On Calabi-Yau Complete Intersections in Toric Varieties

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Abstract

We investigate Hodge-theoretic properties of Calabi-Yau complete intersections V of r semi-ample divisors in d-dimensional toric Fano varieties having at most Gorenstein singularities. Our main purpose is to show that the combinatorial duality proposed by second author agrees with the duality for Hodge numbers predicted by mirror symmetry. It is expected that the complete verification of mirror symmetry predictions for singular Calabi-Yau varieties V in arbitrary dimension demands considerations of so called *string-theoretic Hodge numbers* $h^{p,q}_{\rm st}(V)$. We restrict ourselves to the string-theoretic Hodge numbers $h^{0,q}_{\rm st}(V)$ and $h^{1,q}_{\rm st}(V)$ $(0 \leq q \leq d-r)$ which coincide with the usual Hodge numbers $h^{0,q}(\widehat{V})$ and $h^{1,q}(\widehat{V})$ of a $MPCP$-desingularization $\widehat{V}$ of V.

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1 Introduction

It was conjectured in [1] that the polar duality for reflexive polyhedra induces the mirror involution for Calabi-Yau hypersurfaces in Gorenstein toric Fano varieties. The second author has proposed a more general duality which conjecturally induces the mirror involution also for Calabi-Yau *complete intersections* in Gorenstein toric Fano varieties. The verfication of predictions for Gromov-Witten invariants of Calabi-Yau complete intersections in ordinary and weighted projective spaces [15, 14] as well as in some toric varieties [2] (see also [5, 8, 9, 11]) can be considered as partial confirmations of this generalized mirror construction.

If two *smooth* n-dimensional Calabi-Yau manifolds V and W form a mirror pair, then their Hodge numbers must satisfy the relation

$$h^{p,q}(V) = h^{n-p,q}(W), \tag{1}$$

for all $0 \le p, q \le n$.

However, the combinatorial involutions which were constructed in [1, 6] relate families of *singular* Calabi-Yau varieties. If V is a Calabi-Yau complete intersection of semi-ample divisors in a Gorenstein toric Fano variety $\mathbf{P}$, then there exists always a partial desingularization $\pi : \widehat{V} \to V$ ($MPCP$-desingularization of V) such that:

- $\widehat{V}$ is again a Calabi-Yau complete intersection of semi-ample divisors in a projective toric variety $\widehat{\mathbf{P}}$;
- $\widehat{V}$ and $\widehat{\mathbf{P}}$ have only Gorenstein terminal abelian quotient singularities.

If $\widehat{V}$ is smooth (this is always the case for $n \le 3$), we can use $\widehat{V}$ instead of V for verification of the duality (1). In general, we have to change the cohomology theory and to consider the so called *string-theoretic Hodge numbers* $h^{p,q}_{\rm st}(V)$ for singular V. For mirror pair of *singular* n-dimensional Calabi-Yau varieties V and W, we must have the duality for the string-theoretic Hodge numbers:

$$h^{p,q}_{\rm st}(V) = h^{n-p,q}_{\rm st}(W), \tag{2}$$

for all $0 \le p, q \le n$.

The main properties of the string-theoretic Hodge numbers were considered in [4]. These numbers satisfy the Poincaré duality and we have $h^{p,q}_{\rm st}(V) = h^{p,q}_{\rm st}(\widehat{V})$ for all $0 \le p, q \le n$. Moreover, one has the following properties:

Proposition 1.1 *Let $h^{p,q}(\widehat{V})$ denote the usual (p,q)-Hodge number of $\widehat{V}$. Then*
(i) $h^{p,q}_{\rm st}(V) = h^{p,q}(\widehat{V})$ *for all* $p = 0, 1$ *and* $0 \le q \le n$;
(ii) $h^{p,q}_{\rm st}(V) = h^{p,q}(\widehat{V})$ *for all* $0 \le p, q \le n$ *if* $\widehat{V}$ *is smooth.*

The main purpose of this paper is to verify the duality (2) for string-theoretic $(0,q)$ and $(1,q)$-Hodge numbers of Calabi-Yau complete intersections V and its mirror partner W predicted by the construction in [6]. According to 1.1(i), it is sufficient to check the analgous duality for the usual Hodge numbers of the corresponding $MPCP$-desingularizations $\widehat{V}$ and $\widehat{W}$.

In section 2 we remind necessary facts from the theory of toric varieties. Section 3 is devoted to basic properties of complete intersections in toric varieties. In section 4 we explain the relation between Calabi-Yau complete intersection in Gorenstein toric Fano varieties and nef-partitions of reflexive polyhedra. In section 5 we prove the duality (2) for $(0,q)$-Hodge numbers and give explicit formulas for them. It turned out to be not so easy to derive general formulas for $(1,q)$-Hodge numbers and to prove the duality (2) for $(1,q)$-Hodge numbers in full generality. In sections 6 and 7 we give explicit formulas and prove the duality only for the alternative sum of $(1,q)$-Hodge numbers; i.e., for the Euler characteristics of the sheaves of 1-forms on $\widehat{V}$ and $\widehat{W}$. In section 8, we derive explicit formulas for $(1,q)$-Hodge numbers of Calabi-Yau complete intersections of *ample* divisors. Finally, in section 9 we establish the duality (2) for all $(1,q)$-Hodge numbers of Calabi-Yau complete intersections in projective spaces and their mirror partners.

2 Basic notations and statements

Let M and $N = \mathrm{Hom}(M, \mathbf{Z})$ be dual free abelian groups of rank d, $M_{\mathbf{R}}$ and $N_{\mathbf{R}}$ the real scalar extensions of M and N, $\langle *, * \rangle \; : \; M_{\mathbf{R}} \times N_{\mathbf{R}} \rightarrow \mathbf{R}$ the canonical pairing. We consider M (resp. N) as the maximal lattice in $M_{\mathbf{R}}$ (resp. in $N_{\mathbf{R}}$). We denote by $\mathbf{T}$ the *affine algebraic torus* over $\mathbf{C}$:

$$\mathbf{T} := \mathrm{Spec}\, \mathbf{C}[M] \cong \mathbf{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}].$$

By a *lattice polyhedron* in $M_{\mathbf{R}}$ (resp. in $N_{\mathbf{R}}$) we always mean a convex polyhedron of dimension $\leq d$ whose vertices belong to M (resp. in N). The *relative interior* of Δ is the set of interior points of Δ which is considered as a subset of the minimal $\mathbf{R}$-linear affine subspace containing Δ. For any lattice polyhedron Δ, we denote by $l^*(\Delta)$ the number of lattice points in the relative interior of Δ. We set $b(\Delta) = (-1)^{\dim \Delta} l^*(\Delta)$ and denote by $l(\Delta)$ the number of lattice points in Δ.

A lattice polyhedron Δ defines the *projective toric variety* over $\mathbf{C}$:

$$\mathbf{P}_{\Delta} = \mathrm{Proj}\, S_{\Delta},$$

where S_{Δ} is the monomial subalgebra in the polynomial ring

$$\mathbf{C}[X_0, X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$$

spanned as $\mathbf{C}$-linear space by monomials $X_0^k X_1^{m_1} \cdots X_d^{m_d}$ such that the corresponding lattice point $(m_1, \ldots, m_d) \in M$ belongs to $k\Delta$. We remark that the dimension of $\mathbf{P}_{\Delta}$ equals $\dim \Delta$. If $\dim \Delta = d$, then $\mathbf{P}_{\Delta}$ can be considered as a projective compactification of $\mathbf{T}$. In the latter case, the irreducible components $\mathbf{D}_1, \ldots, \mathbf{D}_n$ of $\mathbf{P}_{\Delta} \setminus \mathbf{T}$ one-to-one correspond to $(d-1)$-dimensional faces $\Theta_1, \ldots, \Theta_n$ of Δ. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the primitive lattice points in N which define the linear equations for the affine hyperplanes containing $\Theta_1, \ldots, \Theta_n$ (in other words, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are primitive integral interior normal vectors to faces $\Theta_1, \ldots, \Theta_n$).

The is another well-known definition of toric varieties $\mathbf{P}_\Delta$ via the normal fan $\Sigma = \{\sigma_B\}$ consisting of all rational polyhedral cones

$$\sigma_B = \mathbf{R}_{\geq 0}\mathbf{e}_{i_1} + \cdots + \mathbf{R}_{\geq 0}\mathbf{e}_{i_s} \subset N_{\mathbf{R}}$$

corresponding to those subsets $B = \{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ for which the intersection $\Theta_{i_1} \cap \cdots \cap \Theta_{i_s}$ is not empty. In this situation, we also use the notation $\mathbf{P}_\Sigma$ for toric varieties associated with Σ.

It is known that every invertible sheaf $\mathcal{L}$ on any toric variety $\mathbf{P}$ admits a $\mathbf{T}$-linearization. By this reason, we shall consider in this paper only $\mathbf{T}$-linearized invertible sheaves on toric varieties. A $\mathbf{T}$-linearization of $\mathcal{L}$ induces the M-grading of the cohomology spaces

$$H^i(\mathbf{P}, \mathcal{L}) = \bigoplus_{m \in M} H^i(\mathbf{P}, \mathcal{L})(m).$$

The convex hull $\Delta(\mathcal{L})$ of all lattice points $m \in M$ for which $H^0(\mathbf{P}, \mathcal{L})(m) \neq 0$ will be called the *supporting polyhedron for global sections of* $\mathcal{L}$.

Recall the following well-known statement [7, 16]:

Theorem 2.1 *There is one-to-one correspondence*

$$\mathcal{L} \cong \mathcal{O}_{\mathbf{P}}(a_1\mathbf{D}_1 + \cdots + a_n\mathbf{D}_n) \leftrightarrow \varphi, \quad a_i = \varphi(\mathbf{e}_i),\ i = 1, \ldots, k,$$

between $\mathbf{T}$-linearized invertible sheaves $\mathcal{L}$ on $\mathbf{P} = \mathbf{P}_\Sigma$ and continious functions $\varphi : N_{\mathbf{R}} \to \mathbf{R}$ which are integral (i.e., $\varphi(N) \subset \mathbf{Z}$) and $\mathbf{R}$-linear on every cone $\sigma \in \Sigma$. Moreover, the supporting polyhedron for the global sections of $\mathcal{L} = \mathcal{L}(\varphi)$ equals

$$\Delta(\mathcal{L}) = \{x \in M_{\mathbf{R}} | \langle x, y \rangle \geq -\varphi(y) \ \text{ for all } y \in N_{\mathbf{R}}\}.$$

By *semi-ample invertible sheaf* $\mathcal{L}$ on a projective toric variety $\mathbf{P}_\Delta$ we always mean an invertible sheaf $\mathcal{L}$ generated by global sections. We have the following [7, 16]:

Proposition 2.2 *$\mathcal{L}$ is semi-ample if and only if the corresponding Σ-piecewise linear φ is upper convex. Moreover, any semi-ample invertible sheaf $\mathcal{L}$ (together with a $\mathbf{T}$-linearization) on a toric variety $\mathbf{P}$ is uniquely determined by its supporting polyhedron $\Delta(\mathcal{L})$:*

$$\mathcal{L} \cong \mathcal{O}_{\mathbf{P}}(a_1\mathbf{D}_1 + \cdots + a_n\mathbf{D}_n), \ \text{ where } \ a_i = -\min_{x \in \Delta(\mathcal{L})} \langle x, \mathbf{e}_i \rangle.$$

Definition 2.3 Let Δ and Δ' be two lattice polyhedra. Then we call a polyhedron Δ' a *Minkowski summand* of Δ if there exist a positive integer μ and a lattice polyhedron Δ'' such that $\mu\Delta = \Delta' + \Delta''$.

