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Chiral rings of vertex algebras of mirror symmetry

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Abstract. We calculate chiral rings of the $N = 2$ vertex algebras constructed from the combinatorial data of toric mirror symmetry and show that they coincide with the description of stringy cohomology conjectured previously in a joint work with A. Mavlyutov. This constitutes an important reality check of the vertex algebra approach to mirror symmetry.

1. Introduction

Mirror symmetry was originally formulated by physicists as a correspondence between IIA models constructed for one family of Calabi-Yau varieties with IIB models constructed for another family. IIA and IIB models are $N = (2, 2)$ superconformal field theories. While the very notion of $N = (2, 2)$ superconformal field theory is not yet properly axiomatized, part of the structure of such a theory is a vertex algebra with an $N = 2$ superconformal structure, which does have a rigorous mathematical meaning. Every smooth Calabi-Yau variety X gives rise to one such algebra, namely the cohomology of its chiral de Rham complex, which is a certain sheaf of vertex algebras on X, constructed in [14]. Unfortunately, the definition of chiral de Rham complex involves only the complex structure of the variety, not the metric or B-field. As such, it lacks instanton corrections and, at best, is only related to the large Kähler structure (radius) limit of the model. In the case of the smooth Calabi-Yau variety, the chiral rings of the IIA and IIB models are expected to give small quantum cohomology of the variety and the cohomology of the exterior powers of its tangent bundle. The chiral rings coming from the chiral de Rham complex capture the latter ring, but only give the usual cohomology product for the former.

Vertex algebras of mirror symmetry are a particular class of vertex algebras with $N = 2$ structure. Their definition in [6] was inspired by the calculation of the cohomology of chiral de Rham complex for the hypersurfaces in toric varieties, which is by far the most common case of mirror symmetry. They are defined *in purely combinatorial terms* as a cohomology of certain explicit differentials on the lattice vertex algebras constructed from the combinatorial data that underlie toric mirror symmetry. In the hypersurface case, a given pair of dual reflexive polytopes (Δ, Δ^*) gives rise to a finite-dimensional family of $N = 2$ vertex algebras $V_{f,g}$ where f and g are parameters that can be thought of as elements of finite-dimensional vector spaces of complex-valued functions on the sets of lattice points of Δ and Δ^* respectively. Parameter f simply encodes the coefficients of the defining equation of the hypersurface X in the ambient toric variety. Parameter g can be thought of as a defining equation of an element X^* of the mirror family of Calabi-Yau varieties, but its geometric meaning in terms of the original Calabi-Yau X is far less clear. In the smooth hypersurface case, after modding out by the symmetries induced by the torus action, the space of g has the dimension equal to $H^{1,1}(X)$. It is *conjecturally* related to the metric and B-field on X. In the more general case of X given by a complete intersection that corresponds to a nef-partition, the more appropriate language is that of dual Gorenstein cones, see [4]. While the algebraic results of this paper are applicable to this more general setting, in what follows we will focus on the hypersurface case of toric mirror symmetry.

The basic premise of the vertex algebra approach to mirror symmetry is that the family of vertex algebras $V_{f,g}$ is identical to the family of vertex algebras of IIA-IIB models in the toric case, under some, yet unknown, identification of the space of parameters g and the complexified Kähler cone of the Calabi-Yau hypersurface. At this moment this premise remains conjectural. Eventually, the hope is to give a mathematically rigorous *geometric* definition of the $N = 2$ vertex algebras of IIA-IIB models for a given metric and B-field on an arbitrary Calabi-Yau manifold and then calculate it in the toric hypersurface case. On the other hand, the family of vertex algebras $V_{f,g}$ is clearly compatible with the combinatorics of mirror involution (i.e. a switch of Δ and Δ^* induces the mirror involution of the $N = 2$ structure). It has been also shown in [6] that in the smooth ambient toric variety case for a given choice of parameters f there is an algebraic family of vertex algebras which has $V_{f,g}$ as its general fiber and has the cohomology of the chiral de Rham complex of X as a special fiber. Here g give the parameter space of the family. The author hopes that the process of going from $V_{f,g}$ to the cohomology of chiral de Rham complex will eventually be interpreted as a mathematically rigorous counterpart of the physical procedure of going to the large radius limit.

This paper shows that vertex algebras $V_{f,g}$ pass two additional important reality checks. First, for a generic choice of parameters (f, g) the $N = 2$ vertex algebra $V_{f,g}$ satisfies the positivity of Hamiltonians property expected of the vertex algebras of IIA-IIB models.

Theorem 6.8. *For strongly non-degenerate f and g the $N = 2$ vertex algebra $V_{f,g}$ is of σ-model type.*

Here we define an $N = 2$ vertex algebra to be of σ-model type if it satisfies the positivity, diagonalizability and finite-dimensionality of eigen-spaces of Hamiltonians H_A and H_B, see Definition 2.4. We remark that in the case of smooth ambient toric variety this result can be established via the degeneration argument. Indeed, the statement is obvious for the cohomology of the chiral de Rham complex of the hypersurface X_f given by a generic f. Since the cohomology of the chiral de Rham complex fits into an algebraic family with general fiber being $V_{f,g}$ (see [6]), and the dimensions of graded components in such family can only jump at special fibers, the result follows. The author thanks the referee for the suggestion to stress this point. The more direct approach of this paper has the advantage of avoiding the smoothness condition which holds true only for a small minority of the reflexive polytopes, especially in higher dimensions. In addition to giving a direct proof of Theorem 6.8, our approach yields a more explicit description of the chiral rings, which was the main motivation of the paper.

Our definition of chiral rings of $N = 2$ vertex algebras of σ-model type follows closely that of [12]. They are defined as zero eigenspaces of H_A and H_B and are called A-ring and B-ring respectively. The main result of this paper is that for a general choice of parameters (f, g) the chiral rings of the algebra $V_{f,g}$ have the expected graded dimension, as given by the (stringy) Hodge numbers of the hypersurface X. Moreover, we give a rather explicit combinatorial description of the fields in the lattice vertex algebra which descend to the fields in the chiral rings of $V_{f,g}$.

Theorem 7.6. *Let f and g be strongly non-degenerate. Then both chiral rings of the $N = 2$ vertex algebra $V_{f,g}$ are naturally isomorphic* **as vector spaces** *to the space*

$$W_{f,g} = \oplus_{\theta \subseteq K} R_1(\theta, f) \otimes_{\mathbb{C}} R_1(\theta^*, g).$$

Here the direct sum is taken over the faces θ of the reflexive Gorenstein cone K associated to Δ, θ^* denotes the dual face of the dual cone and R_1 are vector spaces defined by Batyrev in [1] (see also [8]). We remark that this paper implies that $W_{f,g}$ must possess two product structures, depending on whether we consider it to be the A-ring or the B-ring of the algebra. Surprisingly, we can not at present define these product structures in non-vertex-algebra terms. On the other hand, the A-product structure on the “diagonal part” which corresponds to $\theta = \{0\}$ is well-understood and is illustrated in the famous quintic case, see Example 7.8.

We remark that the spaces $W_{f,g}$ should be thought of as “(small) quantum stringy cohomology spaces”, under the A-ring product. The first explicit occurrence of this idea in the literature seems to be in [13], which contains an easier calculation of the chiral rings of the related family of vertex algebras that correspond to the situation when X is a toric variety rather than a Calabi-Yau. On the other hand, this idea is implicit in [6][Remark 8.6] and has been known to the author since then. Paper [13] also treats some Fano hypersurface examples.

We would like to clarify the relation of our construction with the concept of stringy Hodge numbers of Calabi-Yau hypersurfaces in toric varieties. Toric mirror symmetry is best formulated for *ample* hypersurfaces X (i.e. without partial resolutions of singularities) which are likely to be singular. There are two major reasons for it. First, in dimension higher than three, Calabi-Yau hypersurfaces in toric varieties are unlikely to admit a crepant resolution of singularities. Second, even in dimension three, the crepant resolution of singularities is by no means unique. In fact different crepant resolutions often produce different cohomology rings. According to an idea of Batyrev, in the singular case the usual cohomology with the Hodge structure has to be replaced by some different double-graded vector space called stringy cohomology. While the graded dimension of the stringy cohomology vector space has been defined in great generality (see [3]), it is usually rather unclear how to define the actual vector space. In fact, the author has been suggesting for a while that it may not be possible to construct a single stringy cohomology vector space. Instead, one should strive to introduce a family of such vector spaces that depend on a finite number of parameters. It was observed in [8] that in the case of hypersurfaces in toric varieties, there is a natural construction of a double-graded vector space of correct graded dimension, given by an analog of $W_{f,g}$ above that takes into account the fan of the ambient toric variety. While it was conjectured to be the mysterious stringy cohomology space, it was not given a product structure expected of such a space. The current paper rectifies this problem, even though the product is not constructed explicitly.

While we use some key results of [6] and [8], this paper is mostly self-contained. It is written almost entirely in algebraic terms, in hopes that it will be accessible to the specialists in the theory of vertex algebras. We are hoping to make the case that vertex algebras of mirror symmetry are rich and beautiful objects that warrant further study.

The paper is organized as follows. In Section 2 we recall the definitions of vertex algebras with $N = 2$ structure and their chiral rings. Section 3 contains the definition of the particular class of lattice vertex algebras relevant to this paper. In Section 4 we recall the combinatorial data of toric mirror symmetry and the definition of the vertex algebras of mirror symmetry. Section 5 contains basic definitions and properties of strongly non-degenerate coefficient functions which are a technical tool needed for this paper. Our main results are collected in Sections 6 and 7. Finally, in Section 8 we list some important open questions. We hope that they will not stay open for long.

The author thanks the referee for suggestions on improving the exposition.

2. N = 2 vertex algebras and their chiral rings

In this section we recall the definitions of $N = 2$ vertex algebras and their chiral rings. We first define vertex algebras, according to [10].

Definition 2.1. *A vertex algebra V is first of all a super vector space over $\mathbb{C}$, that is $V = V_0 \oplus V_1$ where elements of V_0 are called bosonic or even and elements from V_1 are called fermionic or odd. In addition, there is a fixed bosonic vector $|0\rangle$ called vacuum vector. The last part of the data that defines a vertex algebra is the so-called state-field correspondence which is a parity preserving linear map from V to* $\mathrm{End} V[[z, z^{-1}]]$

$$a \mapsto Y(a, z) = \sum_{n \in Z} a_{(n)} z^{-n-1}$$

such that for fixed v and a all $a_{(n)}v$ are zero for sufficiently big n. This state-field correspondence must satisfy the following axioms.

