THE ORBIFOLD CHOW RING OF A TORIC DELIGNE-MUMFORD STACK

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Abstract. Generalizing toric varieties, we introduce toric Deligne-Mumford stacks which correspond to combinatorial data. The main result in this paper is an explicit calculation of the orbifold Chow ring of a toric Deligne-Mumford stack. As an application, we prove that the orbifold Chow ring of the toric Deligne-Mumford stack associated to a simplicial toric variety is a flat deformation of (but is not necessarily isomorphic to) the Chow ring of a crepant resolution.

1. Introduction

The orbifold Chow ring of a smooth Deligne-Mumford stack, defined by Abramovich, Graber and Vistoli [AGV], is the algebraic version of the orbifold cohomology ring introduced by Chen and Ruan [CR1] [CR2]. By design, this ring incorporates numerical invariants, such as the orbifold Euler characteristic and the orbifold Hodge numbers, of the underlying singular space. The product structure is defined as the degree zero part of the quantum product; in particular, it involves Gromov-Witten invariants. Inspired by results in [Bat] and [Yas] and predictions in string theory, one expects that, in nice situations, the orbifold Chow ring coincides with the Chow ring of a resolution of singularities. Fantechi and Göttsche [FG] and Uribe [Uri] verify this conjecture when the orbifold is $\mathrm{Sym}^n(S)$ where S is a smooth projective surface with $K_S = 0$ and the resolution is $\mathrm{Hilb}^n(S)$. The initial motivation for this project was to compare the orbifold Chow ring of a simplicial toric variety with the Chow ring of a crepant resolution.

To achieve this goal, we first develop the theory of toric Deligne-Mumford stacks. Modeled on simplicial toric varieties, a toric Deligne-Mumford stack corresponds to a combinatorial object called a *stacky fan*. As a first approximation, this object is a simplicial fan with a distinguished lattice point on each ray in the fan. Hence, there is a natural toric Deligne-Mumford stack associated to every simplicial toric variety. More precisely, a stacky fan $\mathbf{\Sigma}$ consists of a simplicial fan Σ in $\mathbb{Q} \otimes_{\mathbb{Z}} N$ and a map $\beta \colon \mathbb{Z}^n \to N$ where n is the number of rays in Σ and N is a finitely generated abelian group. The stacky fan $\mathbf{\Sigma}$ encodes a group action on a quasi-affine variety and the associated toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is the quotient. We show that many of the basic concepts, such as open and closed toric substacks, line bundles, and maps between toric Deligne-Mumford stacks, correspond to combinatorial notions. We expect many more results

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about toric varieties to lift to the realm of stacks and we hope that toric Deligne-Mumford stacks, like toric varieties, will serve as a useful testing ground for general theories.

Our description for the orbifold Chow ring of a toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ parallels the "Stanley-Reisner" presentation for the Chow ring of a simplicial toric variety. Specifically, the stacky fan $\mathbf{\Sigma}$ gives rise to the *deformed group ring* $\mathbb{Q}[N]^{\mathbf{\Sigma}}$. As a $\mathbb{Q}$-vector space, $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is simply the group algebra of N. Since N is abelian, we write $\mathbb{Q}[N]^{\mathbf{\Sigma}} = \bigoplus_{c \in N} \mathbb{Q}\, y^c$ where y is a formal variable. For $c \in N$, $\bar{c}$ denotes the image of c in $\mathbb{Q} \otimes_{\mathbb{Z}} N$. Multiplication in $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is defined by the equation:

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is } \sigma \in \Sigma \text{ such that } \bar{c}_1 \in \sigma \text{ and } \bar{c}_2 \in \sigma; \\ 0 & \text{otherwise.} \end{cases}$$

If b_i is the image under the map $\beta \colon \mathbb{Z}^n \to N$ of the ith standard basis vector, then we endow $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ with a $\mathbb{Q}$-grading by setting $\deg(y^c) = \sum_{\bar{b}_i \in \sigma} m_i$ where σ is the minimal cone in Σ containing $\bar{c}$, each m_i is a positive rational number and $\bar{c} = \sum_{\bar{b}_i \in \sigma} m_i \bar{b}_i$. Let $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ be the orbifold Chow ring of $\mathcal{X}(\mathbf{\Sigma})$ with rational coefficients. Our main results is:

Theorem 1.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then there is an isomorphism of $\mathbb{Q}$-graded rings:*

$$A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big) \cong \frac{\mathbb{Q}[N]^{\mathbf{\Sigma}}}{\big\langle \sum_{i=1}^{n} \theta(b_i) y^{b_i} : \theta \in \operatorname{Hom}(N, \mathbb{Z}) \big\rangle}\,.$$

In the context of differential geometry, Jiang [Jia] establishes the analogous result for the weighted projective space $\mathbb{P}(1, 2, 2, 3, 3, 3)$.