Using 2.4, one easily obtains:

Proposition 2.4 *A lattice polyhedron Δ' is the supporting polyhedron for global sections of a $\mathbf{T}$-linearized semi-ample invertible sheaf on $\mathbf{P}_\Delta$ if and only if Δ' is a Minkowski summand of Δ.*

In the sequel, we shall use many times the following statement:

Theorem 2.5 *Let D be a nef-Cartier divisor (or, equivalently, $\mathcal{O}_{\mathbf{P}}(D)$ is a semi-ample invertible sheaf) on a projective toric variety $\mathbf{P} = \mathbf{P}_\Delta$ Δ' the lattice polyhedron supporting the global sections $\mathcal{O}_{\mathbf{P}}(D)$. Then*

$$H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(-D)) = 0, \ \textit{if } i \neq \dim \Delta'$$

$$H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(-D)) = l^*(\Delta'), \ \textit{if } i = \dim \Delta' \ .$$

In particular the Euler characteristic $\chi(\mathcal{O}_{\mathbf{P}}(-D))$ equals $b(\Delta')$.

Proof. Let $k = \dim \Delta$. Then the invertible sheaf $\mathcal{O}_{\mathbf{P}}(D)$ defines the canonical morphism

$$\pi_D \ : \ \mathbf{P} \rightarrow \mathbf{V} = \mathrm{Proj} \oplus_{n \geq 0} H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(nD)),$$

where $\mathbf{V} \cong \mathbf{P}_\Delta$ is a k-dimensional projective toric variety, and $\mathcal{O}_{\mathbf{P}}(D) \cong \pi_D^* \mathcal{O}_{\mathbf{V}}(1)$. Therefore,

$$H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(-D)) \cong H^i(\mathbf{V}, \mathcal{O}_{\mathbf{V}}(-1)).$$

Hence $H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(-D)) = 0$ for $i > k$. On the other hand, $H^i(\mathbf{V}, \mathcal{O}_{\mathbf{V}}(-1)) = 0$ for $i < k$, and $H^k(\mathbf{V}, \mathcal{O}_{\mathbf{V}}(-1)) = l^*(\Delta)$ (see [7, 12]). □

A complex d-dimensional algebraic variety W is said to have only *toroidal singularities* if the m-adic completion of the local ring (R, m) corresponding to any point $p \in W$ is isomorphic to the m_σ-completion of a semi-group ring (S_σ, m_σ), where $S_\sigma = \mathbf{C}[\sigma \cap M]$ for some d-dimensional rational convex polyhedral cone $\sigma \subset M_{\mathbf{R}}$ with vertex at $0 \in M$, and the maximal ideal $m_\sigma \subset S_\sigma$ is generated by all non-constant monomials. We formulate without proof the following technical statement which is a generalization of the classical Bertini's theorem:

Theorem 2.6 *Let W be a complex projective algebraic variety with only toroidal singularities, $\mathcal{L}$ a semi-ample invertible sheaf on W, $D \subset W$ the set of zeros of a generic global section of $\mathcal{L}$. Then D again has only toroidal singularities. In particular, D is irreducible if $\dim D > 0$.*

3 Complete intersections

Let $\Delta_1, \ldots, \Delta_r$ be lattice polyhedra in $M_{\mathbf{R}}$. In this section $\mathbf{P}$ denotes the d-dimensional toric variety $\mathbf{P}_{\Delta}$, where $\Delta = \Delta_1 + \cdots + \Delta_r$ (without loss of generality, we assume $\dim \Delta = \dim M_{\mathbf{R}} = d$). By 2.4, the lattice polyhedron Δ_i is the support polyhedron for global sections of some semi-ample invertible sheaf $\mathcal{L}_i$ on $\mathbf{P}$ $(i = 1, \ldots, r)$. We identify $H^0(\mathbf{P}, \mathcal{L}_i)$ with the space of all Laurent polynomials $f_i \in \mathbf{C}[M]$ having Δ_i as the Newton polyhedron.

Definition 3.1 Lattice polyhedra $\Delta_1, \ldots, \Delta_r$ are called k-dependent if there exist $n > 0$ and n-element subset

$$\{\Delta_{i_1}, \ldots, \Delta_{i_n}\} \subset \{\Delta_1, \ldots, \Delta_r\},$$

such that

$$\dim (\Delta_{i_1} + \cdots + \Delta_{i_n}) < n + k - 1.$$

Lattice polyhedra which are not k-dependent will be called k-independent.

Remark 3.2 It follows immediately from definition that the k-independence of lattice polyhedra $\Delta_1, \ldots, \Delta_r$ implies the l-independence for $1 \leq l \leq k$.

Theorem 3.3 *Let $Z_f \subset \mathbf{T}$ be a complete intersection of r affine hypersurfaces $Z_{f_1}, \ldots, Z_{f_r}$ defined by a general system of the equations $f_1 = \cdots = f_r = 0$ where f_i is a general Laurent polynomial with the Newton polyhedra Δ_i $(i = 1, \ldots, r)$. Denote by Z_i the closure of Z_{f_i} in $\mathbf{P}$ $(i = 1, \ldots, r)$. Let*

$$V = Z_1 \cap \cdots \cap Z_r.$$

Then the following statements hold:
(i) *V is non-empty if and only if $\Delta_1, \ldots, \Delta_r$ are* 1*-independent;*
(ii) *if $\Delta_1, \ldots, \Delta_r$ are* 2*-independent, then V is irreducible;*
(iii) *if $\Delta_1, \ldots, \Delta_r$ are k-independent $(k \geq 3)$, then $h^1(\mathcal{O}_V) = \cdots = h^{k-2}(\mathcal{O}_V) =$* 0*.*

Proof. Denote by $\mathcal{K}^*$ the Koszul complex

$$\mathcal{O}_{\mathbf{P}}(-Z_1 - \cdots - Z_r) \rightarrow \cdots \rightarrow \sum_{i<j} \mathcal{O}_{\mathbf{P}}(-Z_i - Z_j) \rightarrow \sum_i \mathcal{O}_{\mathbf{P}}(-Z_i) \rightarrow \mathcal{O}_{\mathbf{P}}.$$

There are two spectral sequences $'E$ and $''E$ converging to the hypercohomology $\mathbf{H}^*(\mathbf{P}, \mathcal{K}^*)$ (cf. [10]):

$$'E_2^{p,q} = H^p(\mathbf{P}, \mathcal{H}^q(\mathcal{K}^*)),$$

$$''E_2^{p,q} = H^q(\mathbf{P}, H^p(\mathcal{K}^*)).$$

Since $\mathcal{K}^*$ is which an acyclic resolution of $\mathcal{O}_V$, we have $\mathcal{H}^q(\mathcal{K}^*) = 0$ for $q \neq r$, and $\mathcal{H}^r(\mathcal{K}^*)) = \mathcal{O}_V$, the first spectral sequence degenerates and we get the isomorphisms

$$\mathbf{H}^{r+p}(\mathbf{P}, \mathcal{K}^*) \cong H^p(\mathbf{P}, \mathcal{O}_V) \cong H^p(V, \mathcal{O}_V).$$

On the other hand, the second spectral sequence does not degenerate in general. The term $''E_2^{p,q}$ is the cohomology of the bicomplex

$$''E_1^{p,q} = \bigoplus_{\{i_1,\ldots,i_{r-p}\}} H^q(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(-Z_{i_1} - \cdots - Z_{i_{r-p}})).$$

We can compute $''E_1^{p,q}$ using Proposition 2.5. The statements (i)-(iii) will follow from the consideration of $''E$.

(i) It is clear that V is empty if and only if Z_f is empty. Assume that $\Delta_1, \ldots, \Delta_r$ are 1-dependent. Then there exists $n \geq 1$ such that $d' = \dim(\Delta_{i_1} + \cdots + \Delta_{i_n}) < n$. This means that we can choose the coordinates $X_1, \ldots, X_d$ on $\mathbf{T}$ in such a way that the polynomials $f_{i_1}, \ldots, f_{i_n}$ depends only on some d' of them. Therefore, we obtain an overdetermined system $f_{i_1} = \cdots = f_{i_n} = 0$; i.e., V is empty.

Assume now that V is empty, i.e., $\mathcal{K}^*$ is acyclic. If some Δ_i is 0-dimensional, then $\Delta_1, \ldots, \Delta_r$ are 1-dependent, and everything is proved. Otherwise, one has the non-zero cycle in $''E_1^{r,0} \cong {''E_2^{r,0}} \cong H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}) \cong \mathbf{C}$ which must be killed by some next non-zero differential

$$d_l \;:\; {''E_l^{r-l,l-1}} \to E_l^{r,0} \; (l \geq 2).$$

Therefore $''E_l^{r-l,l-1} \neq 0$ for some $2 \leq l \leq r$. This implies that there exists an l-element subset $\{i_1, \ldots, i_l\} \subset \{1, \ldots, r\}$ such that

$$H^{l-1}(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(-Z_{i_1} - \cdots - Z_{i_l})) \neq 0.$$

Applying 2.5, we see that there exists an l-element subset

$$\{\Delta_{i_1}, \ldots, \Delta_{i_l}\} \subset \{\Delta_1, \ldots, \Delta_r\}$$

such that

$$\dim(\Delta_{i_1} + \cdots + \Delta_{i_l}) = l - 1,$$

i.e., $\Delta_1, \ldots, \Delta_r$ are 1-dependent.