- **translation covariance:** $\{T, Y(a,z)\}_- = \partial_z Y(a,z)$ *where* $\{,\}_-$ *denotes the usual commutator and* T *is defined by* $T(a) = a_{(-2)}|0\rangle$*;*
- **vacuum:** $Y(|0\rangle, z) = \mathbf{1}_V,\ Y(a,z)|0\rangle_{z=0} = a$*;*
- **locality:** $(z-w)^N\{Y(a,z), Y(b,w)\}_\mp = 0$ *for all sufficiently big* N*, where* $\mp$ *is* $+$ *if and only if both* a *and* b *are fermionic. The equality is understood as an identity of formal power series in* z *and* w*.*

Remark 2.2. We will usually write $a(z)$ in place of $Y(a,z)$. The coefficients $a_{(n)}$ are called *modes* of the field $a(z)$. For every two elements a and b there is an operator product expansion (OPE)

$$a(z)b(w) = \sum_{i=1}^{N} \frac{c^i(w)}{(z-w)^i} + :a(z)b(w):$$

where the meaning of the symbols $\frac{1}{(z-w)^i}$ and $:\ :$ in the above formulas is as in Chapter 2 of [10]. Operator product expansions is a convenient way to encode the (super-)commutators of the modes of a and b.

Definition 2.3. *An* $N = 2$ *vertex algebra is a vertex algebra* V *with the following additional structure. There are fixed bosonic elements* L *and* J *and fixed fermionic elements* G^+ *and* G^- *of* V *such that*

- $L_{(0)} = T$
- $L_{(1)}$ *is diagonalizable and satisfies* $\{L_{(1)}, Y(a,z)\}_- = z\partial_z Y(a,z) + Y(L_{(1)}a, z)$ *for all* $a \in V$*.*
- *Fields* $L(z)$, $J(z)$, $G^\pm(z)$ *satisfy the OPEs*

$$\begin{aligned}
L(z)L(w) &= \tfrac{c/2}{(z-w)^4} + \tfrac{2L(w)}{(z-w)^2} + \tfrac{\partial_w L(w)}{z-w} + :L(z)L(w): \ ,\\
L(z)J(w) &= \tfrac{J(w)}{(z-w)^2} + \tfrac{\partial_w J(w)}{z-w} + :L(z)J(w): \ ,\\
L(z)G^\pm(w) &= \tfrac{(3/2)G^\pm(w)}{(z-w)^2} + \tfrac{\partial_w G^\pm(w)}{z-w} + :L(z)G^\pm(w): \ ,\\
J(z)J(w) &= \tfrac{c/3}{(z-w)^2} + :J(z)J(w): \ ,\\
J(z)G^\pm(w) &= \pm\tfrac{G^\pm(w)}{z-w} + :J(z)G^\pm(w): \ ,\\
G^\pm(z)G^\mp(w) &= \tfrac{2c/3}{(z-w)^3} \pm \tfrac{2J(w)}{(z-w)^2} + \tfrac{2L(w)\pm\partial_w J(w)}{z-w} + :G^\pm(z)G^\mp(w): \ ,\\
G^\pm(z)G^\pm(w) &= :G^\pm(z)G^\pm(w): \ .
\end{aligned}$$

Here c *is a constant and we will call* $\hat{c} = c/3$ *the* central charge *of the* $N = 2$ *vertex algebra.*

Let V be an $N = 2$ vertex algebra. We are particularly interested in the operators $L_{(1)}$ and $J_{(0)}$. OPE of L and J implies that $L_{(1)}$ and $J_{(0)}$ commute with each other. In all the situations that we will consider in this paper they will provide V with a double grading. The following definition is inspired by [12].

Definition 2.4. *We call an* $N = 2$ *vertex algebra* V *an* $N = 2$ vertex algebra of σ-model type *if the eigenspaces* V_α *of* $L_{(1)}$ *are finite-dimensional for all* α *and zero except for* $\alpha \in \frac{1}{2}\mathbb{Z}_{\geq 0}$*, and the operators* $H_A := L_{(1)} - \frac{1}{2}J_{(0)}$ *and* $H_B := L_{(1)} + \frac{1}{2}J_{(0)}$ *have only nonnegative integer eigenvalues.*

Remark 2.5. From the physicists' point of view the conditions above are satisfied for $\bar{\partial}$-quotients of the $N = (2, 2)$ algebras constructed (empirically) from sigma-models on Calabi-Yau manifolds. The central charge of such algebra equals the dimension of the Calabi-Yau manifold. The operators H_A and H_B are the Hamiltonians for the two topological twists of the σ-model, which explains the notations. In the case of smooth Calabi-Yau manifold, the A-ring should equal its small quantum cohomology while the B-ring should equal the cohomology of exterior powers of its tangent bundle.

We now define chiral A-rings and chiral B-rings of $N = 2$ vertex algebras of σ-model type as zero eigenspaces of H_A and H_B respectively. Letters A and B here stand for the rings of the topological quantum field theories of IIA and IIB models. The ring structures are provided by the following proposition. We formulate it for the A-ring with B-ring case being completely analogous.

Proposition 2.6. *Let V be an $N = 2$ vertex algebra of σ-model type. Let $a, b \in V$ be such that $H_A(a) = H_A(b) = 0$. Then all the modes of a and b super-commute and $c(z) := a(z)b(z)$ is a field of V which corresponds to an element c with $H_A(c) = 0$. This provides the kernel of H_A with the structure of an associative super-commutative ring.*

Proof. The proof is essentially contained in [12]. Consider the OPE

$$a(z)b(w) = \sum_{i=1}^{N} \frac{c^i(w)}{(z-w)^i} + :a(z)b(w):$$

of a and b. A standard calculation shows that the H_A grading of c^i is negative, hence $c^i = 0$ for all i. As a result, the modes of $a(z)$ and $b(w)$ super-commute. The field $a(z)b(z)$ corresponds to the element $a_{(-1)}b_{(-1)}|0\rangle$ which is easily shown to have $H_A = 0$, since $a_{(-1)}$ and $b_{(-1)}$ preserve the H_A grading.

Remark 2.7. We recall that mirror involution is the automorphism of $N = 2$ vertex algebra that does not change the vertex algebra V but moves L, J, G^+, G^- to $L, -J, G^-, G^+$. Indeed, this change preserves the OPEs of $N = 2$ algebra. This involution switches A-rings and B-rings.

Remark 2.8. From physical considerations, one often expects to find a vector space isomorphism between the A-ring and the B-ring of a vertex algebra of σ-model type, given by the so-called *spectral flow* (see [12]). We will establish this isomorphism explicitly for the $N = 2$ vertex algebras considered in this paper, see Remark 7.9. It is important to keep in mind that spectral flow *does not preserve the ring structure.*

3. Lattice vertex algebras

In this section we will recall the definition of lattice vertex algebras in a particular case. Let M and N be two dual free abelian groups which will be fixed throughout this section. We denote the natural pairing of $m \in M$ and $n \in N$ by $m \cdot n$ and extend it by linearity to a pairing between $M_{\mathbb{C}}$ and $N_{\mathbb{C}}$.

The infinite Heisenberg algebra associated to $M \oplus N$ is the associative $\mathbb{C}$-algebra with generators m_i^{bos} and n_j^{bos} where $m \in M_{\mathbb{C}}, n \in N_{\mathbb{C}}, i, j \in \mathbb{Z}$. The relations are linear relations on m_i^{bos} for a fixed i and n_j^{bos} for fixed j according to the relations in $M_{\mathbb{C}}$ and $N_{\mathbb{C}}$, as well as the relations

$$\{m_i^{bos}, n_j^{bos}\}_- = i(m \cdot n)\delta_{i+j}^0, \ \{m_i^{bos}, m_j^{bos}\}_- = \{n_i^{bos}, n_j^{bos}\}_- = 0.$$

Here $\{\}_-$ denotes the commutator and δ is the Kronecker symbol.

Analogously, the infinite Clifford algebra associated to $M \oplus N$ has generators m_i^{ferm} and n_j^{ferm} for $m \in M_{\mathbb{C}}$, $n \in N_{\mathbb{C}}$ and $i, j \in \mathbb{Z} + \frac{1}{2}$. In addition to linear relations for fixed i or j, there are anti-commutator relations

$$\{m_i^{ferm}, n_j^{ferm}\}_+ = (m \cdot n)\delta_{i+j}^0, \{m_i^{ferm}, m_j^{ferm}\}_+ = \{n_i^{ferm}, n_j^{ferm}\}_+ = 0.$$

Remark 3.1. The superscripts bos and $ferm$ stand for bosonic and fermionic respectively.

We will consider the following representation of the direct sum of the Clifford and Heisenberg algebras. Let $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ be the tensor product over all integers $i < 0$ of the symmetric algebras of $M_{\mathbb{C}}$ and $N_{\mathbb{C}}$ tensored by the product over all half-integers $i < 0$ of the exterior algebras of $M_{\mathbb{C}}$ and $N_{\mathbb{C}}$. For $i < 0$ the action of the generators m_i^{bos} and n_i^{bos} on $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ of the infinite Heisenberg algebra is defined as the multiplication by m_i^{bos} and n_i^{bos} respectively. For $i > 0$ the action of m_i^{bos} is defined as i times the corresponding differentiation on the symmetric algebra of $N_{\mathbb{C}}$ for the index $-i$, and similarly for n_i^{bos}. The action of the generators of infinite Clifford algebra is similarly defined by either multiplication or differentiation. Finally, the action of the generators m_0^{bos} and n_0^{bos} (which commute with all other generators and with each other) is defined to be zero. As will be apparent later, this is the reason for the subscript $\mathbf{0} \oplus \mathbf{0}$ in our notations for the space. Traditionally, the constant 1 in this product of symmetric and exterior powers is denoted $|\mathbf{0}, \mathbf{0}\rangle$.