Our proof of this theorem involves two steps. By definition, the orbifold Chow ring $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is isomorphic as an abelian group to the Chow ring of the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$. We first express $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ as a disjoint union of certain toric Deligne-Mumford stacks and establish the isomorphism in Theorem 1.1 at the level of $\mathbb{Q}$-graded vector spaces. To compare the ring structures, we also express the moduli space $\mathcal{K}_{0,3}\big(\mathcal{X}(\mathbf{\Sigma}), 0\big)$ of 3-pointed twisted stable maps as a disjoint union of toric Deligne-Mumford stacks. This combinatorial description allows us to compute the virtual fundamental class of $\mathcal{K}_{0,3}\big(\mathcal{X}(\mathbf{\Sigma}), 0\big)$. We are then able to verify that multiplication in the deformed group ring coincides with the product in the orbifold Chow ring.

The paper is organized as follows. In Section 2, we extend Gale duality to maps of finitely generated abelian groups. This duality forms an essential link between stacky fans and toric Deligne-Mumford stacks. Nevertheless, this theory is entirely self-contained, requiring only basic homological algebra, and may be of interest in other situations. The rudimentary theory of toric Deligne-Mumford stacks is developed in Sections 3 and 4. Specifically, we detail the correspondence between stacky fans and toric Deligne-Mumford stacks, we describe the open and closed toric substacks and we express the inertia stacks as disjoint unions of toric Deligne-Mumford stacks. The proof of Theorem 1.1 is given in Sections 5 and 6. Finally in Section 7, we use our main result to compare the orbifold Chow rings of a simplicial toric variety and its crepant resolutions.

Conventions. Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers and consider Chow rings and orbifold Chow rings with rational coefficients.

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2. Gale Duality with Torsion

In this section, we extend Gale duality to finitely generated abelian groups. To orient the reader, we recall the duality theory for vector configurations; see Theorem 6.14 in [Zie]. If $\{b_1 \cdots b_n\}$ is a set of n column vectors which span $\mathbb{Q}^d$, then there exists a dual configuration $[a_1 \cdots a_n] \in \mathbb{Q}^{(n-d)\times n}$ such that $0 \to \mathbb{Q}^d \xrightarrow{[b_1\cdots b_n]^{\mathsf{T}}} \mathbb{Q}^n \xrightarrow{[a_1\cdots a_n]} \mathbb{Q}^{n-d} \to 0$ is a short exact sequence. The set of column vectors $\{a_1, \ldots, a_n\}$ is uniquely determined up to linear coordinates transformation in $\mathbb{Q}^{n-d}$. We generalize this to maps of finitely generated abelian groups.

Let N be a finitely generated abelian group and consider a group homomorphism $\beta\colon \mathbb{Z}^n \to N$. The map β is determined by a finite subset $\{b_1, \ldots, b_n\}$ of N. We write $(-)^\star$ for the functor $\operatorname{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$. The dual map $\beta^\vee\colon (\mathbb{Z}^n)^\star \to \mathrm{DG}(\beta)$ is defined as follows. Choose projective resolutions $\boldsymbol{E}$ and $\boldsymbol{F}$ of the abelian groups $\mathbb{Z}^n$ and N. The map $\beta\colon \mathbb{Z}^n \to N$ lifts to a morphism $\boldsymbol{E} \to \boldsymbol{F}$ and the associated mapping cone $\mathrm{Cone}(\beta)$ fits into an exact sequence $0 \to \boldsymbol{F} \to \mathrm{Cone}(\beta) \to \boldsymbol{E}[1] \to 0$. Since $\boldsymbol{E}$ is projective, we have the exact sequence $0 \to \boldsymbol{E}[1]^\star \to \mathrm{Cone}(\beta)^\star \to \boldsymbol{F}^\star \to 0$ and taking the long exact sequence in cohomology produces the exact sequence:

$$N^\star \xrightarrow{\beta^\star} (\mathbb{Z}^n)^\star \longrightarrow H^1\big(\mathrm{Cone}(\beta)^\star\big) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(N,\mathbb{Z}) \longrightarrow 0\,. \tag{2.0.1}$$

Set $\mathrm{DG}(\beta) := H^1\big(\mathrm{Cone}(\beta)^\star\big)$ and define the dual map $\beta^\vee\colon (\mathbb{Z}^n)^\star \to \mathrm{DG}(\beta)$ to be the second map in (2.0.1); both are well-defined up to natural isomorphism. Since $\mathbb{Z}^n$ is projective, $\beta^\vee$ is in fact the only nontrivial map from $H^i(\boldsymbol{E}^\star)$ to $H^{i+1}\big(\mathrm{Cone}(\beta)^\star\big)$. This abstract definition guarantees that $(-)^\vee$ is a contravariant functor; see Lemma 2.3.