(ii) Assume that $\Delta_1, \ldots, \Delta_r$ are 2-independent. By 2.5, one has

$$''E^{r,1} = {''E_1^{r-s,s-1}} = {''E_1^{r-s,s}} = 0 \;\text{ for } 1 \leq s \leq r.$$

Hence

$$''E_l^{r,1} = {''E_l^{r-s,s-1}} = {''E_l^{r-s,s}} = 0 \;\text{ for } 1 \leq s \leq r,\, l \geq 1.$$

So $\mathbf{H}^r(\mathbf{P}, \mathcal{K}^*) \cong \mathbf{C} \cong H^0(V, \mathcal{O}_V)$. Therefore V is connected. By 2.6, V is irreducible.

(iii) Assume that $\Delta_1, \ldots, \Delta_r$ are k-independent ($k \geq 3$). By the same arguments using 2.5, we obtain $\mathbf{H}^r(\mathbf{P}, \mathcal{K}^*) \cong \mathbf{C}$, and $\mathbf{H}^{r+1}(\mathbf{P}, \mathcal{K}^*) = \cdots = \mathbf{H}^{r+k-2}(\mathbf{P}, \mathcal{K}^*) = 0$. This implies $h^1(\mathcal{O}_V) = \cdots = h^{k-2}(\mathcal{O}_V) = 0$. □

Remark 3.4 Khovanskiî announced ([13] p.41) that the statement 3.3(i) was first discovered and proved by D. Bernshtein using properties of mixed volumes in the following equivalent form:

The affine variety Z_f is empty if and only if there exists an l-dimensional affine subspace of $M_{\mathbf{R}}$ containing affine translates of some $l+1$ polyhedra from $\{\Delta_1, \ldots, \Delta_r\}$.

Corollary 3.5 *Assume that all lattice polyhedra $\Delta_1, \ldots, \Delta_r$ have positive dimension, $l^*(\Delta_1 + \cdots + \Delta_r) = 1$, $d = \dim(\Delta_1 + \cdots + \Delta_r)$, and for any proper subset $\{k_1, \ldots, k_s\} \subset \{1, \ldots, r\}$ one has $l^*(\Delta_{k_1} + \cdots + \Delta_{k_s}) = 0$. Then the following statements hold:*

(i) *V is empty if and only if $d = r - 1$;*

(ii) *V consists of 2 distinct points if and only if $d = r$;*

(iii) *V is a smooth irreducible curve of genus 1 if and only if $d = r + 1$.*

(iv) *V is a nonempty irreducible variety of dimension $d - r \geq 2$ having the property $h^1(\mathcal{O}_V) = \cdots = h^{d-r-1}(\mathcal{O}_V) = 0$ and $h^0(\mathcal{O}_V) = h^{d-r}(\mathcal{O}_V) = 1$ if and only if $d \geq r + 2$.*

Proof. It follows from Proposition 2.5 and our assumptions that $''E_1^{r,0}$ and $''E_1^{0,d}$ have dimension 1 and all remaining spaces $''E_1^{p,q}$ are zero.

(i) If $r = d + 1$, then Z_f is empty by dimension arguments. On the other hand, if Z_f is empty, then $''E_l^{p,q}$ becomes acyclic for $l \gg 0$. Note that the only nontrivial one-dimensional spaces $''E_1^{r,0}$ and $''E_1^{0,d}$ can kill each other only via the non-zero differential

$$d_l \; : \; ''E_l^{0,d} \to ''E_l^{r,0}$$

where $r = d + 1$ and $l = r$, i.e., Z_f is empty if and only if $r = d + 1$.

Assume $r > d + 1$. Then

$$\mathbf{C} \cong ''E_1^{0,d} \cong ''E_\infty^{0,d} \cong \mathbf{H}^d(\mathbf{P}, \mathcal{K}^*).$$

On the other hand, the isomorphism $\mathbf{H}^{r+p}(\mathbf{P}, \mathcal{K}^*) \cong H^p(C, \mathcal{O}_V)$ implies $\mathbf{H}^i(\mathbf{P}, \mathcal{K}^*) = 0$ for $i < r$. Contradiction.

(ii) If $r = d$, then

$$\mathbf{C}^2 \cong ''E_1^{0,d} \oplus ''E_1^{r,0} \cong ''E_\infty^{0,d} \oplus ''E_\infty^{r,0} \cong H^0(V, \mathcal{O}_V).$$

Since V is nonempty, one has $\dim V = 0$, i.e., V consists of 2 distinct points.

(iii)-(iv) For $r \leq d - 1$, we have isomorphisms

$$\mathbf{C} \cong ''E_1^{r,0} \cong H^0(V, \mathcal{O}_V),$$

$$\mathbf{C} \cong ''E_1^{0,d} \cong H^{d-r}(V, \mathcal{O}_V),$$

and

$$0 \cong ''E_1^{p,q} \cong H^{d-r}(V, \mathcal{O}_V) \;\text{ if } p + q \neq r, d.$$

This proves (iii)-(iv). □

4 Calabi-Yau varieties and nef-partitions

Definition 4.1 A lattice polyhedron $\Delta \subset M_{\mathbf{R}}$ is called *reflexive* if $\mathbf{P}_\Delta$ has only Gorenstein singularities and $\mathcal{O}_{\mathbf{P}}(1)$ is isomorphic to the anticanonical sheaf which is considered together with its natural $\mathbf{T}$-linearization. In this case, we call $\mathbf{P}_\Delta$ a *Gorenstein toric Fano variety.*

Remark 4.2 Since the $\mathbf{T}$-linearized anticanonical sheaf on $\mathbf{P}_\Delta$ is isomorphic to $\mathcal{O}_{\mathbf{P}_\Delta}(\mathbf{D}_1+\cdots+\mathbf{D}_n)$, it follows from the above definition that any reflexive polyhedron Δ has $0 \in M$ as the unique interior lattice point. Moreover,

$$\Delta = \{x \in M_{\mathbf{R}} | \langle x, \mathbf{e}_i \rangle \geq -1,\ i = 1, \ldots, n\},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the primitive integral interior normal vectors to codimension-1 faces $\Theta_1, \ldots, \Theta_n$ of Δ. These properties of reflexive polyhedra were used in another their definition [1].

Theorem 4.3 [1] *Let Δ be a reflexive polyhedron as above. Then the convex hull $\Delta^* \subset N_{\mathbf{R}}$ of the lattice vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is again a reflexive polyhedron. Moreover, $(\Delta^*)^* = \Delta$.*

Definition 4.4 The lattice polyhedron Δ^* is called *dual* to Δ.

Using the adjunction formula, 2.2 and 2.4, we immediately obtain:

Proposition 4.5 *Assume that a d-dimensional Gorenstein toric Fano variety $\mathbf{P}_\Delta$ contains a $(d-r)$-dimensional complete intersection V $(r < d)$ of r semi-ample Cartier divisors $Z_1, \ldots, Z_r$ such that the canonical (or, equivalently, dualizing) sheaf of V is trivial. Then there exist lattice polyhedra $\Delta_1, \ldots, \Delta_r$ such that*

$$\Delta = \Delta_1 + \cdots + \Delta_r.$$

Definition 4.6 Let $\Sigma \subset N_{\mathbf{R}}$ be the normal fan defining a Gorenstein toric Fano variety $\mathbf{P}_\Delta$, $\varphi : N_{\mathbf{R}} \to \mathbf{R}$ the integral upper convex Σ-piecewise linear function corresponding to the $\mathbf{T}$-linearized anticanonical sheaf (by 2.1, $\varphi(\mathbf{e}_i) = 1$, $i = 1, \ldots, n$), $\Delta = \Delta_1 + \cdots + \Delta_r$ a decomposition of Δ into a Minkowski sum of r lattice polyhedra Δ_j $(j = 1, \ldots, r)$, $\varphi = \varphi_1 + \cdots + \varphi_r$ the induced decomposition of φ into the sum of integral upper convex Σ-piecewise linear functions φ_j (see 2.2). Then $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ is called a *nef-partition of* Δ if $\varphi_j(\mathbf{e}_i) \in \{0, 1\}$ for $i = 1, \ldots, n$, $j = 1, \ldots, r$.

Remark 4.7 There are reflexive polyhedra Δ which admit decompositions into Minkowski sum of two lattice polyhedra, but do not admit any nef-partition. For example, let $\Delta \subset \mathbf{R}^2$ be the convex hull of 4 lattice points: $(1,0)$, $(-1,0)$, $(0,1)$ and $(0,-1)$. Then Δ is a 2-dimensional reflexive polyhedron which admits a decomposition into Minkowski sum of two 1-dimensional polyhedra Δ_1 and Δ_2, where Δ_1 is the convex hull of $(-1,0)$ and $(0,-1)$, and Δ_2 is the convex hull of $(0,0)$ and $(1,1)$. However, Δ does not admit any nef-partition.

Remark 4.8 The notation in 4.6 are a little bit different from definitions and notations in [3, 6], but they are essentially equivalent.

Definition 4.9 Let $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ be a nef-partition of a reflexive polyhedron Δ. We define the lattice polyhedron $\nabla_j \subset N_{\mathbf{R}}$ $(j = 1, \ldots, r)$ as the convex hull of $0 \in N$ and all lattice vectors $\mathbf{e}_i \in \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ such that $\varphi_j(\mathbf{e}_i) = 1$.

Theorem 4.10 [6] *The Minkowski sum $\nabla = \nabla_1 + \cdots + \nabla_r$ is a reflexive polyhedron. Moreover, $\Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$ is a nef-partition of ∇, and one has also*

$$\nabla^* = \mathrm{Conv}\{\Delta_1, \ldots, \Delta_r\},$$

$$\Delta^* = \mathrm{Conv}\{\nabla_1, \ldots, \nabla_r\}.$$

Definition 4.11 The nef-partition $\Pi(\nabla)$ is called the *dual nef-partition.*

Example 4.12 Let $\Delta_j \subset M_{\mathbf{R}}^{(j)}$ is a reflexive polyhedron $(j = 1, \ldots, r)$, $\nabla_j := \Delta_j^* \subset N_{\mathbf{R}}^{(j)}$ the corresponding dual reflexive polyhedron $(j = 1, \ldots, r)$. We set $M_{\mathbf{R}} := M_{\mathbf{R}}^{(1)} \oplus \cdots \oplus M_{\mathbf{R}}^{(r)}$, $N_{\mathbf{R}} := N_{\mathbf{R}}^{(1)} \oplus \cdots \oplus N_{\mathbf{R}}^{(r)}$. Then $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ is a nef-partition of the reflexive polyhedron $\Delta = \Delta_1 + \cdots + \Delta_r \subset M_{\mathbf{R}}$, and $\Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$ is the dual nef-partition of the reflexive polyhedron $\nabla = \nabla_1 + \cdots + \nabla_r \subset N_{\mathbf{R}}$.