It is well-known that the space $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ has a structure of a vertex algebra with $|\mathbf{0}, \mathbf{0}\rangle$ as its vacuum vector and the parity given by the number of the modes of m^{ferm} and n^{ferm} modulo 2. The fields of this algebra are linear combinations of normal ordered products of the fields

$$\begin{aligned}
m^{bos}(z) &:= \textstyle\sum_{i\in\mathbb{Z}} m_i^{bos} z^{-i-1}\\
n^{bos}(z) &:= \textstyle\sum_{i\in\mathbb{Z}} n_i^{bos} z^{-i-1}\\
m^{ferm}(z) &:= \textstyle\sum_{i\in\mathbb{Z}+\frac{1}{2}} m_i^{ferm} z^{-i-\frac{1}{2}}\\
n^{ferm}(z) &:= \textstyle\sum_{i\in\mathbb{Z}+\frac{1}{2}} n_i^{ferm} z^{-i-\frac{1}{2}}
\end{aligned}$$

and their derivatives with respect to z. Normal ordered product means that the generators with positive modes are always applied first (see [10]). We remark that, in particular, the fields $m^{bos}(z)$ correspond to the elements $m_{-1}^{bos}|\mathbf{0}, \mathbf{0}\rangle$ which we will call m^{bos}, abusing the notations. Similarly, we define n^{bos}, m^{ferm} and n^{ferm} as the elements that correspond to the above fields. We remark that while we have

$m_i^{bos} = m_{(i)}^{bos}$ and $n_i^{bos} = n_{(i)}^{bos}$ where the brackets in the subscripts signify the convention from Definition 2.1, we get $m_i^{ferm} = m_{(i-\frac{1}{2})}^{ferm}$ and $n_i^{ferm} = n_{(i-\frac{1}{2})}^{ferm}$.

We have not yet used the lattice structure of M and N, since we have only worked with their complexifications. We will now define a bigger vertex algebra $\mathrm{Fock}_{M\oplus N}$ which will contain $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ as a subalgebra. As a vector space, $\mathrm{Fock}_{M\oplus N}$ is a tensor product over $\mathbb{C}$ of $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ and the group algebra $\mathbb{C}[M \oplus N]$. We will denote the tensor product of $|\mathbf{0}, \mathbf{0}\rangle$ with the basis element of the group algebra that corresponds to $m \oplus n \in M \oplus N$ by $|m, n\rangle$. We will denote the corresponding subspace of $\mathrm{Fock}_{M\oplus N}$ by $\mathrm{Fock}_{m\oplus n}$, in agreement with our convention for $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$. Every element of $\mathrm{Fock}_{m\oplus n}$ can be thought of as a polynomial in the generators of the infinite Heisenberg and Clifford algebras with negative modes applied to $|m, n\rangle$.

Theorem 3.2. *The vector space* $\mathrm{Fock}_{M\oplus N}$ *has a natural structure of a vertex algebra.*

We omit the proof, since it is standard. However we will describe the fields of this algebra below to fix the notations.

We extend the action of $m^{bos}(z)$ and $n^{bos}(z)$ to $\mathrm{Fock}_{M\oplus N}$ by requiring that m_0^{bos} and n_0^{bos} act by $(m \cdot n')\mathbf{id}$ and $(m' \cdot n)\mathbf{id}$ on $\mathrm{Fock}_{m'\oplus n'}$ for all $m' \in M, n' \in N$. The fermionic fields m^{ferm} and n^{ferm} extend without any changes. As a result, the action of all fields of $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ extends to $\mathrm{Fock}_{M\oplus N}$. To describe the vertex algebra structure on $\mathrm{Fock}_{M\oplus N}$ we also need the so-called *vertex operators* which will be the fields of the vertex algebra that correspond to the elements $|m, n\rangle$.

Definition 3.3 ([10]). *Let* γ_{m+n} *denote the multiplication by* $[m \oplus n]$ *in the semigroup algebra* $\mathbb{C}[M \oplus N]$. *Then the field* $:\mathrm{e}^{\int (m^{bos}+n^{bos})(z)}:$ *is defined by*

$$\gamma_{m+n}(-1)^{m_0^{bos}} z^{m_0^{bos}+n_0^{bos}} \prod_{i<0} \mathrm{e}^{-(m_i^{bos}+n_i^{bos})\frac{z^{-i}}{i}} \prod_{i>0} \mathrm{e}^{-(m_i^{bos}+n_i^{bos})\frac{z^{-i}}{i}}$$

and is called vertex operator *for* $m \oplus n$. *Here* $z^{m_0^{bos}}$ *acts on* $\mathrm{Fock}_{m'\oplus n'}$ *by* $z^{m\cdot n_0}$ *and similarly for* $z^{n_0^{bos}}$ *and* $(-1)^{m_0^{bos}}$.

A general field of the vertex algebra $\mathrm{Fock}_{M\oplus N}$ is a linear combination of normal ordered products of one the fields $\mathrm{e}^{\int (m^{bos}+n^{bos})(z)}$ and the fields of $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$ where the normal ordering is taken with respect to the generators of $\mathrm{Fock}_{\mathbf{0}\oplus\mathbf{0}}$.

Remark 3.4. We will often omit normal ordering from our formulas to simplify the notations, since it will be present in all calculations.

Definition 3.5. *For any subset* S *of* $M \oplus N$ *we define the subspace* Fock_S *of* $\mathrm{Fock}_{M\oplus N}$ *as* $\oplus_{m\oplus n\in S}\mathrm{Fock}_{m\oplus n}$. *If* S *is a subsemigroup, then* Fock_S *is a vertex subalgebra.*

4. Vertex algebras of mirror symmetry

In this section we recall the combinatorics of the toric mirror symmetry and define certain families of $N = 2$ vertex algebras in terms of this combinatorics. These algebras have been first introduced in [6].

The fundamental notion of the toric mirror symmetry is that of a pair of dual reflexive Gorenstein cones. As in the previous section, we fix a pair of dual lattices M and N. We recall that a rational polyhedral cone is the intersection of a finite number of closed rational halfspaces. We will also assume that a cone does not contain any sublattices. The dual of a cone K is defined as the set of all elements of the dual rational vector space that are nonnegative on K.

Definition 4.1 ([4])**.** *Let K and K^* be rational polyhedral cones in M and N respectively that are dual to each other. These cones are called reflexive Gorenstein cones iff there exist lattice elements* $\deg \in K$ *and* $\deg^* \in K^*$ *such that the lattice generators m of all one-dimensional faces of K satisfy* $m \cdot \deg^* = 1$ *and the lattice generators n of all one-dimensional faces of K^* satisfy* $\deg \cdot n = 1$. *The number* $\deg \cdot \deg^*$ *is called* index *of the pair of reflexive Gorenstein cones.*

Definition 4.2. *Let K and K^* be dual reflexive Gorenstein cones. We denote by Δ (resp. Δ^*) the set of points of $m \in K \cap M$ (resp. $K^* \cap N$) that satisfy* $m \cdot \deg^* = 1$ *(resp.* $\deg \cdot n = 1$). *These are polytopes of dimension* $\mathrm{rk} M - 1$.

Remark 4.3. For the readers familiar with the mirror symmetry for hypersurfaces in toric varieties, we remark that in the case of index 1 the polytopes Δ and Δ^* are reflexive (see [2]). In a more general case of a Calabi-Yau complete intersection of k hypersurfaces, associated to a nef-partition, one gets reflexive Gorenstein cones of index k (see [4]). Although not all reflexive Gorenstein cones come from this construction, their number is known to be finite in any given dimension.

To define vertex algebras of mirror symmetry we will need additional data, namely a pair of *coefficient functions* $f : \Delta \to \mathbb{C}$ and $g : \Delta^* \to \mathbb{C}$ which assign a constant to every lattice point of Δ and Δ^*. The terminology is justified by the fact that in the hypersurface case these functions give the coefficients of the defining equation.

Definition 4.4. *A coefficient function f is called* non-degenerate *iff the quotient of the semigroup ring $\mathbb{C}[K]$ by the ideal generated by the elements $z_n = \sum_{m \in \Delta} f(m)(m \cdot n)[m]$ for $n \in N$ is finite-dimensional. Here $[m]$ denotes the element of $\mathbb{C}[K]$ that corresponds to the element $m \in K$. Non-degenerate functions g are defined similarly.*

One can show (see [1] or [5]) that the non-degenerate functions f form a Zariski open subset in the set of all possible f. Moreover, for any basis $\{n_k\}$ of $N_{\mathbb{C}}$ and any non-degenerate f, elements z_{n_k} form a regular sequence in $\mathbb{C}[K]$.

We are now ready to describe the vertex algebras of interest.

Definition 4.5. *Let K and K^* be dual reflexive Gorenstein cones. Let f and g be non-degenerate coefficient functions for K and K^* respectively. Then the* vertex algebra of mirror symmetry $V_{f,g}$ *is the cohomology of the lattice vertex algebra* $\mathrm{Fock}_{M\oplus N}$ *with respect to the operator* $D_{f,g} = D_f + D_g$ *where*

$$D_f := \mathrm{Res}_{z=0} \sum_{m\in\Delta} f(m) m^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)},$$

$$D_g := \mathrm{Res}_{z=0} \sum_{n\in\Delta^*} g(n) n^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)}.$$

We remark that an easy OPE calculation shows that $D_f^2 = D_g^2 = D_f D_g + D_g D_f = 0$, so the above operator is indeed a differential. The vertex algebra structure on $\mathrm{Fock}_{M\oplus N}$ induces a vertex algebra structure on the cohomology by $D_{f,g}$. Moreover, one can endow $V_{f,g}$ with $N = 2$ structure by considering the elements G^+, G^-, J, L that correspond to the fields

$$\begin{aligned}
G^+(z) &= \sum_k (n^k)^{bos}(z)(m^k)^{ferm}(z) - \partial_z \deg^{ferm}(z)\\
G^-(z) &= \sum_k (m^k)^{bos}(z)(n^k)^{ferm}(z) - \partial_z (\deg^*)^{ferm}(z)\\
J(z) &= \sum_k (m^k)^{ferm}(z)(n^k)^{ferm}(z) + \deg^{bos}(z) - (\deg^*)^{bos}(z)\\
L(z) &= \sum_k (m^k)^{bos}(z)(n^k)^{bos}(z) + \frac{1}{2}\sum_k \partial_z (m^k)^{ferm}(z)(n^k)^{ferm}(z)\\
&\quad -\frac{1}{2}\sum_k (m^k)^{ferm}(z)\partial_z (n^k)^{ferm}(z) - \frac{1}{2}\deg^{bos}(z) - \frac{1}{2}(\deg^*)^{bos}(z)
\end{aligned} \tag{1}$$

where $\{m^k\}$ and $\{n^k\}$ are (any) dual bases of $M_{\mathbb{C}}$ and $N_{\mathbb{C}}$.