On the other hand, there is an explicit description of the dual map $\beta^\vee$ and the dual group $\mathrm{DG}(\beta)$. The structure theorem of finitely generated abelian groups implies $N \cong \mathbb{Z}^d \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/q_r\mathbb{Z}$. Hence, $0 \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \to 0$ is a projective resolution of N and the map $\beta\colon \mathbb{Z}^n \to N$ lifts to a map $\mathbb{Z}^n \to \mathbb{Z}^{d+r}$ given by a matrix B. Since $\mathbb{Z}^n$ is projective, the complex with $E_0 = \mathbb{Z}^n$ and $E_i = 0$ for all $i \neq 0$ is a projective resolution of $\mathbb{Z}^n$. With these choices, $\mathrm{Cone}(\beta)$ is the complex $0 \to \mathbb{Z}^{n+r} \xrightarrow{[B\;Q]} \mathbb{Z}^{d+r} \to 0$ and we obtain the sequence (2.0.1) by applying the Snake Lemma to the diagram:

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & (\mathbb{Z}^{d+r})^\star & \longrightarrow & (\mathbb{Z}^{d+r})^\star & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle [B\;Q]^\star} & & \Big\downarrow{\scriptstyle Q^\star} & & \\
0 & \longrightarrow & (\mathbb{Z}^n)^\star & \longrightarrow & (\mathbb{Z}^{n+r})^\star & \longrightarrow & (\mathbb{Z}^r)^\star & \longrightarrow & 0
\end{array}\,.$$

It follows that $\mathrm{DG}(\beta) = (\mathbb{Z}^{n+r})^\star/\operatorname{Im}([B\ Q]^\star)$ and that the map $\beta^\vee$ is the composition of the inclusion map $(\mathbb{Z}^n)^\star \to (\mathbb{Z}^{n+r})^\star$ and the quotient map $(\mathbb{Z}^{n+r})^\star \to \mathrm{DG}(\beta)$.

Example 2.1. The set $\{(2,1),(-3,0)\} \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ yields a map $\beta\colon \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In this case, $Q = \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 2 & -3 \\ 1 & 0 \end{smallmatrix}\right]$. Since the vector $\left[\begin{smallmatrix} 6 & 4 & -3 \end{smallmatrix}\right]^\star$ spans the integer kernel of matrix $\left[\begin{smallmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right]$, we have $\mathrm{DG}(\beta) \cong \mathbb{Z}^3/\operatorname{Im}\big(\left[\begin{smallmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{smallmatrix}\right]^\star\big) \cong \mathbb{Z}$ and $\beta^\vee\colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $\left[\begin{smallmatrix} 6 & 4 \end{smallmatrix}\right]$.

We are especially interested in the map $\beta\colon \mathbb{Z}^n \to N$ when it has a finite cokernel; in other words, the elements $\{b_1, \dots, b_n\}$ generate a finite index subgroup in N. The next result shows that this assumption characterizes when $(-)^\vee$ is a dualizing functor.

Proposition 2.2. *Let $\beta\colon \mathbb{Z}^n \to N$ be a homomorphism of finitely generated abelian groups. The map β is naturally isomorphic to $\beta^{\vee\vee}$ if and only if the cokernel of β is finite. Moreover, if cokernel of β is finite, then the kernel of $\beta^\vee$ is $N^\star$.*

Proof. Suppose that $\operatorname{Coker}(\beta)$ is not finite. The sequence (2.0.1) implies that the $\operatorname{Coker}(\beta^{\vee\vee})$ is $\operatorname{Ext}^1_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{Z})$. Since $\operatorname{Ext}^1_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{Z})$ is finite, we see that β cannot be isomorphic to $\beta^{\vee\vee}$.