By [1], a maximal projective triangulation $\mathcal{T}$ of $\Delta^* \subset N_{\mathbf{R}}$ defines a *MPCP*-desingularization $\pi_{\mathcal{T}} : \widehat{\mathbf{P}}_{\Delta} \to \mathbf{P}_{\Delta}$. The $\mathbf{T}$-invariant divisors on $\widehat{\mathbf{P}}_{\Delta}$ one-to-one correspond to lattice points on the boundary $\partial\Delta^*$.

Definition 4.13 Denote by $\mathcal{V}(\Delta^*)$ the set $\partial\Delta^* \cap N$. If v is a lattice point in $\mathcal{V}(\Delta^*)$, then the corresponding toric divisor on $\widehat{\mathbf{P}}_{\Delta}$ will be denoted by $\mathbf{D}(v)$.

Since the anticanonical sheaf on $\widehat{\mathbf{P}}_{\Delta}$ is semi-ample, $\Delta_1, \ldots, \Delta_r$ are supporting polyhedra for global sections of some semi-ample invertible sheaves $\widehat{\mathcal{L}}_1, \ldots, \widehat{\mathcal{L}}_r$ on $\widehat{\mathbf{P}}_{\Delta}$.

Definition 4.14 Let $\Pi(\Delta)$ be a nef-partition, $Z_f \subset \mathbf{T}$ a complete intersection of r general affine hypersurfaces $Z_{f_1}, \ldots, Z_{f_r}$ defined by a general system of the polynomial equations $f_1 = \cdots = f_r = 0$ where f_i is a general Laurent polynomial with the Newton polyhedra Δ_i $(i = 1, \ldots, r)$. Denote by $\widehat{Z}_i$ the closure of Z_{f_i} in $\widehat{\mathbf{P}}_{\Delta}$ $(i = 1, \ldots, r)$. Define V (resp. $\widehat{V}$) as the closure of Z_f in $\mathbf{P}_{\Delta}$ (resp. in $\widehat{\mathbf{P}}_{\Delta}$; i.e., $\widehat{V} = \widehat{Z}_1 \cap \cdots \cap \widehat{Z}_r$). If $\Pi(\nabla)$ is the dual nef-partition, then the corresponding general complete intersection in $\mathbf{P}_{\nabla}$ (resp. in $\widehat{\mathbf{P}}_{\nabla}$) will be denoted by W (resp. $\widehat{W}$).

By the adjunction formula and 2.6, one immediately obtains:

Proposition 4.15 *In the above notations, assume that V is nonempty and irreducible. Then $\widehat{V}$ is an irreducible $(d - r)$-dimensional projective algebraic variety with at most Gorenstein terminal abelian quotient singularities and trivial canonical class. In particular, $\widehat{V}$ is smooth if $d - r \leq 3$.*

In the next section we prove the following:

Theorem 4.16 *Let $\Pi(\Delta)$ be a nef-partition, $\Pi(\nabla)$ the dual nef-partition, V (resp. W) corresponding to $\Pi(\Delta)$ (resp. to $\Pi(\nabla)$) general complete intersections in $\mathbf{P}_\Delta$ (resp. in $\mathbf{P}_\nabla$). Then the following statements hold:*
(i) *V is nonempty if and only if W is nonempty;*
(ii) *V is irreducible if and only if W is irreducible;*
(iii) *$h^i(\mathcal{O}_V) = h^i(\mathcal{O}_W)$ for $0 \leq i \leq d-r$.*

Corollary 4.17 *Let $\Pi(\Delta)$ be a nef-partition, $\Pi(\nabla)$ the dual nef-partition, $\widehat{V}$ (resp. $\widehat{W}$) corresponding to $\Pi(\Delta)$ (resp. $\Pi(\nabla)$) general complete intersections in $\widehat{\mathbf{P}}_\Delta$ (resp. in $\widehat{\mathbf{P}}_\nabla$). Then $\widehat{V}$ is an irreducible $(d-r)$-dimensional projective algebraic variety with at most Gorenstein terminal abelian quotient singularities and trivial canonical class if and only if $\widehat{W}$ has the same properties.*

5 Semi-simplicity principle for nef-partitions

Let Δ be a d-dimensional reflexive polyhedron in $M_{\mathbf{R}}$, $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ a nef-partition of Δ.

Definition 5.1 We say that $\Pi(\Delta)$ *splits into a direct sum*

$$\Pi(\Delta) = \Pi(\Delta^{(1)}) \oplus \cdots \oplus \Pi(\Delta^{(k)})$$

if there exist convex lattice polyhedra $\Delta^{(1)}, \ldots, \Delta^{(k)} \subset \Delta$ satisfying the conditions
(i) $d = \dim \Delta^{(1)} + \cdots + \dim \Delta^{(k)}$ and

$$\Delta = \Delta^{(1)} + \cdots + \Delta^{(k)};$$

(ii) for $1 \leq i \leq k$, the lattice point 0 is cointained in the relative interior of $\Delta^{(i)}$, $\Delta^{(i)}$ is reflexive, and

$$\Pi(\Delta^{(i)}) = \{\Delta_j \subset \Delta \mid \Delta_j \subset \Delta^{(i)}\}$$

is a nef-partition of $\Delta^{(i)}$.

Definition 5.2 Assume that $\Pi(\Delta)$ splits into a direct sum

$$\Pi(\Delta) = \Pi(\Delta^{(1)}) \oplus \cdots \oplus \Pi(\Delta^{(k)}).$$

Denote by $M_{\mathbf{R}}^{(i)}$ the minimal $\mathbf{R}$-linear subspace of $M_{\mathbf{R}}$ containing $\Delta^{(i)}$ ($i = 1, \ldots, k$). We set also $M^{(i)} = M \cap M_{\mathbf{R}}^{(i)}$ ($i = 1, \ldots, k$). We say that $\Pi(\Delta)$ splits into the direct sum *over* $\mathbf{Z}$ if

$$M = M^{(1)} \oplus \cdots \oplus M^{(k)}.$$

It is easy to see the following:

Proposition 5.3 *Assume that a nef-partition* $\Pi(\Delta)$ *splits over* $\mathbf{Z}$ *into a direct sum* $\Pi(\Delta^{(1)}) \oplus \cdots \oplus \Pi(\Delta^{(k)})$. *Then the Gorenstein toric Fano variety* $\mathbf{P}_{\Delta}$ *is the product of the Gorenstein toric Fano varieties* $\mathbf{P}_{\Delta^{(i)}}$ $(i = 1, \ldots, k)$. *Moreover,*

$$V = V^{(1)} \times \cdots \times V^{(k)},$$

$$\widehat{V} = \widehat{V}^{(1)} \times \cdots \times \widehat{V}^{(k)},$$

where $V^{(i)}$ *(resp.* $\widehat{V}^{(i)}$*) is the Calabi-Yau complete intersections defined by the nef-partition* $\Pi(\Delta^{(i)})$ *in* $\mathbf{P}_{\Delta^{(i)}}$ *(resp. in* $\widehat{\mathbf{P}}_{\Delta^{(i)}}$*),* $i = 1, \ldots, k$.

Remark 5.4 In 4.12 we gave a simplest example of a nef-partition $\Pi(\Delta)$ which splits into a direct sum $\{\Delta_1\} \oplus \cdots \oplus \{\Delta_r\}$ over $\mathbf{Z}$. In general situation, if $\Pi(\Delta)$ splits into a direct sum $\Pi(\Delta) = \Pi(\Delta^{(1)}) \oplus \cdots \oplus \Pi(\Delta^{(k)})$, then $M^{(1)} \oplus \cdots \oplus M^{(k)}$ is a sublattice of finite index in M, but not necessarily the lattice M itself.

Example 5.5 Let $\Delta = \Delta_1 + \Delta_2 \subset \mathbf{R}^4$ where

$$\Delta_1 = \mathrm{Conv}\{(1,0,0,0), (0,1,0,0), (-1,0,0,0), (0,-1,0,0)\};$$

$$\Delta_2 = \mathrm{Conv}\{(0,0,1,0), (0,0,0,1), (0,0,-1,0), (0,0,0,-1)\}.$$

We define the lattice $M \subset \mathbf{R}^4$ as

$$M = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + \mathbf{Z}^4.$$

Then Δ splits into direct sum $\Pi(\Delta) = \{\Delta_1\} \oplus \{\Delta_2\}$. But it is not a splitting over $\mathbf{Z}$, because $M_1 \oplus M_2$ is a sublattice of index 2 in M.

Definition 5.6 We call a nef-partition $\Pi(\Delta)$ *reducible* if there exists a subset

$$\{k_1, \ldots, k_s\} \subset \{1, \ldots, r\} \ (0 < s < r)$$

such that $\Delta_{k_1} + \ldots + \Delta_{k_s}$ contains 0 in its relative interior. Nef-partitions which are not reducible are called *irreducible*.

Remark 5.7 (i) Notice that $\Delta_{k_1} + \ldots + \Delta_{k_s}$ contains 0 in its relative interior if and only if

$$l^*(\Delta_{k_1} + \ldots + \Delta_{k_s}) = 1.$$

(ii) It is clear that an irreducible nef-partition has no nontrivial splitting into a direct sum.

Theorem 5.8 *Any nef-partition* $\Pi(\Delta)$ *has a unique decomposition into direct sum of irreducible nef-partitions.*

Proof. Assume that $\Pi(\Delta)$ is reducible. Choose lattice polyhedra $\Delta_{k_1}, \ldots, \Delta_{k_s}$ such that $\Delta' = \Delta_{k_1} + \ldots + \Delta_{k_s}$ contains 0 in its relative interior. Denote

$$\Lambda' = \mathrm{Conv}(\Delta_{k_1}, \ldots, \Delta_{k_s}).$$

It is clear that Λ' also contains 0 in its relative interior and has the same dimension as $\Delta_{k_1} + \ldots + \Delta_{k_s}$. Denote by $M'_{\mathbf{R}}$ the minimal linear subspace in $M_{\mathbf{R}}$ containing Δ'. Denote by $\psi_1, \ldots, \psi_r$ the piecewise linear functions on $M_{\mathbf{R}}$ which determine the dual nef-partition $\Pi(\nabla)$ (see 4.6). We set $\psi' = \psi_{k_1} + \cdots + \psi_{k_s}$, $\psi'' = \psi_1 + \cdots + \psi_r - \psi' = \psi_{j_1} + \cdots + \psi_{j_{r-s}}$. Then

$$\Lambda' = \{x \in M'_{\mathbf{R}} \mid \psi'(x) \leq 1\}.$$

Since ψ' is a non-negative integral convex piecewise linear function having value 1 at the relative boundary $\partial\Lambda' \subset M'_{\mathbf{R}}$, the polyhedron Λ' is reflexive.