Proposition 4.6. *Elements* $G^{\pm}$, J, L *lie in the kernel of* $D_{f,g}$ *and therefore descend to its cohomology. The central charge of the resulting* $N = 2$ *algebra is* $\mathrm{rk} M - 2\deg\cdot\deg^*$.

Proof. This is a standard calculation in OPEs and is left to the reader. □

5. Strongly non-degenerate coefficient functions

The setup for this section is the following. Let K be a cone in M such that there exists an element $\deg^* \in N = \mathrm{Hom}(M,\mathbb{Z})$ such that the values of the linear function $\deg^* : M \to \mathbb{Z}$ on the minimum lattice generators of one-dimensional cones of K are 1. We will denote the natural pairing $M \times N \to \mathbb{Z}$ by $\cdot$. In the applications of this paper the cone K will be reflexive Gorenstein, but we only need the above weaker condition. We denote by Δ the set of lattice points $m \in K$ that satisfy $m\cdot\deg^* = 1$. As before, $\deg^*$ provides the semigroup ring $\mathbb{C}[K]$ with a grading and every coefficient function $f : \Delta \to \mathbb{C}$ gives a homogeneous element of degree one in $\mathbb{C}[K]$.

We recall that for any basis $\{n_i\}$ of $N_{\mathbb{C}}$ and any non-degenerate f the elements z_{n_i} form a regular sequence. We now introduce a technical notion of strongly non-degenerate coefficient functions, which will be used in the next section. Let n_0 be an element of N. It provides $\mathbb{C}[K]$ with additional grading $\cdot n_0$ which may or may not be nonnegative. Let $\{n_i\}$, $i = 1, \dots, \mathrm{rk}N$ be a basis of N. We define $z_{i,j}$ to be the j-th graded component of the element $z_i := z_{n_i}$ with respect to the $\cdot n_0$ grading. We consider all values of $\cdot n_0$ that occur for some $m \in \Delta$, but some of the $z_{i,j}$ may well be zero. We will denote the set of possible values of $m \cdot n_0$ by $I(n_0)$.

We then consider the Koszul complex associated to the sequence of elements $z_{i,j} \in \mathbb{C}[K]$ with $j \in I(n_0)$. It is given by

$$\wedge(\oplus_i \oplus_{j \in I(n_0)} \mathbb{C}e_{i,j}) \otimes_{\mathbb{C}} \mathbb{C}[K] \tag{2}$$

where $\wedge$ means the exterior algebra and the differential is given by $d = \sum_{i,j} \lrcorner(e_{i,j}) z_{i,j}$. Clearly, the cohomology $W(f; n_0)$ of this complex is independent from the choice of the basis $\{n_i\}$.

Proposition 5.1. *For every non-degenerate f and every n_0 the space $W(f; n_0)$ is finite-dimensional.*

Proof. Denote by j_{min} the smallest element in $I(n_0)$. The vector space $E = \oplus_i \oplus_{j \in I(n_0)} \mathbb{C}e_{i,j}$ is the direct sum of the subspace E_1 generated by $e_i = e_{i,j_{min}}$ and the subspace E_2 generated by $e'_{i,j} = e_{i,j} - e_{i,j_{min}}$, $j > j_{min}$. The direct sum decomposition $E = E_1 \oplus E_2$ induces the decomposition

$$\wedge^r E = \oplus_{k+l=r} \wedge^k E_1 \otimes_{\mathbb{C}} \wedge^l E_2.$$

We use this decomposition to realize the Koszul complex associated to the sequence of elements $z_{i,j} \in \mathbb{C}[K]$ as the total complex of a double complex. Namely, we can decompose the differential d on $\wedge E \otimes_{\mathbb{C}} \mathbb{C}[K]$ as $d = d_1 + d_2$ where

$$d_1 = \sum_i \lrcorner(e_i) z_i, \ d_2 = \sum_{i, j > j_{min}(n_0)} \lrcorner(e'_{i,j}) z_{i,j}.$$

Here we have used $z_i = \sum_j z_{i,j}$.

In view of the spectral sequence for the double complex, to show that $W(f; n_0)$ is finite-dimensional, it is enough to show that the cohomology of $\wedge E_1 \otimes_{\mathbb{C}} \wedge E_2 \otimes_{\mathbb{C}} \mathbb{C}[K]$ with respect to d_1 is finite-dimensional. This cohomology is isomorphic to the tensor product over $\mathbb{C}$ of $\wedge E_2$ and the cohomology of $\wedge E_1 \otimes_{\mathbb{C}} \mathbb{C}[K]$ with respect to d_1. The latter is simply the Koszul complex for $\{z_i\}$, so the statement follows from the assumption that f is non-degenerate.

Definition 5.2. *A non-degenerate f is called* strongly non-degenerate *if for each n_0 the dimension of $W(f, n_0)$ is minimum among all possible choices of the coefficient functions.*

Proposition 5.3. *Strongly non-degenerate functions form a Zariski open subset of all coefficient functions.*

Proof. Indeed, while there are infinitely many different choices of n_0 they lead to only finitely many choices for $\{z_{i,j}\}$, since Δ has only finitely many points. For each such choice the minimality of the dimension of the quotient is a Zariski open condition.

Observe that we have not used the double grading by $\cdot \deg^*$ and $\cdot n_0$ in the definition of strongly non-degenerate f. However, $W(f, n_0)$ does inherit a double grading from $\mathbb{C}[K]$ since the Koszul complex is taken for a sequence of homogeneous elements. There is a unique, up to a shift, way to endow the Koszul complex with the structure of the double-graded module over $\mathbb{C}[K]$, in a way to make its differential grading-preserving.

Proposition 5.4. *Let f be strongly non-degenerate. Then for every n_0 the dimension of each double-graded piece of $W(f, n_0)$ is the smallest among all possible choices of f.*

Proof. It is clear that the dimension of each graded component $W(f, n_0)$ is at least as big as it is for a general choice of f. However, the sum of the dimensions is the same as for the general coefficient function, which means that all graded dimensions are minimum.

We will also need to consider *partial* semigroup rings[1]. Namely, let $\mathcal{T}$ be a regular triangulation of Δ. The regularity condition is best described in terms of the corresponding decomposition Φ of K into simplicial cones. It means that there is a real valued continuous function h on K which is linear on each cone and satisfies the convexity relation

$$h(x + y) \geq h(x) + h(y)$$

for all $x, y \in K$ where the equality holds iff x and y lie in the same cone of Φ. Moreover, h can be picked to have integer values on the lattice points of K. We can define the *partial semigroup ring* $\mathbb{C}[K]^\Phi$ by redefining the product of monomials $[m][m_1]$ to be $[m + m_1]$ if m and m_1 are in the same cone of Φ and zero otherwise. It has been shown in [8] that $\mathbb{C}[K]^\Phi$ is a Cohen-Macaulay ring. Moreover, if we introduce the notion of non-degenerate coefficient function for $\mathbb{C}[K]^\Phi$, such functions form a Zariski open subset (see [8]). As a consequence, the theory developed in this section transfers immediately to $\mathbb{C}[K]^\Phi$ in place of $\mathbb{C}[K]$. We will write $^\Phi$ to indicate the partial semigroup theory.

We will be interested in the relation between the dimensions of the double-graded components of $W(f^\Phi, n_0)^\Phi$ and $W(f, n_0)$.

Proposition 5.5. *Let f be a strongly non-degenerate coefficient function for $\mathbb{C}[K]$ and let f^Φ be any non-degenerate coefficient function for $\mathbb{C}[K]^\Phi$. Then for a fixed n_0 and a fixed value of the double grading, the dimension of the double-graded component of $W(f, n_0)$ does not exceed that of the corresponding component of $W(f^\Phi, n_0)^\Phi$.*

[1] These have been called deformed semigroup rings in our earlier papers.

Proof. The regularity of triangulation allows us to see $\mathbb{C}[K]^{\Phi}$ as a certain limit of $\mathbb{C}[K]$. Namely, consider a family of multiplication structures $*_q$ on the vector space $\mathbb{C}[K]$ by defining

$$[m] *_q [m_1] = [m + m_1]q^{h(m+m_1)-h(m)-h(m_1)}$$

where h is the function from the definition of regularity. For $q \neq 0$ the resulting ring is isomorphic to the ring $\mathbb{C}[K]$ under the rescaling $[m] \to [m]q^{h(m)}$. At $q = 0$ the resulting ring is precisely $\mathbb{C}[K]^{\Phi}$.

Consider the family of Koszul complexes with $f_q = f^{\Phi}$. Since f^{Φ} is non-degenerate for $\mathbb{C}[K]^{\Phi}$, the dimension of the cohomology at $q = 0$ is finite. Since this construction is compatible with the double grading which has finite-dimensional components, it implies that the dimensions of the graded components of the cohomology of the Koszul differential at $q \neq 0$ do not exceed those for $q = 0$. On the other hand, the Koszul complex at $q \neq 0$ is isomorphic to the Koszul complex for $\mathbb{C}[K]$ and the coefficient function $f_q(m) = f^{\Phi}(m)q^{-h(m)}$. Finally, by Proposition 5.4, the dimensions of the graded pieces of the cohomology of the Koszul complex for f do not exceed those for any other coefficient function, so in particular they do not exceed those for f_q.

6. Main Theorem

The goal of this section is to show that for general choices of f and g the vertex algebra $V_{f,g}$ is of σ-model type. We will also describe the subspaces of the fields of the lattice vertex algebra which descend to the chiral rings of $V_{f,g}$, although we defer the more explicit calculation of A- and B-rings of the algebra until the next section.