Conversely, assume that the cokernel of β is finite. To compute the map $\beta^{\vee\vee}$, we first construct a projective resolution of $\mathrm{DG}(\beta) = (\mathbb{Z}^{n+r})^\star/\operatorname{Im}([B\ Q]^\star)$. Applying the Snake Lemma to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^r \arrow[r] \arrow[d, equal] & \mathbb{Z}^{n+r} \arrow[r] \arrow[d, "{[B\ Q]}"] & \mathbb{Z}^n \arrow[r] \arrow[d, "\beta"] & 0 \\ 0 \arrow[r] & \mathbb{Z}^r \arrow[r, "Q"] & \mathbb{Z}^{d+r} \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

establishes that $\operatorname{Coker}([B\ Q]) = \operatorname{Coker}(\beta)$ and $\operatorname{Ker}([B\ Q]) = \operatorname{Ker}(\beta)$. Hence, the complex $0 \to \operatorname{Ker}(\beta) \to \mathbb{Z}^{n+r} \xrightarrow{[B\ Q]} \mathbb{Z}^{d+r} \to 0$ is a projective resolution of $\operatorname{Coker}(\beta)$. Since $\operatorname{Ext}^i_{\mathbb{Z}}(\operatorname{Coker}(\beta), \mathbb{Z})$ can be compute from this resolution and $\operatorname{Coker}(\beta)^\star = 0$, we deduce that $[B\ Q]^\star$ is injective and $0 \to (\mathbb{Z}^{d+r})^\star \xrightarrow{[B\ Q]^\star} (\mathbb{Z}^{n+r})^\star \to 0$ is a projective resolution of $\mathrm{DG}(\beta)$.

Since the dual map $\beta^\vee$ is the composition of the inclusion map $(\mathbb{Z}^n)^\star \to (\mathbb{Z}^{n+r})^\star$ and the quotient map $(\mathbb{Z}^{n+r})^\star \to \mathrm{DG}(\beta)$, it follows that $\mathrm{DG}(\beta^\vee) = (\mathbb{Z}^{n+d+r})^{\star\star}/\operatorname{Im}\left[\begin{smallmatrix} I_n & B^\star \\ 0 & Q^\star \end{smallmatrix}\right]^\star$ and $\beta^{\vee\vee}$ is the composition of inclusion $(\mathbb{Z}^n)^{\star\star} \to (\mathbb{Z}^{n+d+r})^{\star\star}$ and the quotient map $(\mathbb{Z}^{n+d+r})^{\star\star} \to \mathrm{DG}(\beta^\vee)$. Because $\mathbb{Z}^m$ is naturally isomorphic to $(\mathbb{Z}^m)^{\star\star}$, it follows that $\mathrm{DG}(\beta^\vee)$ is naturally isomorphic to $(\mathbb{Z}^{d+r}/\operatorname{Im}(Q)) = N$ and $\beta^{\vee\vee}$ is naturally isomorphic to β. Lastly, our resolution of $\mathrm{DG}(\beta)$ also implies that $H^0\big(\operatorname{Cone}(\beta)^\star\big) = 0$ and thus the long exact sequence which gives (2.0.1) proves the second part of the proposition. $\square$

The functor $(-)^\vee$ is also well-behaved in short exact sequences.

Lemma 2.3. *Given a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{n_1} \arrow[r] \arrow[d, "\beta_1"] & \mathbb{Z}^{n_2} \arrow[r] \arrow[d, "\beta_2"] & \mathbb{Z}^{n_3} \arrow[r] \arrow[d, "\beta_3"] & 0 \\ 0 \arrow[r] & N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & 0 \end{tikzcd} \tag{2.3.2}$$

in which the rows are exact and the columns have finite cokernels, there is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{Z}^{n_3})^\star \arrow[r] \arrow[d, "\beta_3^\vee"] & (\mathbb{Z}^{n_2})^\star \arrow[r] \arrow[d, "\beta_2^\vee"] & (\mathbb{Z}^{n_1})^\star \arrow[r] \arrow[d, "\beta_1^\vee"] & 0 \\ 0 \arrow[r] & \mathrm{DG}(\beta_3) \arrow[r] & \mathrm{DG}(\beta_2) \arrow[r] & \mathrm{DG}(\beta_1) \arrow[r] & 0 \end{tikzcd} . \tag{2.3.3}$$

Proof. For $1 \leq i \leq 3$, choose $\boldsymbol{E}_i := (\mathbb{Z}^{n_i})[0]$ as a projective resolution of $\mathbb{Z}^{n_i}$. Using the Horseshoe Lemma, the bottom row of (2.3.2) lifts to an exact sequence of projective resolutions $0 \to \boldsymbol{F}_1 \to \boldsymbol{F}_2 \to \boldsymbol{F}_3 \to 0$. Hence, the diagram (2.3.2) yields to a commutative diagram of cochain complexes with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \boldsymbol{E}_1 \arrow[r] \arrow[d] & \boldsymbol{E}_2 \arrow[r] \arrow[d] & \boldsymbol{E}_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \boldsymbol{F}_1 \arrow[r] & \boldsymbol{F}_2 \arrow[r] & \boldsymbol{F}_3 \arrow[r] & 0 \end{tikzcd} .$$