Since ψ' is convex and non-negative

$$C' = \{x \in M_{\mathbf{R}} \mid \psi'(x) = 0\}$$

is a convex subset of $M_{\mathbf{R}}$ containing all non-negative linear combinations of vertices of $\Delta_{j_1}, \ldots, \Delta_{j_{r-s}}$. One has $M'_{\mathbf{R}} \cap C' = 0$. Assume that 0 is not contained in the relative interior of C'. Then, by separateness theorem for convex sets, there exists a non-zero element $y' \in N_{\mathbf{R}}$ such that $\langle M'_{\mathbf{R}}, y' \rangle = 0$, and $\langle C', y' \rangle \geq 0$. Therefore $\langle v, y' \rangle \geq 0$ for all vertices v of $\Delta_1, \ldots, \Delta_r$. Contradiction. Thus 0 is contained in the relative interior of C'. Since $\mathbf{R}_{\geq 0}C' \subset C'$, C' is a linear subspace and $M_{\mathbf{R}} = M'_{\mathbf{R}} \oplus C'$. We put $M''_{\mathbf{R}} = C'$.

Define

$$\Lambda'' = \{x \in M''_{\mathbf{R}} \mid \psi''(x) \leq 1\}.$$

Then $\Lambda'' = \mathrm{Conv}(\Delta_{j_1}, \ldots, \Delta_{j_{r-s}})$, and $\Delta'' := \Delta_{j_1} \oplus \cdots \oplus \Delta_{j_{r-s}})$ contains 0 in its relative interior. Therefore Λ'' is also reflexive.

Thus we obtain $\nabla^* = \mathrm{Conv}(\Lambda', \Lambda'')$, $\dim \Lambda' + \dim \Lambda'' = d$, and

$$\Pi(\Delta') = \{\Delta_{k_1}, \ldots, \Delta_{k_s}\},$$

$$\Pi(\Delta'') = \{\Delta_{j_1}, \ldots, \Delta_{j_{r-s}}\}$$

are nef-partitions of the reflexive polyhedra Δ' and Δ'' corresponding to restricitions of $\psi_{k_1}, \ldots, \psi_{k_s}$ (resp. of $\psi_{j_1}, \ldots, \psi_{j_{r-s}}$) on $M'_{\mathbf{R}}$ (resp. on $M''_{\mathbf{R}}$). Now the statement of Theorem 5.8 follows by induction. □

Proof of Theorem 4.16. Note that V is nonempty if and only if $h^0(V, \mathcal{O}_V) \neq 0$. By 2.6, V is irreducible if and only if $h^0(V, \mathcal{O}_V) = 1$. Therefore, (i) and (ii) follow from (iii). By 3.5 and 5.7(i), the spectral sequence $''E^{p,q}$ which computes the cohomology of $\mathcal{O}_V$ degenerates at $''E_2^{p,q}$ for all irreducible nef-partitions. By 5.3, we obtain the degeneration of $''E^{p,q}$ at $''E_2^{p,q}$ for all nef-partitions $\Pi(\Delta)$ which split over $\mathbf{Z}$ into a sum of irreducible nef-partitions.

According to Theorem 5.8, in general situation we have a finite covering

$$\pi \; : \; \mathbf{P}_{\Delta} \to \mathbf{P}_{\Delta^{(1)}} \times \cdots \times \mathbf{P}_{\Delta^{(k)}}$$

where the degree of π equals the index of $M^{(1)} \oplus \cdots \oplus M^{(k)}$ in M, and $\Pi(\Delta^{(1)}), \ldots, \Pi(\Delta^{(k)})$ are irreducible nef-partitions. Let

$$\tilde{\pi} \,;\, V \to \tilde{V} := V^{(1)} \times \cdots \times V^{(k)}$$

the finite covering induced by π. Then we obtain the canonical homomorphism of the spectral sequences:

$$\tilde{\pi}^* \;:\; {}''\tilde{E}^{p,q} \to {}'' E^{p,q}.$$

By 5.7(i), we have the canonical isomorphism

$${}''\tilde{E}_1^{p,q} \cong\!\!\to {}'' E_1^{p,q}.$$

Since ${}''\tilde{E}^{p,q}$ degenerates at ${}''\tilde{E}_1^{p,q}$ (see the above arguments), ${}''E^{p,q}$ also degenerates at ${}''E_2^{p,q}$. Therefore, $h^i(\mathcal{O}_V) = h^i(\mathcal{O}_{\tilde{V}})$ $(i = 1, \ldots, d-r)$. Analogously, $h^i(\mathcal{O}_W) = h^i(\mathcal{O}_{\tilde{W}})$ $(i = 1, \ldots, d-r)$. Thus, we have reduced (iii) to already known case. $\square$

Corollary 5.9 *Let $I = \{1, \ldots, r\}$. Denote by $|J|$ the cardinality of a subset $J \subset I$. Define*

$$E(\Delta, t) = \sum_{i=0}^{d-r} h^i(\mathcal{O}_V) t^i.$$

Then

$$E(\Delta, t) = \sum_{J \subset I} l^*(\sum_{j \in J} \Delta_j) t^{\dim\left(\sum_{j\in J} \Delta_j\right) - |J|}.$$

By Serre duality, and using the natural isomorphisms $H^i(V, \mathcal{O}_V) \cong H^i(\widehat{V}, \mathcal{O}_{\widehat{V}})$ $(i = 1, \ldots, d-r)$, we also obtain:

Corollary 5.10 *Assume that $\widehat{V}$ is nonempty and irreducible. Then*

$$h^i(\widehat{V}, \mathcal{O}_{\widehat{V}}) = h^{d-r-i}(\widehat{W}, \mathcal{O}_{\widehat{W}}).$$

6 $\chi(\Omega^1)$ for complete intersections

Now we want to calculate the Euler characteristic of $\Omega^1_{\widehat{V}}$ of a Calabi-Yau complete intersection $\widehat{V} = \widehat{Z}_1 \cap \cdots \cap \widehat{Z}_r$ in a $MPCP$-desingularization $\widehat{\mathbf{P}} := \widehat{\mathbf{P}}_\Delta$ of a Gorenstein toric Fano variety $\mathbf{P} := \mathbf{P}_\Delta$.

One has the standard exact sequence for a complete intersection:

$$0 \to \mathcal{O}_{\widehat{V}}(-\widehat{Z}_1) \oplus \cdots \oplus \mathcal{O}_{\widehat{V}}(-\widehat{Z}_r) \to \Omega^1_{\widehat{\mathbf{P}}} \,|_{\widehat{V}} \to \Omega^1_{\widehat{V}} \to 0.$$

On the other hand, for toric varieties with only quotient singularities there exists the exact sequence [7, 16]:

$$0 \to \Omega^1_{\widehat{\mathbf{P}}} \to \mathcal{O}^d_{\widehat{\mathbf{P}}} \to \bigoplus_{v \in \mathcal{V}(\Delta^*)} \mathcal{O}_{\mathbf{D}(v)} \to 0.$$

By transversality of $\widehat{Z}_1, \ldots, \widehat{Z}_r$ to all strata on $\widehat{\mathbf{P}}$ we can restrict the last exact sequence on $\widehat{V}$ without obtaining additional Tor-sheaves:

$$0 \to \Omega^1_{\widehat{\mathbf{P}}} \mid_{\widehat{V}} \to \mathcal{O}^d_{\widehat{V}} \to \bigoplus_{v \in \mathcal{V}(\Delta^*)} \mathcal{O}_{\mathbf{D}(v) \cap \widehat{V}} \to 0.$$

Consequently, we obtain:

Proposition 6.1

$$\chi(\Omega^1_{\widehat{V}}) = \chi(\mathcal{O}^d_{\widehat{V}}) - \sum_{j=1}^{r} \chi(\mathcal{O}_{\widehat{V}}(-\widehat{Z}_j)) - \sum_{v \in \mathcal{V}(\Delta^*)} \chi(\mathcal{O}_{\mathbf{D}(v) \cap \widehat{V}}).$$

In order to compute $\chi(\mathcal{O}_{\widehat{V}}(-\widehat{Z}_j))$, we consider the Koszul resolution

$$0 \to \mathcal{O}_{\widehat{\mathbf{P}}}(-\widehat{Z}_1 - \cdots - \widehat{Z}_r) \to \cdots$$

$$\cdots \to \sum_{j<k} \mathcal{O}_{\widehat{\mathbf{P}}}(-\widehat{Z}_j - \widehat{Z}_k) \to \sum_j \mathcal{O}_{\widehat{\mathbf{P}}}(-\widehat{Z}_j) \to \mathcal{O}_{\widehat{\mathbf{P}}} \to \mathcal{O}_{\widehat{V}} \to 0.$$

Tensoring it by $\mathcal{O}_{\widehat{\mathbf{P}}}(-\widehat{Z}_i)$ and using 2.5, we get

Proposition 6.2

$$\chi(\mathcal{O}_{\widehat{V}}(-\widehat{Z}_i)) = -\sum_j b(\Delta_i + \Delta_j) + \sum_{j<k} b(\Delta_i + \Delta_j + \Delta_k) - \cdots$$

$$\cdots + (-1)^r b(\Delta_i + \Delta_1 + \cdots + \Delta_r).$$

For the computation of $\chi(\mathcal{O}_{\mathbf{D}(v) \cap \widehat{V}})$ we need the following:

Proposition 6.3 *Let v an arbitrary lattice point in $\mathcal{V}(\Delta^*)$, $\Gamma(v)$ the minimal face of Δ^* containing v. Then $\Gamma(v)$ is a face of a polyhedron ∇_i for some i ($1 \leq i \leq r$).*

Proof. If v is a vertex of Δ^*, then the statement is evident, because $\Delta^* = \mathrm{Conv}\{\nabla_1, \ldots, \nabla_r\}$.