The following proposition describes the action on $\mathrm{Fock}_{M\oplus N}$ of the operators $L_{(1)}$ and $J_{(0)}$ of the $N = 2$ structure defined in Section 4.

Proposition 6.1. *Consider an element*

$$v = \prod_p (m^p)^{bos}_{-i_p} \prod_q (n^q)^{bos}_{-j_q} \prod_r (m^r)^{ferm}_{-i'_r} \prod_s (n^s)^{ferm}_{-j'_s} |\hat{m}, \hat{n}\rangle$$

where $m^p, m^r \in M_{\mathbb{C}}$ and $n^q, n^s \in N_{\mathbb{C}}$. Then

$$\begin{aligned} L_{(1)}v &= (\hat{m} \cdot \hat{n} + \tfrac{1}{2} \deg \cdot \hat{n} + \tfrac{1}{2}\hat{m} \cdot \deg^* \\ &\quad + \textstyle\sum_p i_p + \sum_q j_q + \sum_r i'_r + \sum_s j'_s)v \\ J_{(0)}v &= (\deg \cdot \hat{n} - \hat{m} \cdot \deg^* + \textstyle\sum_r 1 - \sum_s 1)v. \end{aligned}$$

Proof. The zeroeth modes of $\deg^{bos}(z)$ and $(\deg^*)^{bos}(z)$ act on $\mathrm{Fock}_{\hat{m}\oplus\hat{n}}$ by multiplication by $\deg \cdot \hat{n}$ and $\hat{m} \cdot \deg^*$ respectively, which accounts for the terms with deg and $\deg^*$ in the above formulas. The rest is a standard calculation in free bosonic and fermionic fields (see [10, Sections 3.5, 3.6]).

Corollary 6.2. *In the notations of the above proposition, the operator H_A is given by*

$$H_A v = (\hat{m}\cdot(\deg^* + \hat{n}) + \sum_p i_p + \sum_q j_q + \sum_r (i'_r - \frac{1}{2}) + \sum_s (j'_s + \frac{1}{2}))v.$$

Proof. Operator H_A was defined as $L_{(1)} - \frac{1}{2}J_{(0)}$, so it remains to use Proposition 6.1.

We first give some estimates of the cohomology of $\mathrm{Fock}_{K\oplus N}$ by D_f. While H_A has some negative eigen-values on $\mathrm{Fock}_{K\oplus N}$ due to the $\hat{m}\cdot(\deg^* + \hat{n})$ term in Corollary 6.2, it turns out to be nonnegative on the D_f cohomology, at least for a strongly non-degenerate f.

Proposition 6.3. *For a strongly non-degenerate f the D_f-cohomology of* $\mathrm{Fock}_{K\oplus N}$ *has only eigenspaces of $H_A \geq 0$. Moreover, the $H_A = 0$ eigenspace comes from* $\mathrm{Fock}_{K\oplus(K^*-\deg^*)}$.

Proof. Clearly, D_f does not change the N-grading. Let us fix for a moment $n_0 \in N$ and work in $V_{n_0} = \mathrm{Fock}_{K\oplus n_0}$. If n_0 lies in $K^* - \deg^*$ then H_A is nonnegative on V_{n_0}. As a result, it is enough to show that for $n_0 \notin K^* - \deg^*$ the values of H_A on D_f cohomology of V_{n_0} are at least 1.

From now on we will assume that $n_0 \notin K^* - \deg^*$. The space V_{n_0} is filtered by the spaces W_k defined as the span of all elements v of Proposition 6.1 with $\sum_p 1 - \sum_q 1 \geq k$, i.e. the number of bosonic modes of coming from lattices M minus the number of bosonic modes coming from lattice N is at least k. Since D_f is a linear combination of products of some modes of m^{ferm} and some modes of $\mathrm{e}^{\int m^{bos}}$, it preserves the filtration. Together with $\cdot\deg^*$ grading, this provides V_{n_0} with the structure of the filtered complex, with D_f as its differential.

Notice that the filtration by W_k is also compatible with H_A grading, so it is enough to consider a fixed eigenvalue of H_A. For every value of $\cdot\deg^*$, there are only finitely many possible $m \in K$. As a result, Corollary 6.2 shows that the dimension of the corresponding eigenspace of H_A in the fixed $\cdot\deg^*$ graded component of $\mathrm{Fock}_{m\oplus n_0}$ is finite. As a result, the $\{W_k\}$ filtration is finite, which assures the convergence of the spectral sequence of the filtered complex. So to show that a certain H_A-component of D_f cohomology of V_{n_0} is zero, it is enough to show that the cohomology of the differential d induced by D_f on the H_A-component of W_k/W_{k+1} is zero for all k.

Let us consider the action of d on $\oplus_k W_k/W_{k+1}$. Recall that

$$D_f = \oint dz \sum_{m\in\Delta} f(m) m^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)}.$$

The only mode of $\mathrm{e}^{\int m^{bos}(z)}$ that acts non-trivially on $\oplus_k W_k/W_{k+1}$ is given by the shift γ_m. As a result, the action of d on $\oplus_k W_k/W_{k+1}$ is given by

$$d = \sum_{m\in\Delta} f(m) m^{ferm}_{l_m} \gamma_m$$

where l_m depends on m and n_0 only. In fact,

$$l_m = m \cdot (n_0 + \deg^*) - \frac{1}{2} = m \cdot n_0 + \frac{1}{2}. \tag{3}$$

It can be derived, for instance, from the fact that D_f commutes with H_A and therefore does not change the H_A grading.

As before, we denote by $I(n_0)$ the set of all possible values of $m \cdot n_0$ for $m \in \Delta$. The action of d on $\oplus_k W_k/W_{k+1}$ really comes from its action on the space

$$U_{n_0} = \Big(\wedge (\oplus_{j \in I(n_0)} F_j)\Big) \otimes_{\mathbb{C}} \mathbb{C}[K].$$

Here F_j is $(M_{\mathbb{C}})^{ferm}_{j+\frac{1}{2}}$ for $j + \frac{1}{2} < 0$ and $(N_{\mathbb{C}})^{ferm}_{-j-\frac{1}{2}}$ otherwise. Namely, the space $\oplus_k W_k/W_{k+1}$ is the tensor product of U_{n_0} and the polynomial algebra in all other non-positive modes of $m^{bos}, n^{bos}, m^{ferm}, n^{ferm}$, and the action of d comes from the U_{n_0} component of this tensor product. Hence, the cohomology of d is the tensor product of its cohomology on U_{n_0} and this other space. Clearly, extra non-positive modes can not decrease the value of H_A so it is enough to show that the cohomology of d on U_{n_0} has the desired properties with respect to H_A grading. At this point one can forget about vertex algebras and talk simply about the action of $d = \sum_{m \in \Delta} f(m)[m] \otimes m^{ferm}_{l_m}$ on the space U_{n_0}. The action of $m^{ferm}_{j+\frac{1}{2}}$ is either multiplication or contraction, depending on whether or not $j + \frac{1}{2} < 0$.

The key observation is that this complex is precisely isomorphic to the Koszul complex considered in Section 5. To see this, we consider the isomorphism between $\otimes_{j \in I(n_0)} \wedge F_j$ and $\wedge(\oplus_i \oplus_{j \in I(n_0)} \mathbb{C} e_{i,j})$ of (2) given as follows. Let $\{n_i\}$ be a basis of N and let $\{m_i\}$ be the dual basis of M. For $j + \frac{1}{2} > 0$ we can identify F_j with $\oplus_i \mathbb{C} e_{i,j}$ by mapping $(n_i)^{ferm}_{-j-\frac{1}{2}}$ to $e_{i,j}$. Then the part of the differential d

$$\sum_{m \in \Delta, l_m = j + \frac{1}{2}} f(m)[m] \otimes m^{ferm}_{l_m} \tag{4}$$

corresponds exactly to the part of the Koszul differential

$$\sum_i \lrcorner e_{i,j} z_{i,j}. \tag{5}$$

For $j + \frac{1}{2} < 0$ we identify $\wedge F_j \cong \wedge (M_{\mathbb{C}})^{ferm}_{j+\frac{1}{2}}$ with $\wedge(\oplus_i \mathbb{C} e_{i,j})$ by mapping

$$(m_{i_0})^{ferm}_{j+\frac{1}{2}} \wedge \ldots \wedge (m_{i_s})^{ferm}_{j+\frac{1}{2}}$$

to

$$(\lrcorner e_{i_0,j}) \ldots (\lrcorner e_{i_s,j}) \Big(e_{1,j} \wedge \ldots \wedge e_{\mathrm{rk} N, j}\Big).$$

This has an effect of changing the multiplication action of $m^{ferm}_{j+\frac{1}{2}}$ into the contraction action by a linear combination of $e_{i,j}$, and once again (4) translates into (5). We will now use Proposition 5.5.

Recall that n_0 does not lie in $K^* - \deg^*$, which means that at least one of the l_m is negative. Let us fix this m and call it $\hat{m}$. Consider the regular triangulation of Δ so that $\hat{m}$ is contained in any nonboundary simplex. Such a triangulation can be constructed from the following collection of heights on lattice points of Δ. Assign $\hat{m}$ height 0 and assign all other lattice points of Δ heights that are generic and close to 1. Clearly, every simplex of maximum dimension of the resulting triangulation of Δ contains $\hat{m}$. Moreover, every nonboundary simplex of this triangulation contains $\hat{m}$. Otherwise, it can be extended to a nonboundary simplex of codimension at most one with the same property. Such a codimension one simplex is a boundary of two simplices of maximum dimension, only one of which can contain $\hat{m}$. We denote by Φ the fan on K that corresponds to this triangulation. This triangulation allows us to redefine U_{n_0} by redefining the product structure on $\mathbb{C}[K]$ as in Section 5. We will denote the corresponding space by $U_{n_0}^{\Phi}$. The new differential will still be denoted by d, abusing the notations slightly. By Proposition 5.5, it is enough to show the vanishing of $H_A \leq 0$ cohomology of $U_{n_0}^{\Phi}$. In fact, it is enough to show this for one specific value of f, namely the one that has values 1 for all vertices of Δ and values 0 otherwise.

