j}}{t^{l_j}}$ with $\mu(l_j) = 0$ for $j > 1$ and $\mu(l_1) = 1$. We may assume that $v_i = v + \sum_j a_{ij} l_j$ for some integers a_{ij}. We have

$$\frac{\partial}{\partial t^{l_1}} t^{v_i - v} = a_{i1} t^{-l_1} t^{v_i - v} = \mu(v_i - v) t^{-l_1} t^{v_i - v}$$

We then have

$$\begin{aligned}\int_\gamma \frac{\sum_i \mu(v_i) x_i t^{v_i}}{f} w = \int_\gamma \frac{\sum_i \mu(v_i - v) x_i t^{v_i}}{f} w &= \int_\gamma t^{l_1} \frac{\partial}{\partial t^{l_1}} \ln(f/t^v) w \\ &= \int_\gamma d\Big(\ln(f/t^v) \wedge_{j=2}^d \frac{dt^{l_j}}{t^{l_j}} \Big) = 0.\end{aligned}$$

$\square$

Remark 5.3. We can use a similar calculation for an arbitrary c instead of reverting to Batyrev's paper. Namely, for any μ

$$\begin{aligned}
&\sum_i \mu(v_i)x_i\partial_i\Phi_c + \mu(c)\Phi(c)\\
=&\int_\gamma (-1)^{\deg c}(\deg c)!\frac{\sum_i \mu(v_i)x_it^{c+v_i}}{f^{\deg c+1}}w\\
-&\int_\gamma (-1)^{\deg c}(\deg c-1)!\frac{\sum_i \mu(c)x_it^{c+v_i}}{f^{\deg c+1}}w\\
=&(-1)^{\deg c}(\deg c-1)!\int_\gamma \frac{\sum_i \mu((\deg c)v_i - c)x_it^{c+v_i}}{f^{\deg c+1}}w.
\end{aligned}$$

As before, we write $w = \wedge_{j=1}^d \frac{dt^{l_j}}{t^{l_j}}$ with $\mu(l_1) = 1$, $\mu(l_{\geq 2}) = 0$. It remains to observe that the integrand is up to a constant

$$d\Big(\frac{t^{c-(\deg c)v}}{(f/t^v)^{\deg c}} \wedge_{j=2}^d \frac{dt^{l_j}}{t^{l_j}}\Big)$$

because in a calculation similar to $c = 0$ case we get

$$-(\deg c)\mu(v_i - v) + \mu(c - (\deg c)v) = \mu(c - (\deg c)v_i).$$

The further details are left to the reader.

Corollary 5.4. *The solutions to the better behaved GKZ with $\beta = 0$ are linear combinations of the solutions obtained by the above proposition. Moreover, it is effectively a 1-to-1 correspondence and the integral structure on the homology of $T \setminus Z_f$ produces an integral structure on the space of solutions to the better behaved GKZ.*

In particular, this corollary explains why the monodromy of the better behaved GKZ system is integral.

6. Further comments and open problems

The properties of the better behaved version of the GKZ system show that this system is better suited than the usual GKZ if any type of functorial considerations are to be invoked. Although we do not prove it in this paper, the mirror symmetric identification of the Fourier-Mukai transform and the analytic continuation formulae for the solutions to the GKZ system discussed in [BH] continues to hold in the better behaved GKZ case. In fact, the context of this paper is more natural since the rank of the space of local solutions to the better behaved GKZ one side matches the rank of the orbifold cohomology/stacky K-theory on the other side. This was not in the case in our previous work. It would be an interesting problem to study an appropriately defined category of better behaved GKZ systems and its functorial properties, part of which would mirror the properties of category of toric DM stacks. In the same realm, we expect that, in an appropriate sense, the system $\mathrm{GKZ}(\mathcal{A}, K; \beta)$ is dual to the system $\mathrm{GKZ}(\mathcal{A}, K^\circ; -\beta)$.

More importantly, we believe that the better behaved GKZ system lends itself to a process of categorification, which as a first step, is expected to provide a non-commutative categorical resolution of a Gorenstein toric singularity. Such a categorification will have to pass all the toric mirror symmetric checks, and, as such, would have a transcendental component which is missing in the algebraic proposals of non-commutative resolutions currently available in the literature. We hope to come back to this problem in a future paper.

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