Assume that $\Gamma(v)$ is a convex hull of $k > 1$ faces of polyhedra ∇_i. Let $v_1, \ldots, v_m$ be vertices of $\Gamma(v)$. By assumption, $m \geq k$. Without loss of generality, we assume that $v_1 \in \nabla_1, \ldots, v_k \in \nabla_k$. Since v belongs to relative interior of $\Gamma(v)$, there exist positive numbers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = 1$ and $\lambda_1 v_1 + \cdots + \lambda_m v_m = v$. Choose an arbitrary vertex w of the dual face $\Gamma^*(v) \subset \Delta$. Since $\Delta = \Delta_1 + \cdots + \Delta_r$, there exist $w_i \in \Delta_i$ such that $w = w_1 + \cdots + w_r$. It follows from definition of dual nef-partitions (cf. [6]) that

$$\langle w_i, v_i \rangle \geq -1, \ \ 1 \leq i \leq k$$

and

$$\langle w_i, v_j \rangle \geq 0, \ \ i \neq j, \ 1 \leq i \leq k, \ 1 \leq i \leq k.$$

On the other hand,

$$\langle w, v \rangle = -1 = \sum_{i=1}^{m} \sum_{j=1}^{r} \lambda_i \langle w_j, v_i \rangle.$$

Since $\lambda_i > 0$, the last equality is possible only if all vertices $v_1, \ldots, v_m$ belong to *the same* polyhedron ∇_i for some i $(1 \leq i \leq r)$. $\square$

Corollary 6.4

$$l(\Delta^*) = l(\nabla_1) + \cdots + l(\nabla_r) - r + 1.$$

Definition 6.5 Denote by $\Delta_j(v)$ the face of Δ_j which defines the Newton polyhedron for the equation of $\widehat{Z}_j$ in the toric divisor $\mathbf{D}(v) \subset \widehat{\mathbf{P}}_\Delta$.

It is easy to prove the following:

Proposition 6.6 *Assume that the face $\Gamma(v) \subset \Delta^*$ is also a face of ∇_i. Let $\Gamma^*(v)$ be the dual face of Δ. Then*

$$\Delta_i(v) = \{x \in \Delta_i \mid \langle x, v \rangle = -1\},$$

$$\Delta_j(v) = \{x \in \Delta_j \mid \langle x, v \rangle = 0 \ \ \textit{if } j \neq i\},$$

and

$$\Delta_1(v) + \cdots + \Delta_r(v) = \Gamma^*(v).$$

Tensoring the Koszul resolution of $\mathcal{O}_{\widehat{V}}$ by $\mathcal{O}_{\mathbf{D}(v)}$, we obtain:

Proposition 6.7

$$\chi(\mathcal{O}_{\mathbf{D}(v) \cap C}) = -\sum_j b(\Delta_j(v)) + \sum_{j<k} b(\Delta_j(v) + \Delta_k(v)) - \cdots$$

$$\cdots + (-1)^r b(\Delta_1(v) + \cdots + \Delta_r(v)).$$

Let $\nabla_i^0 = \nabla \cap \mathcal{V}(\Delta^*)$ $(i = 1, \ldots, r)$, and $I := \{1, \ldots, r\}$. If we rewrite

$$\sum_{v \in \mathcal{V}(\Delta^*)} \chi(\mathcal{O}_{\mathbf{D}(v) \cap C}) = \sum_{i=1}^{r} \sum_{v \in \nabla_i^0} \sum_{J \subset I} (-1)^{|J|+1} b(\sum_{j \in J} \Delta_j(v)) =$$

$$= \sum_{i=1}^{r} \sum_{v \in \nabla_i^0} \sum_{i \notin J \subset I} (-1)^{|J|+1} b(\sum_{j \in J} \Delta_j(v)) + \sum_{i=1}^{r} \sum_{v \in \nabla_i^0} \sum_{i \in J \subset I} (-1)^{|J|+1} b(\sum_{j \in J} \Delta_j(v)),$$

then 6.1 and 6.2 imply:

Theorem 6.8

$$\chi(\Omega^1_{\widehat{V}}) = \chi(\mathcal{O}^d_{\widehat{V}}) + \sum_{i=1}^{r} \sum_{J \subset I} (-1)^{|J|+1} b(\Delta_i + \sum_{j \in J} \Delta_j) +$$

$$+ \sum_{i=1}^{r} \sum_{v \in \nabla_i^0} \sum_{i \notin J \subset I} (-1)^{|J|+1} b(\sum_{j \in J} \Delta_j(v)) +$$

$$+ \sum_{i=1}^{r} \sum_{v \in \nabla_i^0} \sum_{i \in J \subset I} (-1)^{|J|+1} b(\sum_{j \in J} \Delta_j(v)).$$

7 Mirror duality for $\chi(\Omega^1)$

We prove that the involution $\Pi(\Delta) \leftrightarrow \Pi(\nabla)$ chahge the sign of $\chi(\Omega^1)$ by $(-1)^{d-r}$ (as it would follow from the expected duality $h^{1,q}(\widehat{V}) = h^{1,d-r-q}(\widehat{W})$, $0 \leq q \leq d-r$):

Theorem 7.1

$$\chi(\Omega^1_{\widehat{V}}) = (-1)^{d-r} \chi(\Omega^1_{\widehat{W}}).$$

For the proof we need some preliminary statements.

Proposition 7.2 *Let $i \in J \subset I$, $\Delta_i^0 = \Delta_i \cap \mathcal{V}(\nabla^*)$. Then a nonzero lattice point w belongs to the relative interior of the polyhedron $\Lambda = \Delta_i + \sum_{j \in J} \Delta_j$ if and only if $w \in \Delta_i^0$ and the zero point $0 \in N$ belongs to the relative interior of $\sum_{j \notin J} \nabla_j(w)$. Moreover, if this happens, then*

$$\dim \left(\Delta_i + \sum_{j \in J} \Delta_j \right) + \dim \left(\sum_{j \notin J} \nabla_j(w) \right) = d.$$

Proof. First of all, let's check that the interior lattice point w must belong to Δ_i.

This means checking $\langle w, \nabla_k \rangle \geq 0$ for $k \neq i$ and $\langle w, \nabla_i \rangle \geq -1$, which follows easily from the fact that w is a lattice point that belongs to $(1-\epsilon)\Lambda$ for some small positive ϵ.

The polyhedron $\Lambda - w$ is defined by the inequalities

$$\langle x, v \rangle \geq -2 - \langle w, v \rangle, \; v \in \nabla_i,$$

$$\langle x, v \rangle \geq -1 - \langle w, v \rangle, \; v \in \nabla_j, \; j \in J, \; j \neq i,$$

$$\langle x, v \rangle \geq 0 - \langle w, v \rangle, \; v \in \nabla_j, \; j \notin J.$$

Since $\langle w, v \rangle \geq -1$ for $v \in \nabla_i$, and $\langle w, v \rangle \geq 0$ for $v \notin \nabla_i$, only the inequalities $\langle x, v \rangle \geq 0$ for $v \in \nabla_j$, $\langle w, v \rangle = 0$ $(j \notin J)$ give rise to nontrivial restrictions for the intersection of a small neighbourhood of $0 \in M_{\mathbf{R}}$ with $\Lambda - w$. Therefore, it remains to consider the halfspaces defined by the inequalities $\langle x, v \rangle \geq 0$ where $v \in \nabla_j(w)$ $(j \notin J)$. Denote by L_w the convex cone in $M_{\mathbf{R}}$ defined by the inequalities

$$\langle x, v \rangle \geq 0, \quad \text{for all } v \in \nabla_j(w) \text{ and } j \notin J.$$

Then 0 lies in the relative interior of $\Lambda - w$ if and only if L_w is a linear subspace of $M_{\mathbf{R}}$. On the other hand, the cone

$$C_w = \sum_{j \notin J} \mathbf{R}_{\geq 0} \nabla_j(w)$$

is dual to L_w. Moreover, C_w is a linear subspace in $N_{\mathbf{R}}$ if and only if 0 is contained in the relative interior of $\sum_{j \notin J} \nabla_j(w)$. It remains to note that a convex cone is a linear subspaces if and only if the dual cone is a linear subspace. In the latter case, $\dim L_w + \dim C_w = d$, i.e., $\dim \left(\Delta_i + \sum_{j \in J} \Delta_j \right) + \dim \left(\sum_{j \notin J} \nabla_j(w) \right) = d$. □

Corollary 7.3

$$\sum_{i=1}^{r} \sum_{J \subset I} (-1)^{|J|+1} b(\Delta_i + \sum_{j \in J} \Delta_j) = (-1)^{d-r} \left(\sum_{i=1}^{r} \sum_{w \in \Delta_i^0} \sum_{i \notin J \subset I} (-1)^{|J|+1} b(\sum_{j \in J} \nabla_j(w)) \right).$$

Proof. Denote by $(-1)^{\dim \Theta} b^0(\Theta)$ the number of *nonzero* lattice points in the relative interior of a lattice polyhedron Θ. Let $i \in J \subset I$, $J' = J \setminus \{i\}$. Note that 0 is in the relative interior of $\Delta_i + \sum_{j \in J'} \Delta_j$ if and only if 0 is in the relative interior of $\Delta_i + \sum_{j \in J} \Delta_j$. Since $|J| = |J'| + 1$ and

$$\dim \left(\Delta_i + \sum_{j \in J'} \Delta_j \right) = \dim \left(\Delta_i + \sum_{j \in J} \Delta_j \right),$$

we can have

$$\sum_{i=1}^{r} \sum_{J \subset I} (-1)^{|J|+1} b(\Delta_i + \sum_{j \in J} \Delta_j) = \sum_{i=1}^{r} \sum_{i \in J \subset I} (-1)^{|J|+1} b^0(\Delta_i + \sum_{j \in J} \Delta_j).$$

It remains to apply Proposition 7.2 and the property $|J| + |I \setminus J| = r$. □

Proposition 7.4 *Let $i \in J \subset I$, and $v \in \nabla_i^0$ is a lattice point. Then a lattice point w belongs to the relative interior of the polyhedron $\Lambda = \sum_{j\in J} \Delta_j(v)$ if and only $w \in \Delta_i^0$ and the lattice point v belongs to the relative interior of $\nabla_i(w) + \sum_{j\notin J} \nabla_j(w)$. Moreover, if this happens, then*

$$\dim\left(\sum_{j\in J} \Delta_j(v)\right) + \dim\left(\nabla_i(w) + \sum_{j\notin J} \nabla_j(w)\right) = d-1.$$

Proof. We only need to prove the implication in one direction and the formula for the dimensions.

By 6.6, $\langle \Delta_j(v), v\rangle = 0$ for $j \in J$, $j \neq i$ and $\langle \Delta_i(v), v\rangle = -1$. Therefore, $\langle w, v\rangle = -1$, in particular, $w \neq 0$. Now we would like to prove that $w \in \Delta_i$. Because of $\Lambda \subset \Delta$, this amounts to checking $\langle w, \nabla_k\rangle \geq 0$ for $k \neq i$. Suppose there exists a vertex v' of ∇_k such that $\langle w, v'\rangle = -1$. Because w lies in the relative interior of Λ, we get $\langle \Lambda, v'\rangle = -1$. However, this is impossible, because $\Delta_j(v)$ contain zero if $j \neq i$, which leads to $\Delta_i(v) \subseteq \Lambda$.