We denote by $\Phi(k)$ the set of nonboundary cones of Φ of dimension k. We can write $U_{n_0}^{\Phi}$ as the cohomology of the complex $\mathcal{E}$

$$0 \to E^{\mathrm{rk}M} \to E^{\mathrm{rk}M-1} \to \ldots \to 0$$

where

$$E^k = \oplus_{C\in\Phi(k)}\mathbb{C}[C] \otimes \prod_{l_m} \wedge\mathbb{C}^{\mathrm{rk}M}$$

and the differential d' is constructed from the restriction maps with signs coming from some choices of orientation on the cones. Indeed, for every $m \in K$ the part of this complex with this M-grading comes from nonboundary cones of Φ that contain m. It is isomorphic to a complex that calculates reduced homology of a sphere times $\prod_{l_m} \wedge\mathbb{C}^{\mathrm{rk}M}$. More importantly, the action of d on the cohomology of $U_{n_0}^{\Phi}$ is induced from the following action on the above complex. The action of d on $\mathbb{C}[C] \otimes \prod_{l_m} \wedge\mathbb{C}^{\mathrm{rk}M}$ is

$$\sum_{i=1}^{\dim C} [m] \otimes m_{i,l_i} \tag{6}$$

where m_i are the generators of one-dimensional faces of C, l_i is a shorthand for l_{m_i} and m_{i,l_i} act by either multiplication or by contraction. It is easy to see that d and the differential d' of $\mathcal{E}$ can be combined to get a double complex.

Since the cohomology of $\mathcal{E}$ with respect to d' is concentrated at the first column, its cohomology with respect to the total differential $d + d'$ is the same as the cohomology of $U_{n_0}^{\Phi}$ with respect to d. On the other hand, there is a spectral sequence that converges from the cohomology of the total space of $\mathcal{E}$ with respect to d to the d-cohomology of $U_{n_0}^{\Phi}$. As a result, it is sufficient to check that the d-cohomology of $\mathbb{C}[C] \otimes \prod_{l_m} \wedge\mathbb{C}^{\mathrm{rk}M}$ by the differential of (6) has zero H_A eigenspaces for nonpositive eigenvalues. If we recall how the H_A grading is defined, it is easy to see that it is sufficient to consider the case of $\dim C = \mathrm{rk} M$. Indeed, in the general case we can see that the complex is a product of a complex that involves the exterior algebras of $\mathbb{C}$-span of C and its dual and a vector space of nonnegative H_A grading. Analogously, we can ignore all $\wedge \mathbb{C}^{\mathrm{rk} M}$ for l_m that are not among l_i.

Let's denote the space $\mathbb{C}[C] \otimes \prod_{l_i} \wedge \mathbb{C}^{\mathrm{rk} M}$ by V. Denote by $\{n_i\}$ the basis of $N_{\mathbb{Q}}$ dual to $\{m_i\}$. As a graded space with a differential, V is isomorphic to $W \otimes \mathbb{C}[C] \times \prod_i \wedge \mathbb{C} e_i$ where W is some non-negatively graded vector space with the differential coming from d and e_i stands for either m_i or n_i depending on the sign of l_i. This complex is isomorphic to W times the Koszul complex for the regular sequence $\{[m_i]\}$. As a result, the d-cohomology is given by W tensored with $\oplus_{m \in \mathrm{Box}(C)} \mathbb{C}[m]$, which is the quotient of $\mathbb{C}[C]$ by the ideal generated by $\{[m_i]\}$. Box(C) is defined as the set of all $m \in M$ that have coordinates in [0, 1) in the basis $\{m_i\}$. However, there is an additional shift in the grading due to the fact that isomorphism with the Koszul complex occurs only after switching from the multiplication to the contraction. This adds $\sum_{l_i<0}(-l_i - \frac{1}{2})$ to the eigenvalues of H_A.

If $m = \sum_i a_i m_i$, then the value of H_A on the element $[m]$ is

$$\sum_i a_i m_i \cdot (n_0 + \deg^*) - \sum_{l_i<0} (l_i + \frac{1}{2}) \geq \sum_{l_i<0} (a_i - 1)(l_i + \frac{1}{2}) \tag{7}$$

where we have used $m_i \cdot (n_0 + \deg^*) = l_i + \frac{1}{2}$. This number is positive since by our construction there is at least one $l_i \leq -\frac{3}{2}$, namely the one that corresponds to $\hat{m}$.

Remark 6.4. A slightly weaker version of Proposition 6.3 is proved in Section 9 of [6]. However, the degeneration argument presented there only shows this statement for a generic rather than general f, i.e. one may have to exclude countably many Zariski closed subvarieties. The argument of this paper is more direct and assumes only that f is strongly non-degenerate. We don't know if the statement holds for arbitrary non-degenerate f, although we have no examples to the contrary.

We can generalize Proposition 6.3 to other eigen-values of H_A.

Proposition 6.5. *Let f be strongly non-degenerate. Then for every $k > 0$ the corresponding eigenspace of H_A on D_f-cohomology of* $\mathrm{Fock}_{K \oplus N}$ *comes from* $\mathrm{Fock}_{K \oplus (K^* - r \deg^*)}$ *for some sufficiently big r.*

Proof. We follow the proof of Proposition 6.3. At the very last step we observe that if r is big enough, then l_i that corresponds to $\hat{m}$ will be sufficiently negative to assure that $\sum_{l_i<0}(a_i - 1)(l_i + \frac{1}{2})$ in (7) is bigger than k.

Theorem 6.6. *Let f and g be strongly non-degenerate. Then $D_{f,g}$-cohomology of* $\mathrm{Fock}_{M \oplus N}$ *has only nonnegative integer eigenvalues of H_A. Moreover, the $H_A = 0$ eigenspace comes from* $\mathrm{Fock}_{K \oplus (K^* - \deg^*)}$. *The operator H_B also has only nonnegative integer eigenvalues on $V_{f,g}$ and its kernel comes from* $\mathrm{Fock}_{(K - \deg) \oplus K^*}$.

Proof. First of all, it suffices to look at the $D_{f,g}$-cohomology of $V = \mathrm{Fock}_{K\oplus N}$. Indeed, the proof of Proposition 8.2 of [6] for the *partial* lattice vertex algebra $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ transfers easily to our case.

Now consider $v \in \mathrm{Ker}(D_{f,g})$ such that $H_A v = \alpha v$ for $\alpha < 0$. Differentials D_f and D_g give V the structure of the double complex, where the double grading is provided by $(\deg\cdot, \cdot\deg^*)$. Differentials D_f and D_g increase this grading by $(0, 1)$ and $(1, 0)$ respectively. Let $v = \sum_{a,b} v_{a,b}$ be the double-graded decomposition of v. It is enough to consider the case of $a+b = k$ for a fixed k, so we have $v = \sum_i v_i$ where the double grading of v_i is $(k-i, i)$. The condition $D_{f,g}v = 0$ implies that $D_g v_{i+1} = D_f v_i$. Let j be the maximum i such that $v_i \neq 0$. We have $D_f v_j = 0$, so by Proposition 6.3, there exists a w such that $D_f w = v_j$. Then $v - D_{f,g}w$ is again in the kernel of $D_{f,g}$, but it now has a smaller value of j. By applying this procedure sufficiently many times, we can get $j < 0$, which implies that all v_i are zero, since $\cdot\deg^*$ is nonnegative on V.

The $H_A = 0$ statement is proved similarly. Let $D_{f,g}v = H_B v = 0$, where $v = \sum_i v_i$ is as above. Let j be the maximum value of i such that $v_i \notin \mathrm{Fock}_{K\oplus(K^*-\deg^*)}$. Then we have $D_f v_j = D_g v_{j+1} \in \mathrm{Fock}_{K\oplus(K^*-\deg^*)}$. Since D_f preserves the N-grading, the N-graded part v'_j of v_j that is supported outside of $K^* - \deg^*$ satisfies $D_f v'_j = 0$. Then by Proposition 6.3 there exists $w \in V$ and such that $v'_j = D_f w$. The difference $v - D_{f,g}w$ will have smaller j, which eventually leads to the situation where all $v_i \in \mathrm{Fock}_{K\oplus(K^*-\deg^*)}$.

It is clear from Corollary 6.2 that the eigenvalues of H_A are integers. Finally, the statements for H_B are obtained by switching K and K^*.

Remark 6.7. Similarly, we can use Proposition 6.5 to show that for each k $H_B = k$ eigenspace of $V_{f,g}$ comes from $\mathrm{Fock}_{K\oplus(K^*-r\deg^*)}$ for a sufficiently big r.

Theorem 6.8. *For strongly non-degenerate f and g the $N = 2$ vertex algebra $V_{f,g}$ is of σ-model type.*

Proof. It is clear that H_A and H_B are diagonalizable on $V_{f,g}$. By Theorem 6.6 the eigenvalues of H_A and H_B are nonnegative integers. It has been shown in [7, Lemma 4.5] that the common eigenspaces of $L_{(1)}$ and $J_{(0)}$ of $V_{f,g}$ are finite-dimensional. Since $L_{(1)} \pm \frac{1}{2}J_{(0)} \geq 0$, there are only finitely many such common eigenspaces for a fixed value of $L_{(1)}$, which proves that the eigenspaces of $L_{(1)}$ are finite-dimensional. It is also clear from Proposition 6.1 that the eigenvalues of $L_{(1)}$ are in $\frac{1}{2}\mathbb{Z}$, which finishes the proof.

7. Relation to the conjectural description of stringy cohomology

In this section we examine the structure of the chiral ring of $V_{f,g}$ in more detail and connect it to the description of *quantum stringy cohomology* suggested in [8].

Let K and K^* be dual reflexive Gorenstein cones and let f and g be strongly non-degenerate coefficient functions. Consider the ideal in the semigroup $\mathbb{C}[K \oplus K^*]$ generated by $[m\oplus n]$, $m\cdot n > 0$. Denote the quotient by this ideal by $\mathbb{C}[L]$. Consider the space $V = \wedge(N_{\mathbb{C}}) \otimes \mathbb{C}[L]$ where, as usual, $\wedge$ means the exterior algebra. We recall the following lemma.

