

Towards the Mirror Symmetry for Calabi-Yau Complete Intersections in Gorenstein Toric Fano Varieties

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Abstract

We propose a combinatorical duality for lattice polyhedra which conjecturally gives rise to the pairs of mirror symmetric families of Calabi-Yau complete intersections in toric Fano varieties with Gorenstein singularities. Our construction is a generalization of the polar duality proposed by Batyrev for the case of hypersurfaces.

1 Introduction

Mirror Symmetry discovered by physicists for Calabi-Yau manifolds still remains a surprising puzzle for mathematicians. Some insight on this phenomenon was received from the investigation of Mirror Symmetry for some examples of Calabi-Yau varieties which admit simple birational models embedded in toric varieties. In this context, Calabi-Yau manifolds obtained by the resolution of singularities of complete intersections in toric varieties are the most general examples.

In the paper of Batyrev and van Straten [2], there was proposed a method for conjectural construction of mirror families for Calabi-Yau complete intersections in toric varieties. Unfortunately, their method fails to provide such a nice duality as it is in the case of hypersurfaces [1]. The purpose of these notes is to propose a generalized duality which conjecturally gives rise to the mirror involution for complete intersections.

I am pleased to thank prof. Batyrev who has edited my original notes.

2 Basic definitions and notations

Let M and $N = \text{Hom}(M, \mathbf{Z})$ be dual free abelian groups of rank d , $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ be their real scalar extensions and

$$\langle \cdot, \cdot \rangle : M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$$

be the canonical pairing. For any convex polyhedron P in $M_{\mathbf{R}}$ (or in $N_{\mathbf{R}}$), we denote its set of vertices by P^0 .

Definition 2.1 Let P be a d -dimensional convex polyhedron in $M_{\mathbf{R}}$ such that P contains zero point $0 \in M_{\mathbf{R}}$ in its interior. Then

$$P^* = \{y \in N_{\mathbf{R}} \mid \langle x, y \rangle \geq -1\}$$

is called *polar*, or *dual polyhedron*.

Definition 2.2 A convex polyhedron P in $M_{\mathbf{R}}$ is called a *lattice polyhedron* if $P^0 \subset M$.

Definition 2.3 (cf. [1]) Let Δ be a d -dimensional lattice polyhedron in $M_{\mathbf{R}}$ such that Δ contains 0 in its interior. Then Δ is called *reflexive* if Δ^* is also a lattice polyhedron.

Definition 2.4 Let P be a d -dimensional convex polyhedron in $M_{\mathbf{R}}$ such that P contains zero point $0 \in M_{\mathbf{R}}$ in its interior. We define the d -dimensional fan $\Sigma[P]$ as the union of the zero-dimensional cone $\{0\}$ together with the set of all cones

$$\sigma[\theta] = \{0\} \cup \{x \in M_{\mathbf{R}} \mid \lambda x \in \theta \text{ for some } \lambda \in \mathbf{R}_{>0}\}$$

supporting faces θ of P .

Next four definitions of this section play the main role in our construction.

Definition 2.5 Let $\Delta \in M_{\mathbf{R}}$ be a reflexive polyhedron. Put $E = \{e_1, \dots, e_n\} = \Delta^0$. A representation of $E = E_1 \cup \dots \cup E_r$ as the union of disjoint subsets E_1, \dots, E_r is called *nef-partition of E* if there exist integral convex $\Sigma[\Delta]$ -piecewise linear functions $\varphi_1, \dots, \varphi_r$ on $M_{\mathbf{R}}$ such that $\varphi_i(e_j) = 1$ if $e_j \in E_i$, and $\varphi_i(e_j) = 0$ otherwise.

Remark 2.6 The term *nef-partition* is motivated by the fact that such a partition induces a representation of the anticanonical divisor $-K$ on the Gorenstein toric Fano variety \mathbf{P}_{Δ^*} as the sum of r Cartier divisors which are nef.

Definition 2.7 Let $E = E_1 \cup \dots \cup E_r$ be a nef-partition. Define r convex polyhedra $\Delta_1, \dots, \Delta_r \subset M_{\mathbf{R}}$ as

$$\Delta_i = \text{Conv}(\{0\} \cup E_i), \quad i = 1, \dots, r.$$

Remark 2.8 From Definition 2.7 we immediately obtain that $\Delta_i \cap \Delta_j = \{0\}$ if $i \neq j$ and $\Delta = \text{Conv}(\Delta_1 \cup \dots \cup \Delta_r)$.