The polyhedron $\Lambda - w$ is defined by the conditions

$$\langle x, v'\rangle \geq -1 - \langle w, v'\rangle,\ v' \in \nabla_j,\ j \in J,$$

$$\langle x, v'\rangle \geq 0 - \langle w, v'\rangle,\ v' \in \nabla_j,\ j \notin J,$$
$$\langle x, v\rangle = -1 - \langle w, v\rangle = 0.$$

If we are interested in the neighbourhood of $0 \in M$, we are left with the inequalities $\langle x, v'\rangle \geq 0$, where v' is either a vertex of $\nabla_j(w)$ ($j \notin J$), or v' is the vertex of $\nabla_i(w)$, or $v' = -v$. The zero point is in the interior of $\Lambda - w$, hence the convex cone L_w defined by these inequalities is a linear subspace. As a result, the dual to L_w cone

$$C_w = \mathbf{R}_{\geq 0}\nabla_i(w) - \mathbf{R}_{\geq 0}v + \sum_{j\notin J} \mathbf{R}_{\geq 0}\nabla_j(w).$$

is also a linear subspace.

Now we use 6.6 and the equality $\langle w, v\rangle = -1$ to conclude that $\{y \in C_w | \langle w, y\rangle = 0\}$ is a linear subspace of dimension one less which equals

$$\mathbf{R}_{\geq 0}\left(\nabla_i(w) - v\right) + \sum_{j\notin J} \mathbf{R}_{\geq 0}\nabla_j(w).$$

Because $(\nabla_i(w) - v)$ and $\nabla_j(w)$ contain zero, this implies that their sum has the same dimension and contains zero in its interior, which proves the first statement of the proposition.

The formula for the dimensions follows from the duality of L_w and C_w. □

Corollary 7.5

$$\sum_{i=1}^{r} \sum_{v\in\nabla_i^0} \sum_{i\in J\subset I} (-1)^{|J|+1} b(\sum_{j\in J} \Delta_j(v)) =$$

$$= (-1)^{d-r} \left(\sum_{i=1}^{r} \sum_{w \in \Delta_i^0} \sum_{i \in J' \subset I} (-1)^{|J'|+1} b(\sum_{j \in J'} \nabla_j(w)) \right).$$

Proof. We set $J' = \{I \setminus J\} \cup \{i\}$. It remains to apply Proposition 7.4 and the property $|J| + |J'| = r + 1$. □

Proof of Theorem 7.1: It remains to combine the statements 5.10, 7.3 and 7.5 with the formula in 6.8. □

8 Complete intersections of ample divisors

Our next purpose is to give explicit formulas for $(*, 1)$-Hodge numbers for $MPCP$-resolution $\widehat{V}$ of a Calabi-Yau complete intersection of ample divisors. Notice that in this case nef-partition is always irreducible.

First, we glue together the following two exact sequences

$$0 \to \mathcal{O}_{\widehat{V}}(-\widehat{Z}_1) \oplus \cdots \oplus \mathcal{O}_{\widehat{V}}(-\widehat{Z}_r) \to \Omega^1_{\widehat{\mathbf{P}}} |_{\widehat{V}} \to \Omega^1_{\widehat{V}} \to 0,$$

$$0 \to \Omega^1_{\widehat{\mathbf{P}}} |_{\widehat{V}} \to \mathcal{O}^d_{\widehat{V}} \to \bigoplus_{v \in \mathcal{V}(\Delta^*)} \mathcal{O}_{\mathbf{D}(v) \cap \widehat{V}} \to 0$$

and obtain the complex

$$\mathcal{Q}^* \ : \ 0 \to \mathcal{O}_{\widehat{V}}(-\widehat{Z}_1) \oplus \cdots \oplus \mathcal{O}_{\widehat{V}}(-\widehat{Z}_r) \to \mathcal{O}^d_{\widehat{V}} \to \bigoplus_{v \in \mathcal{V}(\Delta^*)} \mathcal{O}_{\mathbf{D}(v) \cap \widehat{V}} \to 0$$

whose cohomology $\mathcal{H}^i$ can be nontrivial only if $i = 1$, and

$$\mathcal{H}^1(\mathcal{Q}^*) \cong \Omega^1_{\widehat{V}},$$

i.e., the hypercohomology of $\mathcal{Q}^*$ coincides with the cohomology of $\Omega^1_{\widehat{V}}[1]$.

Proposition 8.1 *Assume that $\widehat{Z}_1, \ldots, \widehat{Z}_r$ are nef- and big-divisors; i.e., Δ_i is a Minkowski summand of Δ and $\dim \Delta_i = \dim \Delta = d$ $(i = 1, \ldots, r)$. Then*

$$H^k(\mathcal{O}_{\widehat{V}}(-\widehat{Z}_i)) = 0 \ \textit{ for } i \neq d - r$$

and

$$h^{d-r}(\mathcal{O}_{\widehat{V}}(-\widehat{Z}_i)) = \sum_{J \subset I} (-1)^{r-|J|} l^*(\Delta_i + \sum_{j \in J} \Delta_j).$$

Proof. Consider the Koszul resolution of $\mathcal{O}_{\widehat{V}}(-\widehat{Z}_i)$. Then the corresponding second spectral sequence degenerates in $''E_2$, because our assumptions on $\widehat{Z}_1, \ldots, \widehat{Z}_r$ imply $''E_1^{p,q} = 0$ for $q \neq d$. The latter immediately gives all statements. □

Corollary 8.2 *Let $\widehat{V}$ be a complete intersection such that $d-r \geq 3$. Then, under assumptions in* 8.1*, the second spectral sequence corresponding to the complex $\mathcal{Q}^*$ degenerates in $''E_2^*$, and one obtains the following relations*

$$h^{d-r-1}(\Omega^1_{\widehat{V}}) = \sum_{i=1}^{r} h^{d-r}(\mathcal{O}_{\widehat{V}}(-\widehat{Z}_i)) - d-$$

$$- \sum_{v\in\mathcal{V}(\Delta^*)} \left(h^{d-r-1}(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) - h^{d-r-2}(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}})\right),$$

$$h^k(\Omega^1_{\widehat{V}}) = \sum_{v\in\mathcal{V}(\Delta^*)} h^{k-2}(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) \quad \text{for } 2 \leq k \leq d-r-2,$$

$$h^1(\Omega^1_{\widehat{V}}) = \sum_{v\in\mathcal{V}(\Delta^*)} h^0(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) - d,$$

$$h^0(\Omega^1_{\widehat{V}}) = h^{d-r}(\Omega^1_{\widehat{V}}) = 0.$$

Proposition 8.3 *Assume that $\widehat{Z}_1, \dots, \widehat{Z}_r$ are proper pullbacks of ample divisors on $\mathbf{P}_\Delta$; i.e., Δ_i and Δ are Minkowski summand of each other $(i = 1, \dots, r)$. Choose an element of $v \in \mathcal{V}(\Delta^*)$. Let s be dimension of the minimal face $\Theta \subset \Delta^*$ containing v. Then the faces $\Delta_1(v), \dots, \Delta_r(v)$ depend only on Θ (we denote these faces by $\Theta_1^*, \dots, \Theta_r^*$), and the following statements hold:*

(i) *If $d-r-s-1 > 0$, then*

$$h^{d-r-s-1}(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) = \sum_{J\subset I} (-1)^{r-|J|} l^*(\sum_{j\in J} \Theta_j^*),$$

$$h^0(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) = 1,$$

and

$$h^i(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) = 0 \quad \text{for all } i \neq 0, d-r-s-1.$$

(ii) *If $d-r-s-1 = 0$, then*

$$h^0(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) = 1 + \sum_{J\subset I} (-1)^{r-|J|} l^*(\sum_{j\in J} \Theta_j^*),$$

and

$$h^i(\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}) = 0 \quad \text{for all } i \neq 0.$$

(iii) *If $d-r-s-1 < 0$, then $\mathbf{D}(v) \cap \widehat{V}$ is empty.*

Proof. Since $\widehat{Z}_1, \dots, \widehat{Z}_r$ are proper pulbacks of ample divisors, the polyhedron Δ^* is combinatorially dual to each of $r+1$ polyhedra $\Delta, \Delta_1, \dots, \Delta_r$. By this combinatorial duality, the faces $\Delta_i(v) \subset \Delta_i$ are dual to the face Θ and their arbitary sums have the same dimension $d-s-1$. In the sequel, we denote $\Delta_i(v)$ simply by Θ_i^*.

Consider the Koszul resolution of $\mathcal{O}_{\mathbf{D}(v)\cap\widehat{V}}$. Then the corresponding second spectral sequence degenerates in $''E_2$, because $''E_1^{p,q} = 0$ for $q \neq d-s-1, 0$, $''E_1^{0,p} = 0$ for $p \neq r$, and $''E_1^{0,r} \cong \mathbf{C}$. Now the statements (i)-(iii) are obvious. □

Corollary 8.4 *Let $\widehat{V}$ be a complete intersection such that $d - r \geq 3$. Then, under assumptions in* 8.3, *one obtains*

$$h^{d-r-1}(\Omega^1_{\widehat{V}}) = \sum_{i=1}^{r}\left(\sum_{J\subset I}(-1)^{r-|J|}l^*(\Delta_i + \sum_{j\in J}\Delta_j)\right) - d-$$

$$-\sum_{\substack{\dim\Theta=0\\ \Theta\subset\Delta^*}}\left(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\Theta_j^*)\right) + \sum_{\substack{\dim\Theta=1\\ \Theta\subset\Delta^*}} l^*(\Theta)\cdot\left(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\Theta_j^*)\right),$$

$$h^k(\Omega^1_{\widehat{V}}) = \sum_{\substack{\dim\Theta=d-r-k-1\\ \Theta\subset\Delta^*}} l^*(\Theta)\cdot\left(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\Theta_j^*)\right) \quad \textit{for } 2\leq k\leq d-r-2,$$

$$h^1(\Omega^1_{\widehat{V}}) = \text{Card}\{\text{ lattice points in faces of dimension} \leq d-r-1\} - d+$$

$$+\sum_{\substack{\dim\Theta=d-r-1\\ \Theta\subset\Delta^*}} l^*(\Theta)\cdot\left(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\Theta_j^*)\right),$$

$$h^0(\Omega^1_{\widehat{V}}) = h^{d-r}(\Omega^1_{\widehat{V}}) = 0.$$

Corollary 8.5 *Assume that $r = 1$ and $d \geq 4$. Then the Hodge numbers $h^{p,1}(\widehat{V})$ have the following values*

$$h^{0,1}(\widehat{V}) = h^{d-1,1}(\widehat{V}) = 0,$$

$$h^{1,1}(\widehat{V}) = l(\Delta^*) - d - 1 - \left(\sum_{\substack{\Theta^*\subset\Delta^*\\ \text{codim}\,\Theta^*=1}} l^*(\Theta^*)\right) + \left(\sum_{\substack{\Theta^*\subset\Delta^*\\ \text{codim}\,\Theta^*=2}} l^*(\Theta^*)\cdot l^*(\Theta)\right),$$

$$h^{d-2,1}(\widehat{V}) = l(\Delta) - d - 1 - \left(\sum_{\substack{\Theta^*\subset\Delta^*\\ \text{codim}\,\Theta^*=d}} l^*(\Theta)\right) + \left(\sum_{\substack{\Theta^*\subset\Delta^*\\ \text{codim}\,\Theta^*=d-1}} l^*(\Theta^*)\cdot l^*(\Theta)\right),$$

$$h^{p,1}(\widehat{V}) = \sum_{\substack{\Theta^*\subset\Delta^*\\ \text{codim}\,\Theta^*=p+1}} l^*(\Theta^*)\cdot l^*(\Theta); \quad \textit{for } 2\leq p\leq d-3\ .$$