Lemma 7.1. [8] *The space V is equipped with a differential d given by*

$$d := \sum_m f(m) \lrcorner m \otimes [m] + \sum_n g(n)(\wedge n) \otimes [n]$$

where $[m]$ and $[n]$ means multiplication by the corresponding monomials in $\mathbb{C}[K] \otimes \mathbb{C}[K^*]$ acting on the module $\mathbb{C}[L]$, $\wedge n$ means multiplication by n in the exterior algebra and $\lrcorner$ means contraction in the exterior algebra.

Remark 7.2. In view of the isomorphism $\wedge(N_{\mathbb{C}}) \cong \wedge(M_{\mathbb{C}})$ one could switch the roles of M and N in the construction of V and d without altering the resulting cohomology vector space. This is related to *spectral flow* isomorphism between the A- and B-rings of $V_{f,g}$, see Remark 7.9.

It has been shown in [8, Section 10] that the cohomology of V with respect to d is isomorphic as a vector space to the conjectural description of (quantum) stringy cohomology, in the case of a hypersurface or a complete intersection induced by a nef-partition. We will now observe that this space is also isomorphic to the B-ring of $V_{f,g}$.

Proposition 7.3. *Cohomology H of V with respect to d is naturally isomorphic to the B-ring of $V_{f,g}$.*

Proof. By Theorem 6.6 and the definition of the B-ring of $V_{f,g}$, it comes from $\mathrm{Fock}_{(K-\deg)\oplus K^*}$. Since $D_{f,g}$ commutes with H_B, the B-ring of $V_{f,g}$ is equal to the $D_{f,g}$ cohomology of the $H_B = 0$ eigenspace of $\mathrm{Fock}_{(K-\deg)\oplus K^*}$.

By Proposition 6.1 and definition of H_B,

$$H_B v = ((\hat{m} + \deg) \cdot \hat{n} + \sum_p i_p + \sum_q j_q + \sum_r (i_r + \frac{1}{2}) + \sum_s (j_s - \frac{1}{2}))v.$$

As a result, the $H_B = 0$ eigenspace of $\mathrm{Fock}_{(K-\deg)\oplus K^*}$ is spanned by products of $n^{ferm}_{-\frac{1}{2}}$ and $|\hat{m} - \deg, \hat{n}\rangle$ with $\hat{m} \cdot \hat{n} = 0$. So this eigenspace is isomorphic to V, and it remains to show that the action of d in Lemma 7 is precisely the action of $D_{f,g}$.

Let us calculate the action of $\oint m^{ferm}(z) \mathrm{e}^{\int m^{bos}(z)}\, dz$ on $w|\hat{m} - \deg, \hat{n}\rangle$ where w is a product of $n^{ferm}_{-\frac{1}{2}}$. Since the image will again satisfy $H_B = 0$, it has to be of the same form. As a result, the only mode of m^{ferm} that can give a nonzero contribution will be $m^{ferm}_{\frac{1}{2}}$ that acts by a contraction on w. Similarly, we must have $m \cdot \hat{n} = 0$, since otherwise we move out of Fock_L. Since we are taking $\oint$, the count of the degree of z forces the corresponding mode from $\mathrm{e}^{\int m^{bos}(z)}$ to be a shift γ_m.

The action of $\oint n^{ferm}(z) \mathrm{e}^{\int n^{bos}(z)}\, dz$ on $w|\hat{m} - \deg, \hat{n}\rangle$ is calculated similarly. We need to have $\hat{m} \cdot n = 0$, and the mode of n^{ferm} has to be $n^{ferm}_{-\frac{1}{2}}$, which acts by multiplication. Then the count of the degree of z forces us to have γ_n for the mode of $\mathrm{e}^{\int n^{bos}(z)}$.

We have thus shown that $D_f + D_g$ and d have identical actions on V, which finishes the proof.

In fact, by the results of [8], we can quite explicitly calculate chiral rings in terms of the spaces $R_1(\theta, f)$ which we will now define. Let f and g be non-degenerate coefficient functions. For every face θ of the Gorenstein cone K we will abuse notations and denote the restriction of f to θ by the same letter f. Then f is non-degenerate for θ as well. We consider the ideal I_f of $\mathbb{C}[\theta]$ generated by the restrictions of the elements z_i to $\mathbb{C}[\theta]$. Let $\mathbb{C}[\theta^\circ]$ be the ideal of $\mathbb{C}[\theta]$ which corresponds to the interior of θ.

Definition 7.4. *The natural inclusion* $\mathbb{C}[\theta^\circ] \to \mathbb{C}[\theta]$ *induces a (non-injective) map*

$$\mathbb{C}[\theta^\circ]/I_f\mathbb{C}[\theta^\circ] \to \mathbb{C}[\theta]/I_f\mathbb{C}[\theta]$$

and we denote its image by $R_1(\theta, f)$.

Remark 7.5. Spaces $R_1(\theta, f)$ have been considered very early in the context of mirror symmetry, see [1]. The graded dimension of $R_1(\theta, f)$ has been calculated in [8]. In the case of simplicial θ it can be obtained by a simple inclusion-exclusion formula in terms of various S-polynomials of the faces of θ. More generally, it also involves (intersection cohomology) G-polynomials of the partially ordered set of faces of θ.

For a face $\theta \subseteq K$ we denote by θ^* the dual face of K^*, defined as the set of elements of K^* which are zero on θ.

Theorem 7.6. *Let f and g be strongly non-degenerate. Then both chiral rings of the $N = 2$ vertex algebra $V_{f,g}$ are naturally isomorphic* **as vector spaces** *to the space*

$$W_{f,g} = \oplus_{\theta\subseteq K} R_1(\theta, f) \otimes_{\mathbb{C}} R_1(\theta^*, g).$$

Proof. We combine [8, Theorem 10.2] with Proposition 7.3 for the B-ring case. The A-ring case differs by a switch of M and N and thus follows from the B-ring case.

Remark 7.7. In fact, the isomorphism between the A- or B-ring and $W_{f,g}$ is pretty well-understood. We will state it for the A-ring. Let $\Phi_\theta(z)$ be the field $(m_1)^{ferm}(z) \ldots (m_{\dim\theta})^{ferm}(z)$ of $\mathrm{Fock}_{M\oplus N}$ where $m_1, \ldots, m_{\dim\theta}$ is a basis of $\theta_{\mathbb{C}}$. This is well-defined up to a constant factor. Then the part of the A-ring that corresponds to θ in $W_{f,g}$ comes from linear combinations of the fields of the form

$$\Phi_\theta(z)\mathrm{e}^{\int m^{bos}(z)+n^{bos}(z)-(\deg^*)^{bos}(z)}$$

for $m \in \theta^\circ$ and $n \in (\theta^*)^\circ$. The proof of this statement follows from the argument of [8, Theorem 10.2]. It is rather non-trivial, as we will see from the example below.

Example 7.8. Let Δ_1^* be the reflexive polytope in $\mathbb{Z}^4$ with vertices $\hat{n}_1 = (1, 0, 0, 0)$, $\hat{n}_2 = (0, 1, 0, 0)$, $\hat{n}_3 = (0, 0, 1, 0)$, $\hat{n}_4 = (0, 0, 0, 1)$, and $\hat{n}_5 = (-1, -1, -1, -1)$. The only other lattice point in Δ_1^* is $\hat{n}_0 = (0, 0, 0, 0)$. We remark that this corresponds to the famous quintic example. We consider the corresponding cone $K^* \subseteq \mathbb{Z}^4 \oplus \mathbb{Z}$ and introduce $n_i = (\hat{n}_i, 0)$. The convex hull of $\{n_i\}$ will be denoted by Δ^*. Notice that $n_0 = \deg^*$. It is rather straightforward to see that in this case $R_1(\theta^*, g)$

is zero unless $\theta^* = \{0\}$ or $\theta^* = K^*$. The space R_1 for zero-dimensional face is one-dimensional, so according to Theorem 7.6 the A- and B-rings are isomorphic as vector spaces to $R_1(K, f) \oplus R_1(K^*, g)$.

We will only describe the $R_1(K^*, g)$ part inside the A-ring, which turns out to be a four-dimensional vector space. It corresponds to $\theta = \{0\}$, so according to Remark 7.7, we need to look at linear combinations of the fields of $\mathrm{Fock}_{M\oplus N}$ of the form

$$\mathrm{e}^{\int n^{bos}(z) - n_0^{bos}(z)} \tag{8}$$

for $n \in (K^*)^\circ$. It is clear that all such fields commute with the differential $D_{f,g}$ and therefore descend to $V_{f,g}$. It is also clear that they satisfy $H_A = 0$ and hence descend to elements of the A-ring.

Let us fix a non-degenerate coefficient function g, i.e. six complex numbers $g(n_i)$, $i = 0, \ldots, 5$. For every $m \in M$ and every $n \in K^*$ the field $m^{ferm}(z)\mathrm{e}^{\int n^{bos}(z)}$ anticommutes with D_f and gives

$$\sum_{i=0}^{5} g(n_i)(m \cdot n_i)\mathrm{e}^{\int n_i^{bos}(z) + n^{bos}(z)}$$

when acted upon by $D_{f,g}$. As a result, if we identify the space of fields of the form (8) with $\mathbb{C}[(K^*)^\circ]$, we see that the map to the A-ring passes through the quotient $\mathbb{C}[(K^*)^\circ]/I_g\mathbb{C}[(K^*)^\circ]$. In this particular example, this means that it is enough to consider the linear combinations of the fields $\mathrm{e}^{\int kn_0^{bos}(z)}$ for $k = 0, 1, 2, 3, 4$. We leave some easy commutative algebra calculations to the reader.

It is much more difficult to see that the map from $\mathbb{C}[(K^*)^\circ]$ to the A-ring passes through $R_1(K^*, g)$. This means that in fact $\mathrm{e}^{\int 4n_0^{bos}(z)}$ maps to zero in the A-ring. Denote by m_i, $i = 1, \ldots, 5$ the elements of the cone K that correspond to vertices of the dual polytope Δ. More specifically, they will be given by $(4, -1, -1, -1, 1)$, $(-1, 4, -1, -1, 1)$, $(-1, -1, 4, -1, 1)$, $(-1, -1, -1, 4, 1)$ and $(-1, -1, -1, -1, 1)$ respectively. Their scalar products with n_j are $(m_i \cdot n_j) = 5\delta_i^j$ for $j > 0$ and $(m_i \cdot n_0) = 1$.