Definition 2.9 Let $E = E_1 \cup \dots \cup E_r$ be a nef-partition. Define r convex polyhedra $\nabla_1, \dots, \nabla_r \subset N_{\mathbf{R}}$ as

$$\nabla_i = \{y \in N_{\mathbf{R}} \mid \langle x, y \rangle \geq -\varphi_i(x)\}, \quad i = 1, \dots, r.$$

Remark 2.10 It is obvious that $\{0\} \in \nabla_1 \cap \dots \cap \nabla_r$. By Definition 2.1, one has

$$\Delta^* = \{y \in N_{\mathbf{R}} \mid \langle x, y \rangle \geq -\varphi(x)\},$$

where $\varphi = \varphi_1 + \dots + \varphi_r$. Therefore $\nabla_1 \cup \dots \cup \nabla_r \subset \Delta^*$. Notice that $\nabla_1, \dots, \nabla_r$ are also lattice polyhedra. This fact follows from the following standard statement.

Proposition 2.11 *Let Σ be any complete fan of cones in $M_{\mathbf{R}}$, φ_0 a convex Σ -piecewise linear function on $M_{\mathbf{R}}$. Then*

$$Q_0 = \{y \in N_{\mathbf{R}} \mid \langle x, y \rangle \geq -\varphi_0(x)\}$$

is a convex polyhedron whose vertices are restrictions of φ_0 on cones of maximal dimension of Σ .

Corollary 2.12 *The convex functions $\varphi_1, \dots, \varphi_r$ have form*

$$\varphi_i(x) = -\min_{y \in \nabla_i} \langle x, y \rangle.$$

In particular, we have

$$-\min_{x \in \Delta_j^0, y \in \nabla_i^0} \langle x, y \rangle = \delta_{ji}$$

and

$$\langle \Delta_j, \nabla_i \rangle \geq -\delta_{ji}.$$

Definition 2.13 Define the lattice polyhedron $\nabla \in N_{\mathbf{R}}$ as

$$\nabla = \text{Conv}(\nabla_1 \cup \cdots \cup \nabla_r).$$

Remark 2.14 Remark 2.10 shows that $\nabla \subset \Delta^*$.

3 The combinatorial duality

Proposition 3.1 $\Delta^* = \nabla_1 + \cdots + \nabla_r$.

Proof. The statement follows from the equality $\sum_i \varphi_i = \varphi$, from Remark 2.10 and Proposition 2.11. \square

Proposition 3.2 $\nabla^* = \Delta_1 + \cdots + \Delta_r$.

Proof. Let $x = x_1 + \cdots + x_r$ be a point of $\Delta_1 + \cdots + \Delta_r$ ($x_i \in \Delta_i$), and $y = \lambda_1 y_1 + \cdots + \lambda_r y_r$, ($\lambda_1 + \cdots + \lambda_r = 1$, $\lambda_i \geq 0$, $y_i \in \nabla_i$) be a point in ∇ . By 2.12,

$$\langle x, y \rangle \geq \sum_{i=1}^r \lambda_i \langle x_i, y_i \rangle \geq -\sum_{i=1}^r \lambda_i = -1.$$

Hence $\Delta_1 + \cdots + \Delta_r \subset \nabla^*$.

Let $y \in (\Delta_1 + \cdots + \Delta_r)^*$. Put

$$\lambda_i = -\min_{x \in \Delta_i} \langle x, y \rangle.$$

Since $0 \in \Delta_i$, all λ_i are nonnegative. Since $\langle \sum_i \Delta_i, y \rangle \geq -1$, we have $\sum_i \lambda_i \leq 1$. Consider the convex function $\varphi_y = \sum_i \lambda_i \varphi_i$. For all $x \in M_{\mathbf{R}}$, we have

$$-\varphi_y(x) = \sum_{i=1}^r \lambda_i \varphi_i(x) \leq \langle x, y \rangle.$$

By Proposition 2.11, y is contained in the convex hull of all points in $N_{\mathbf{R}}$ which are equal to restrictions of φ_y on cones of maximal dimension of $\Sigma[\Delta]$. By definition of φ_y , any such a point is a sum $\sum_i \lambda_i p_i$ where $p_i \in \nabla_i$. Hence $y \in \nabla$. Thus we have proved that $(\Delta_1 + \cdots + \Delta_r)^* \subset \nabla$. \square