Proposition 8.6 *Let $\widehat{V}$ be a complete intersection of ample divisors $\widehat{Z}_1, \ldots, \widehat{Z}_r$ such that $d - r \geq 3$. Then one obtains*

$$h^{d-r-1}(\Omega^1_{\widehat{V}}) = \sum_{i=1}^{r} \left(\sum_{J \subset I} (-1)^{r-|J|} l^*(\Delta_i + \sum_{j \in J} \Delta_j) \right) - d-$$

$$- \sum_{\substack{\dim \Theta = 0 \\ \Theta \subset \Delta^*}} \left(\sum_{J \subset I} (-1)^{r-|J|} l^*(\sum_{j \in J} \Theta^*_j) \right),$$

$$h^k(\Omega^1_{\widehat{V}}) = 0 \ \ \textit{for } k \neq 1, d-r-1 \ ,$$

$$h^1(\Omega^1_{\widehat{V}}) = \mathrm{Card}\{ \text{ lattice points in faces of dimension } \leq d-r-1\} - d.$$

Proof. If $\widehat{Z}_1, \ldots, \widehat{Z}_r$ are ample, then $\mathbf{P}_\Delta$ has at most terminal singularities; i.e., $l^*(\Theta) = 0$ for all faces of Δ^* of positive dimension. This immediately implies the formulas. □

Example 8.7 *Complete intersection V_{d_1,d_2} of two hypersurfaces in $\mathbf{P}^5$.*

For two cases below, we have

$$\sum_{\substack{\operatorname{codim} \Xi = 1 \\ \Xi \subset \Delta}} l^*(\Xi) = 30.$$

Case 1. $d_1 = d_2 = 3$. Then $l(\Delta_1) = l(\Delta_2) = 56$, $l^*(2\Delta_1) = l^*(2\Delta_2) = 1$. Therefore,

$$h^{2,1}(V_{3,3}) = (112 - 7) - (30 + 1 + 1) + 0 = 73.$$

Case 2. $d_1 = 2$, $d_2 = 4$. Then $l(\Delta_1) = 21$, $l(\Delta_2) = 126$, $l^*(2\Delta_1) = 0$, $l^*(2\Delta_2) = 21$. Therefore,

$$h^{2,1}(V_{2,4}) = (147 - 7) - (30 + 21 + 0) + 0 = 89.$$

Example 8.8 *Complete intersection $V_{2,2,3}$ in $\mathbf{P}^6$.*

$$h^{2,1}(V_{2,2,3}) = (28 + 28 + 84 - 9) - (7 \cdot 6 + 1 + 1 + 7 + 7) + 0 = 73.$$

Example 8.9 *Complete intersection $V_{2,2,2,2}$ in $\mathbf{P}^7$.*

$$h^{2,1}(V_{2,2,2,2}) = (4 \cdot 36 - 11) - (8 \cdot 7 + 3 \cdot 4) + 0 = 65.$$

9 Complete intersections in $\mathbf{P}^d$

Consider the case when a nef-partition $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ defines a Calabi-Yau complete intersection $V = V_{d_1,\ldots,d_r}$ in projective space $\mathbf{P}^d$. This means that the polyhedra $\Delta_i = d_i\Lambda$ $(i = 1, \ldots, r)$ are d_i-multiples of a regular d-dimensional simplex Λ and $d_1 + \cdots + d_r = d$. Our purpose is to compute the $(*, 1)$-Hodge numbers of the mirror Calabi-Yau complete intersection $\widehat{W} = \widehat{Y}_1 \cap \cdots \cap \widehat{Y}_r$ defined by the dual nef-partition $\Pi(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$. In this case, ∇_i $(i = 1, \ldots, r)$ is a regular d_i-dimensional simplex. If for some i we have $d_i = 1$, then the Calabi-Yau complete intersection in $\mathbf{P}^d$ reduces to a Calabi-Yau complete intersection in $\mathbf{P}^{d-1}$. So we can assume that $d_i \geq 2$ for all $i = 1, \ldots r$.

Proposition 9.1 *In the above situation, one has*

$$h^k(\mathcal{O}(-\widehat{Y}_i - \sum_{j\in J} \widehat{Y}_j)) = 0 \quad \textit{for all } k$$

unless $J = I$, *or* $J \cup \{i\} = I$. *Moreover*

$$h^k(\mathcal{O}(-\widehat{Y}_i - \sum_{j\in I} \widehat{Y}_j)) = 0 \quad \textit{for } k \neq d \ ,$$

$$h^d(\mathcal{O}(-\widehat{Y}_i - \sum_{j\in I} \widehat{Y}_j)) = l(\nabla_i);$$

and

$$h^k(\mathcal{O}(- \sum_{j\in I} \widehat{Y}_j)) = 0 \quad \textit{for } k \neq d \ ,$$

$$h^d(\mathcal{O}(- \sum_{j\in I} \widehat{Y}_j)) = 1.$$

Proof. Note that the polyhedra ∇_i, ∇_j $(j \in J)$ are regular simplices spanning *linearly independent* subspaces unless $i \in J$, $J = I$, or $J \cup \{i\} = I$. Therefore, $\nabla_i + \sum_{j\in J} \nabla_j$ is a regular simplex having no lattice points in its relative interior unless $i \in J$, $J = I$, or $J \cup \{i\} = I$. If $i \in J$, then $\nabla_i + \sum_{j\in J} \nabla_j$ has no lattice points in its relative interior since $l^*(2\nabla_i) = 0$ for all $i \in I$ $(d_i = \dim \nabla_l \geq 2)$. Hence the first statement follows from 2.5.

If $J = I$, or $j \cup \{i\} = I$, then $\dim (\nabla_i + \sum_{j\in J} \nabla_j) = d$. By 2.5, $h^k(\mathcal{O}(-\widehat{Y}_i - \sum_{j\in I} \widehat{Y}_j)) = 0$ for $k \neq d$. The remaining statements follow from $l^*(\nabla_i + \nabla_1 + \cdots + \nabla_r) = l(\nabla_i)$ and $l^*(\nabla_1 + \cdots + \nabla_r) = 1$. □

Corollary 9.2

$$h^k(\mathcal{O}_{\widehat{W}}(-\widehat{Y}_i)) = 0 \ \textit{ for } k \neq d - r$$

and

$$h^{d-r}(\mathcal{O}_{\widehat{W}}(-\widehat{Y}_i)) = l(\nabla_i) - 1.$$

Proof. Tensoring the Koszul resolution $\mathcal{K}^*$ of $\mathcal{O}_{\widehat{W}}$ by $\mathcal{O}(-\widehat{Y}_i)$, we obtain the degenerated spectral sequence "$E^{p,q}$ from which one immediately obtains the statement. □

Proposition 9.3 *For any vertex w of ∇^* and any subset $J \subset I$, one has*

$$l^*(\sum_{j\in J} \nabla_j(w)) = 0,$$

unless $\dim \sum_{j\in J} \nabla_j(w) = 0$.

Proof. We consider two cases:

Case 1. $J \neq I$. We know that $\sum_{j\in J} \nabla_j(w)$ is always a face of $\sum_{j\in J} \nabla_j$. On the other hand, the linear subspaces spanned by ∇_j are linearly independent. Since all ∇_i are regular simplices, there is no face of $\sum_{j\in J} \nabla_i$ of positive dimension containing a lattice point in its relative interior.

Case 2. $J = I$. If there exists $i \in I$ such that $\dim \nabla_i(w) = 0$, then setting $J' = J \setminus \{i\}$ we reduce all to Case 1.

The polyhedron ∇ has $d+2$ lattice points, but only those lattice points $v \in \nabla$ which satisfy the condition $\langle w, v\rangle \in \{0, -1\}$ might appear in $\nabla_j(w)$. Hence there exists a vertex of ∇ which does not belong to any of $\nabla_i(w)$. Therefore linear subspaces spanned by $\nabla_i(w)$ are linearly independent. We again obtain the same statement, since all $\nabla_i(w)$ are regular simplices. □

Using the Koszul resolution, we obtain

Corollary 9.4

$$h^k(\mathcal{O}_{\mathbf{D}(w)\cap\widehat{W}}) = 0 \quad \textit{for } k > 0$$

Therefore the second spectral sequence of the 3-term complex $\mathcal{Q}^*$ degenerates in $''E_2$ and we have

Proposition 9.5 *Assume that $d - r \geq 3$. Then*

$$h^k(\Omega^1_{\widehat{W}}) = 0 \ \ \textit{for } 2 \leq d - r - 2 \ ,$$

and

$$h^{d-r-1}(\Omega^1_{\widehat{W}}) = -d + \sum_{i=1}^{r}(l(\nabla_i) - 1) = 1 = h^1(\Omega^1_V).$$

Using the duality for the Euler characteristic from Theorem 7.1

$$-h^1(\Omega^1_{\widehat{W}}) + (-1)^{d-r-1}h^{d-r-1}(\Omega^1_{\widehat{W}}) = (-1)^{d-r}\left(-h^1(\Omega^1_V) + (-1)^{d-r-1}h^{d-r-1}(\Omega^1_V)\right),$$

we obtain

$$h^1(\Omega^1_{\widehat{W}}) = h^{d-r-1}(\Omega^1_V).$$

Therefore, we get the complete duality for $(1, q)$-Hodge numbers:

Theorem 9.6 *Let V be a Calabi-Yau complete intersection of r hypersurfaces in $\mathbf{P}^d$ and $d - r \geq 3$, $\widehat{W}$ a MPCP-desingularization of the Calabi-Yau complete intersection $W \subset \mathbf{P}_\nabla$ Then*

$$h^q(\Omega^1_{\widehat{W}}) = h^{d-r-q}(\Omega^1_V) \quad \textit{for } 0 \leq q \leq d - r.$$

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