The functors $(-)^\star$ and $\mathrm{Cone}(-)$ produce a commutative diagram with exact rows and columns:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \boldsymbol{E}_3[1]^\star \arrow[r] \arrow[d] & \boldsymbol{E}_2[1]^\star \arrow[r] \arrow[d] & \boldsymbol{E}_1[1]^\star \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Cone}(\beta_3)^\star \arrow[r] \arrow[d] & \mathrm{Cone}(\beta_2)^\star \arrow[r] \arrow[d] & \mathrm{Cone}(\beta_1)^\star \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \boldsymbol{F}_3^\star \arrow[r] \arrow[d] & \boldsymbol{F}_2^\star \arrow[r] \arrow[d] & \boldsymbol{F}_1^\star \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} . \tag{2.3.4}$$

Since $\mathrm{Coker}(\beta_i)$ is finite and $\boldsymbol{E}_i = (\mathbb{Z}^{n_i})[0]$, both $H^j\big(\mathrm{Cone}(\beta_i)^\star\big) = 0$ and $H^j(\boldsymbol{E}_i[1]^\star) = 0$ for all $j \neq 1$ and $1 \leq i \leq 3$. Hence, taking the cohomology of (2.3.4) yields (2.3.3). □

3. Toric Deligne-Mumford Stacks

The purpose of this section is to associate a smooth Deligne-Mumford stack to certain combinatorial data. This construction is inspired by the quotient construction for toric varieties; for example see [Cox].

Let N be a finitely generated abelian group of rank d. We write $\overline{N}$ for the lattice generated by N in the d-dimensional $\mathbb{Q}$-vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural map $N \to \overline{N}$ is denoted by $b \mapsto \bar{b}$. Let Σ be a rational *simplicial* fan in $N_{\mathbb{Q}}$; every cone $\sigma \in \Sigma$ is generated by linearly independent vectors. Let $\rho_1, \dots, \rho_n$ be the rays (one-dimensional cones) in Σ. We assume that $\rho_1, \dots, \rho_n$ span $N_{\mathbb{Q}}$ and we fix an element $b_i \in N$ such that $\bar{b}_i$ generates the cone ρ_i for $1 \leq i \leq n$. The set $\{b_1, \dots, b_n\}$ defines a homomorphism $\beta \colon \mathbb{Z}^n \to N$ with finite cokernel. The triple $\mathbf{\Sigma} := \big(N, \Sigma, \beta\big)$ is called a *stacky fan*.

The stacky fan $\mathbf{\Sigma}$ encodes a group action on a quasi-affine variety Z. To describe this action, let $\mathbb{C}[z_1, \dots, z_n]$ be the coordinate ring of $\mathbb{A}^n$. The quasi-affine variety Z is the open subset defined by the reduced monomial ideal $J_\Sigma := \big\langle \prod_{\rho_i \not\subseteq \sigma} z_i : \sigma \in \Sigma \big\rangle$; in other words, $Z := \mathbb{A}^n - \mathbb{V}(J_\Sigma)$. The $\mathbb{C}$-valued points of Z are the $z \in \mathbb{C}^n$ such that the cone generated by the set $\{\rho_i : z_i = 0\}$ belongs to Σ. We equip Z with an action of the group $G := \operatorname{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$ as follows. By applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the dual map $\beta^\vee \colon (\mathbb{Z}^n)^\star \to \mathrm{DG}(\beta)$ (see Section 2), we obtain a homomorphism $\alpha \colon G \to (\mathbb{C}^*)^n$. The natural action of $(\mathbb{C}^*)^n$ on $\mathbb{A}^n$ induces an action of G on $\mathbb{A}^n$. Since $\mathbb{V}(J_\Sigma)$ is a union of coordinate subspaces, Z is G-invariant.

The quotient stack $\mathcal{X}(\mathbf{\Sigma}) := [Z/G]$ is the Artin stack associated to the groupoid $s, t \colon Z \times G \rightrightarrows Z$ where s is the projection onto the first factor and t is given by the G-action on Z. If S is a scheme, then the objects in $[Z/G](S)$ are principal G-bundles $E \to S$ with a G-equivariant map $E \to Z$ and the morphisms are isomorphisms which preserve the map to Z. Since Z is smooth, $\mathcal{X}(\mathbf{\Sigma})$ is a smooth algebraic stack; see Remark 10.13.2 in [LM]. The next result shows that $\mathcal{X}(\mathbf{\Sigma})$ is in fact a Deligne-Mumford stack. We call $\mathcal{X}(\mathbf{\Sigma})$ the *toric Deligne-Mumford stack* associated to the stacky fan $\mathbf{\Sigma}$.