Consider the fields

$$\begin{aligned}
R_1(z) &= \frac{1}{5g_1} m_1^{ferm}(z)\mathrm{e}^{\int (n_2+n_3+n_4+n_5-n_0)^{bos}(z)} \\
R_2(z) &= \frac{g_0}{5^2 g_1 g_2} m_2^{ferm}(z)\mathrm{e}^{\int (n_3+n_4+n_5)^{bos}(z)} \\
R_3(z) &= \frac{g_0^2}{5^3 g_1 g_2 g_3} m_3^{ferm}(z)\mathrm{e}^{\int (n_4+n_5+n_0)^{bos}(z)} \\
R_4(z) &= \frac{g_0^3}{5^4 g_1 g_2 g_3 g_4} m_4^{ferm}(z)\mathrm{e}^{\int (n_5+2n_0)^{bos}(z)} \\
R_5(z) &= \frac{g_0^4}{5^5 g_1 g_2 g_3 g_4 g_5} m_5^{ferm}(z)\mathrm{e}^{\int 3n_0^{bos}(z)}.
\end{aligned}$$

We observe that $D_f R_i(z) = 0$ for all $i = 1, \ldots, 5$. Indeed, when we write OPE of $m^{ferm}(z)\mathrm{e}^{\int m^{bos}(z)}$ with $R_1(w)$ for some $m \in \Delta$, we will only get a pole at $z = w$ for the bosonic part when $m = m_1$. However, in this case the fermions contribute a zero of order one which cancels the singularity. There are no poles in the bosonic parts of OPE for any of the other R_i, since the exponents lie in K^*.

On the other hand, when we calculate $D_g R_i(z)$ only two terms of D_g will matter, namely the terms for n_i and n_0, in view of the scalar products $m_i \cdot n_j$. This yields

$$D_g R_i(z) = \frac{g_0^{i-1}}{5^{i-1} g_1 \cdots g_{i-1}} e^{\int (n_i + \ldots + n_5 + (i-2)n_0)^{bos}(z)} + \frac{g_0^{i}}{5^{i} g_1 \cdots g_{i}} e^{\int (n_{i+1} + \ldots + n_5 + (i-1)n_0)^{bos}(z)}.$$

As a result, $D_{f,g} \sum_{i=1}^{5} (-1)^{i-1} R_i(z)$ gives $(1 + \frac{g_0^5}{5^5 g_1 \cdot g_5}) e^{\int 4n_0^{bos}(z)}$ where we have used the fact that $\sum_{i=1}^{5} n_i = 5n_0$. This shows that $e^{\int 4n_0^{bos}(z)}$ maps to zero in $V_{f,g}$. It is amusing to notice that the coefficient in front is zero exactly when the dual to the quintic has additional singularities, i.e. when g is degenerate.

As a result of the above calculation we see that the diagonal part of the quantum cohomology ring of the quintic is isomorphic to $\mathbb{C}[t]/t^4$, where t is the image of $e^{\int n_0^{bos}(z)}$.

Remark 7.9. We can observe that the part of $W_{f,g}$ that correspond to θ comes from

$$\Phi_\theta(z) e^{\int m^{bos}(z) + n^{bos}(z) - (\deg^*)^{bos}(z)},$$

for the A-ring and comes from

$$\Psi_{\theta^*}(z) e^{\int m^{bos}(z) + n^{bos}(z) - \deg^{bos}(z)},$$

for the B-ring, where Ψ is defined analogously to Φ. This amounts to an action of the field

$$S(z) = \Phi_K(z) e^{\int \deg^{bos}(z) - (\deg^*)^{bos}(z)}.$$

We can compare this with the spectral flow defined physically in [12], by noticing that *formally* $S(z) = e^{\int J(z)}$, in view of the boson-fermion correspondence for the fermionic part of $\mathrm{Fock}_{M \oplus N}$.

To end this section, we remark on the relation to non-quantum stringy cohomology spaces of the Calabi-Yau hypersurfaces in toric varieties. These have been defined in [8] as follows. Let Δ and Δ^* be dual reflexive polytopes in M_1 and $N_1 = M_1^*$ and let K and K^* be the corresponding reflexive Gorenstein cones in the lattices $M = M_1 \oplus \mathbb{Z}\deg$ and $N = N_1 \oplus \mathbb{Z}\deg^*$. Let Σ_1 be a regular fan in N_1 whose one-dimensional cones contain all vertices of Δ^*. We can define a decomposition Σ of N into a union of semigroups by extending Σ' into the $\mathbb{Z}\deg^*$ direction. When intersected with K^*, Σ becomes a fan, which we will denote by the same letter. Consider the *partial semigroup ring* $\mathbb{C}[M \oplus N]^\Sigma$. It is isomorphic as a vector space to the ring $\mathbb{C}[M \oplus N]$ but the multiplication product is redefined by

$$[m \oplus n][\hat{m} \oplus \hat{n}] = \begin{cases} [(m + \hat{m}) \oplus (n + \hat{n})], & \text{if } n, \hat{n} \in C, \text{ for some } C \in \Sigma \\ 0, & \text{otherwise.} \end{cases}$$

We redefine the shifts $\gamma_{m\oplus n}$ as multiplication in the partial semigroup ring. Then the stringy cohomology for the given two coefficient functions is again the cohomology of the differential of Lemma 7.

One can use the redefined shifts $\gamma_{m\oplus n}$ to construct *partial lattice vertex algebra* $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ (see [6]). Then one can still define coefficient functions and the differential $D_{f,g}$. We denote the cohomology of $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ by $D_{f,g}$ by $V^{\Sigma}_{f,g}$. We claim that its chiral rings again equal the stringy cohomology. Indeed, the calculations for the A-rings are unchanged, since the differential D_f is not affected by Σ, and Proposition 6.3 still holds. It is slightly more difficult to show that the B-ring of $V^{\Sigma}_{f,g}$ can be calculated by the differential $D_{f,g}$ on the $H_B = 0$ part of $\mathrm{Fock}^{\Sigma}_{(K-\deg)\oplus K^*}$. Indeed, we can not simply follow the proof of Proposition 6.3 with the roles of K and K^* interchanged, because we need to be able to degenerate $\mathbb{C}[K^*]^{\Sigma}$ according to a fan Φ, which is impossible. However, one can show that $V^{\Sigma}_{f,g}$ is isomorphic to $D_{f,g}$ cohomology of the space $\mathrm{Fock}_{(K-\deg)\oplus N}$ by modifying the argument of [6, Proposition 8.2] slightly. Then one can prove that H_B is nonnegative on $V_{f,g}$ and $(H_B = 0)$-part of D_f-cohomology of $\mathrm{Fock}^{\Sigma}_{(K-\deg)\oplus N}$ comes from $\mathrm{Fock}^{\Sigma}_{(K-\deg)\oplus K^*}$, by following the arguments of Proposition 6.3. More specifically, we now assume that $n_0 \notin K^*$, which means that there exists a vertex $\hat{m}$ of Δ with $\hat{m}\cdot n < 0$. The formula (3) for l_m is unchanged. We still basically have a Koszul complex, and can perform the degeneration argument with respect to a triangulation Φ. Instead of $\mathrm{Box}(C)$ we will be now dealing with $\mathrm{Box}(C) - \deg$. The inequality (7) will then translate into

$$\begin{aligned} H_B &= (\sum_i a_i m_i)\cdot n_0 - \sum_{l_i<0}(l_i - \frac{1}{2}) \\ &= (\sum_i a_i m_i)\cdot n_0 - \sum_{l_i<0} m_i\cdot n_0 \geq \sum_{l_i<0}(a_i - 1)m_i\cdot n_0 \end{aligned}$$

which will be positive due to the term for $m_i = \hat{m}$. The transition from D_f-cohomology to $D_{f,g}$ cohomology in Theorem 6.6 is unchanged for the partial semigroup case.

Remark 7.10. The dimensions of the chiral rings are clearly unchanged after passing to $V^{\Sigma}_{f,g}$, in view of the calculation of [8] that shows that the dimensions of R_1 are unchanged.

Remark 7.11. We have not mentioned the double grading on A-rings and B-rings of $V_{f,g}$ and $V^{\Sigma}_{f,g}$, which corresponds to the Hodge structure $H = \oplus H^{p,q}$ on the stringy cohomology. The Hodge components are the common eigenspaces of $J_{(0)}$ and the grading $\deg\cdot + \cdot\deg^*$ which clearly descends to the cohomology of $D_{f,g}$. It appears that in some vague sense, this grading is what's left of anti-holomorphic fields of the vertex algebra in the definition of Kapustin and Orlov [11] after taking $\bar{\partial}$ cohomology. However we do not know how to construct a Kapustin-Orlov vertex algebra responsible for mirror symmetry.

8. Open questions

Hopefully, the reader is convinced that the $N = 2$ vertex algebras $V_{f,g}$ provide rich examples of vertex algebras. However, many of their properties are still not well-understood. We list here several open questions of algebraic nature related to this construction, but this list is by no means complete.

Question 8.1. Are $V_{f,g}$ actually different as *vertex algebras*? This is important, since a negative answer would perhaps provide a non-trivial connection on the chiral rings of $V_{f,g}$. Of course, one has to look at generic pairs (f, g).

Question 8.2. Is there a way to see GKZ system of differential equations [9] in the context of $V_{f,g}$? This question seems related to the question of a connection on the space of $V_{f,g}$-s considered as a bundle over the space of the parameters (f, g).

Question 8.3. Are algebras $V_{f,g}$ generated by a finite number of fields? While some examples suggest this, we are not yet able to prove it in general.

Question 8.4. It is not hard to show that for a given k there is a Zariski open set in the space of coefficient functions (f, g) such that the $L_{(1)}$ eigenspace of $V_{f,g}$ has a constant dimension. Is it true that there is a Zariski open set that works for all k simultaneously? This is the question of whether $V_{f,g}$ form a flat family of vertex algebras over an algebraic parameter space.

Question 8.5. The results of this paper provide the quantum and non-quantum stringy cohomology spaces of [8] with a ring structure. Is it possible to describe this structure without a reference to vertex algebras? Note that it is quite easy to see this product structure for the "diagonal part" as in Example 7.8, but the general case is at the moment open.

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