Since ∇ and $\Delta_1 + \cdots + \Delta_r$ are lattice polyhedra, we obtain:

Corollary 3.3 *The polyhedron ∇ is reflexive.*

Proposition 3.4 *Let $E' = \{e'_1, \dots, e'_r\} = \nabla^0$, $E'_i = \nabla_i^0$ ($i = 1, \dots, r$). Then subsets $E'_1, \dots, E'_r \subset E'$ give rise to a nef-partition of E' .*

Proof. First, we prove that $\nabla_i \cap \nabla_j = \{0\}$ for $i \neq j$. Assume that $e'_p \in \nabla_i \cap \nabla_j$. Using 2.12, we obtain that e'_p has non-negative values at all vertices e_1, \dots, e_n of Δ . On the other hand, e'_p has zero value at the interior point $0 \in \Delta$. Hence e'_p must be zero. This means that $E'_i \cap E'_j = \emptyset$ for $i \neq j$.

Let e'_p be a vertex of ∇_i . We prove that e'_p is also a vertex of ∇ . By 2.12, there exists a vertex $e_s \in \Delta_j^0$ such that $\langle e_s, e'_p \rangle = -1$. Moreover,

$$-1 = \min_{y \in \nabla} \langle e_s, y \rangle = \min_{y \in \nabla_i^0} \langle e_s, y \rangle.$$

So e'_p is also a vertex of ∇ .

Define the functions

$$\psi_i : N_{\mathbf{R}} \rightarrow \mathbf{R}, \quad i = 1, \dots, r;$$

$$\psi_i(y) = -\min_{x \in \Delta_i} \langle x, y \rangle.$$

Obviously, ψ_1, \dots, ψ_r are convex. By 2.12, $\psi_i(e'_p) = 1$ if $e'_p \in \nabla_i$, and $\psi_i(e'_p) = 0$ otherwise. We prove that restrictions of ψ_i on cones of $\Sigma[\nabla]$ are linear. It is sufficient to consider restrictions of ψ_i on cones $\sigma[\theta]$ of maximal dimension where $\theta = \nabla \cap \{y \mid \langle v, y \rangle = -1\}$ is a $(d-1)$ -dimensional face of ∇ corresponding to a vertex $v \in \nabla^* = \Delta_1 + \dots + \Delta_r$. Let $v = v_1 + \dots + v_i + \dots + v_r$, where v_i denotes a vertex of Δ_i . If we take another vertex $v'_i \neq v_i$ of Δ_i , then the sum $v = v_1 + \dots + v'_i + \dots + v_r$ represents another vertex of ∇^* . Clearly, $\langle v, y \rangle \leq \langle v', y \rangle$ for any $y \in \sigma[\theta]$, i.e., $\langle v_i, y \rangle \leq \langle v'_i, y \rangle$. Hence the restriction of ψ_i on $\sigma[\theta]$ is $-\langle v_i, y \rangle$. \square

Corollary 3.5

$$\Delta_i = \{x \in m_{\mathbf{R}} \mid \langle x, y \rangle \geq -\psi_i(y)\}, \quad i = 1, \dots, r.$$

Thus we have proved that the set of reflexive polyhedra with nef-partitions has a natural involution

$$\iota : (\Delta; E_1, \dots, E_r) \rightarrow (\nabla; E'_1, \dots, E'_r).$$

On the other hand, every nef-partition of a reflexive polyhedron Δ defines r base point free linear systems of numerically effective Cartier divisors $|D_1|, \dots, |D_r|$ such that the sum $D_1 + \dots + D_r$ is the anticanonical divisor on the Gorenstein toric Fano variety \mathbf{P}_{Δ^*} .

Conjecture 3.6 *The duality between nef-partitions of reflexive polyhedra Δ and ∇ gives rise to pairs of mirror symmetric families of Calabi-Yau complete intersections in Gorenstein toric Fano varieties \mathbf{P}_{Δ^*} and \mathbf{P}_{∇^*} .*

References

- [1] V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, Preprint Universität-GHS-Essen, 1992.
- [2] V. Batyrev, D. van Straten, *Generalized hypergeometric functions and rational curves on Calabi-Yau complete intersections in toric varieties*, Preprint 1993.