Proposition 3.1. *The quotient $\mathcal{X}(\mathbf{\Sigma})$ is a Deligne-Mumford stack.*

Proof. By Corollary 2.2 in [Edi] (or Example 7.17 in [Vis]), it is enough to show that the stablizisers of the geometric points of Z are finite and reduced. Lemma 3.2 shows that the map $Z \times G \to Z \times Z$ defined by $(z, g) \mapsto (z, z \cdot g)$ is a finite morphism. It follows that each stabilizer is finite group scheme. Since we are working in characteristic zero, all finite group schemes are reduced. □

Lemma 3.2. *The map $Z \times G \to Z \times Z$ with $(z, g) \mapsto (z, z \cdot g)$ is a finite morphism.*

Proof. The morphism of affine schemes $\alpha \colon G \to (\mathbb{C}^*)^n$ corresponds to the map of rings $\mathbb{C}[(\mathbb{Z}^n)^\star] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \to \mathbb{C}[\mathrm{DG}(\beta)]$. Since the cokernel of $\beta^\vee$ is finite, the ring $\mathbb{C}[\mathrm{DG}(\beta)]$ is integral over $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and $G \to \operatorname{Im}(\alpha)$ is a finite morphism. Hence, it suffices to prove that $\xi \colon \operatorname{Im}(\alpha) \times Z \to Z \times Z$ is also a finite morphism. Because $\operatorname{Ker}(\beta^\vee) \cong N^\star$, we have $\operatorname{Im}(\alpha) = \operatorname{Spec}\big(\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}] / \langle \prod_{i=1}^n t_i^{\theta(b_i)} - 1 : \theta \in N^\star \rangle\big)$.

We next show that $\xi\colon \operatorname{Im}(\alpha)\times Z \to Z\times Z$ is an affine morphism. For each $\sigma\in\Sigma$, set $z_{\widehat{\sigma}} := \prod_{\rho_i\not\subseteq\sigma} z_i$ and let $U_\sigma := \mathbb{C}^n - \mathbb{V}(z_{\widehat{\sigma}})$. The coordinate ring of the open affine subset U_σ is $\mathbb{C}[z_1,\dots,z_n,z_{\widehat{\sigma}}^{-1}]$ and the collection $\{U_\sigma : \sigma\in\Sigma\}$ covers Z. Therefore, $\{U_\sigma\times U_{\sigma'} : \sigma,\sigma'\in\Sigma\}$ is an open affine cover of $Z\times Z$ and $U_\sigma\times U_{\sigma'} = \operatorname{Spec} B_{\sigma,\sigma'}$ where $B_{\sigma,\sigma'} = \mathbb{C}[z_1,\dots,z_n,z_{\widehat{\sigma}}^{-1},z'_1,\dots,z'_n,(z'_{\widehat{\sigma}'})^{-1}]$. Since coordinate subspaces are G-invariant, $\xi^{-1}(U_\sigma\times U_{\sigma'})$ is the affine set

$$G\times(U_\sigma\cap U_{\sigma'}) = \operatorname{Spec} A_{\sigma,\sigma'} = \operatorname{Spec}\left(\frac{\mathbb{C}\big[t_1^{\pm1},\dots,t_n^{\pm1},z_1,\dots,z_n,z_{\widehat{\sigma}}^{-1},z_{\widehat{\sigma}'}^{-1}\big]}{\big\langle\prod_{i=1}^n t_i^{\theta(b_i)}-1 \,:\, \theta\in N^\star\big\rangle}\right).$$

The restriction of ξ to this affine set corresponds to the map of rings $\zeta\colon B_{\sigma,\sigma'}\to A_{\sigma,\sigma'}$ given by $z_i\mapsto z_i$ and $z'_i\mapsto t_iz_i$ for $1\le i\le n$.

To prove that ξ is finite, we show that $A_{\sigma,\sigma'}$ is a finitely generated $B_{\sigma,\sigma'}$-module. Clearly, the $z_i\in A_{\sigma,\sigma'}$ and $(z_{\widehat{\sigma}})^{-1}$ are integral over $B_{\sigma,\sigma'}$. Since we have

$$t_i = \zeta\Big((z_{\widehat{\sigma}})^{-1}z'_i\prod_{\substack{\rho_j\not\subseteq\sigma'\\ j\neq i}} z_j\Big) \text{ and } t_i^{-1} = \zeta\Big((z'_{\widehat{\sigma}'})^{-1}z_i\prod_{\substack{\rho_j\not\subseteq\sigma'\\ j\neq i}} z'_j\Big),$$

both t_i for $\bar b_i\notin\sigma$ and t_i^{-1} for $\bar b_i\notin\sigma'$ are integral over $B_{\sigma,\sigma'}$. Thus, $t_i^{\pm1}$ is integral when $\bar b_i\notin\sigma\cup\sigma'$. The Separation Lemma (see Section 1.2 in [Ful]) implies there is a $\theta\in N^\star$ such that $\theta(b_i)>0$ if $\bar b_i\in\sigma$ and $\bar b_i\notin\sigma'$; $\theta(b_i)<0$ if $\bar b_i\notin\sigma$ and $\bar b_i\in\sigma'$; and $\theta(b_i)=0$ if $\bar b_i\in\sigma\cap\sigma'$. Hence, the relation $\prod_i t_i^{\theta(b_i)}=1$ can be rewritten as $t_i^{\theta(b_i)}=\prod_{j\neq i}t_j^{-\theta(b_j)}$ and our assumptions on θ imply that the right hand side is integral over $B_{\sigma,\sigma'}$. It follows that $t_i^{\pm1}$ is integral over $B_{\sigma,\sigma'}$ when $\bar b_i\notin\sigma\cap\sigma'$. Because $\sigma\cap\sigma'$ is simplicial, $\bar b_i\in\sigma\cap\sigma'$ implies that the relations $\{\prod_i t_i^{\theta(b_i)}=1 : \theta\in N^\star\}$ allow one to express a power of $t_i^{\pm1}$ as a product of $t_j^{\pm1}$ for $\bar b_j\notin\sigma\cap\sigma'$. This shows that $t_i^{\pm1}$ for $1\le i\le n$ is integral over $B_{\sigma,\sigma'}$. Lastly, we have $(z_{\widehat{\sigma}'})^{-1}=\zeta((z'_{\widehat{\sigma}'})^{-1})\prod_{\rho_i\not\subseteq\sigma'}t_i$ which implies $A_{\sigma,\sigma'}$ is integral over $B_{\sigma,\sigma'}$ and completes the proof. □

Remark 3.3. In [Laf], a "toric stack" is defined to be the quotient of a toric variety by its torus. Since such a quotient is never a Deligne-Mumford stack, $\mathcal{X}(\mathbf{\Sigma})$ is *not* a "toric stack".

Remark 3.4. The definition of $\mathcal{X}(\mathbf{\Sigma})$ does not depend on the fan Σ being simplicial. However, $\mathcal{X}(\mathbf{\Sigma})$ is a Deligne-Mumford stack if and only if the fan Σ is simplicial.

As the next example indicates, our construction produces some classic Deligne-Mumford stacks.

Example 3.5. Let Σ be the complete fan in $\mathbb{Q}$ and consider the subset $\big\{(2,1),(-3,0)\big\}$ of $N:=\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. This data defines a stacky fan $\mathbf{\Sigma}$. From Example 2.1, we know $\beta^\vee\colon\mathbb{Z}^2\to\operatorname{DG}(\beta)\cong\mathbb{Z}$ is given by the matrix $[\,6\ 4\,]$. Furthermore, $Z:=\mathbb{A}^2-\{(0,0)\}$ and $\lambda\in G\cong\mathbb{C}^*$ acts by $(z_1,z_2)\mapsto(\lambda^6z_1,\lambda^4z_2)$. In this case, $\mathcal{X}(\mathbf{\Sigma})$ is precisely the moduli stack of elliptic curves $\overline{\mathcal{M}}_{1,1}$; see Page 126 in [DR].

To illustrate that a toric Deligne-Mumford stack depends on the set $\{b_i\}$, we include the following:

Example 3.6. Let Σ be the complete fan in $\mathbb{Q}$ which implies $Z := \mathbb{A}^2 - \{(0,0)\}$ and let $N := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. If $\beta_1 \colon \mathbb{Z}^2 \to N$ corresponds to the set $\{(1,0),(-1,1)\}$ and $\mathbf{\Sigma}_1 = \big(N, \Sigma, \beta_1\big)$, then the dual map $\beta_1^\vee \colon \mathbb{Z}^2 \to \mathrm{DG}(\beta) \cong \mathbb{Z}$ is given by the matrix $\left[\begin{smallmatrix} 3 & 3 \end{smallmatrix}\right]$ and $\lambda \in G_1 \cong \mathbb{C}^*$ acts by $(z_1, z_2) \mapsto (\lambda^3 z_1, \lambda^3 z_2)$. On the other hand, if $\beta_2 \colon \mathbb{Z}^2 \to N$ corresponds to the set $\{(1,0),(-1,0)\}$, then $\beta_2^\vee \colon \mathbb{Z}^2 \to \mathrm{DG}(\beta) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ is given by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $(\lambda_1, \lambda_2) \in G_2 \cong \mathbb{C}^* \times \boldsymbol{\mu}_3$ acts by $(z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_1 z_2)$. Therefore, for the stacky fan $\mathbf{\Sigma}_2 = (N, \Sigma, \beta_2)$, $\mathcal{X}(\mathbf{\Sigma}_2)$ is the quotient of $\mathbb{P}^1$ by a trivial action of the $\mathbb{Z}/3\mathbb{Z}$ and $\mathcal{X}(\mathbf{\Sigma}_1) \ncong \mathcal{X}(\mathbf{\Sigma}_2)$.

The last result in this section makes the relationship between toric Deligne-Mumford stacks and toric varieties more explicit. Recall that a *coarse moduli space* of a Deligne-Mumford stack $\mathcal{X}$ is an algebraic space X with a morphism $\pi \colon \mathcal{X} \to X$ such that

- for all algebraically closed fields k, the map $\pi(k) \colon \mathcal{X}(k) \to X(k)$ is a bijection;
- given any algebraic space X' and any morphism $\pi' \colon \mathcal{X} \to X'$, there is a unique morphism $\chi \colon X \to X'$ such that $\pi' = \chi \circ \pi$.

Proposition 3.7. *The toric variety $X(\Sigma)$ is the coarse moduli space of $\mathcal{X}(\mathbf{\Sigma})$.*

Proof. By Proposition 4.2 in [Edi], it is enough to show that the toric variety $X(\Sigma)$ is the universal geometric quotient of Z by G. Under the additional assumptions that $N = \overline{N}$ and that the $b_i = \bar{b}_i$ are the unique minimal lattice points generating the rays in Σ, this is Theorem 2.1 in [Cox]. The reader can verify that the proof presented in [Cox] extends to our situation without any significant changes. $\square$

4. Closed and Open Substacks

In this section, we explain how the stacky fan $\mathbf{\Sigma}$ encodes certain closed and open substacks of $\mathcal{X}(\mathbf{\Sigma})$. These are the stack versions of the closed and open toric subvarieties of a toric variety. We also express the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ as a disjoint union of certain closed substacks.

To describe the connection between the combinatorics of the stacky fan $\mathbf{\Sigma}$ and the substacks of $\mathcal{X}(\mathbf{\Sigma})$, we use the theory of groupoids; see [Moe] for an introduction. Recall that a homomorphism of groupoids $\Theta \colon (R' \rightrightarrows U') \longrightarrow (R \rightrightarrows U)$ is called a *Morita equivalence* if

(1) the square

$$\begin{tikzcd} R' \arrow[r, "{(s,t)}"] \arrow[d, "\Theta"'] & U' \times U' \arrow[d, "\Theta \times \Theta"] \\ R \arrow[r, "{(s,t)}"] & U \times U \end{tikzcd}$$

is Cartesian, and

(2) the morphism $t \circ \mathrm{pr}_1 \colon U' \times_{\Theta, U, s} R \to U$ is locally surjective (i.e. U has an open covering $\{U_i \to U\}$ such that each $U_i \to U$ factors through U).

The key observation is that two groupoids are Morita equivalent if and only if the associated stacks are isomorphic.

Fix a cone σ in the fan Σ. Let N_σ be the subgroup of N generated by the set $\{b_i : \rho_i \subseteq \sigma\}$ and let $N(\sigma)$ be the quotient group N/N_σ. By extending scalars, the quotient map $N \to N(\sigma)$ becomes the surjection $N_{\mathbb{Q}} \to N(\sigma)_{\mathbb{Q}}$. The quotient fan Σ/σ in $N(\sigma)_{\mathbb{Q}}$ is the set $\{\tilde{\tau} = \tau + (N_\sigma)_{\mathbb{Q}} : \sigma \subseteq \tau \text{ and } \tau \in \Sigma\}$ and the link of σ is the set $\operatorname{link}(\sigma) := \{\tau : \tau + \sigma \in \Sigma, \tau \cap \sigma = 0\}$. For each ray ρ_i in $\operatorname{link}(\sigma)$, we write $\tilde{\rho}_i$ for the ray in Σ/σ and $\tilde{b}_i$ for the image of b_i in $N(\sigma)$. To ensure that the quotient fan satisfies our hypothesis for constructing toric Deligne-Mumford stacks, we require the following:

Condition 4.1. The rays $\tilde{\rho}_i$ span $N(\sigma)_{\mathbb{Q}}$.

Note that if Σ is a complete fan, then every cone σ satisfies Condition 4.1.

Let ℓ be the number of rays in $\operatorname{link}(\sigma)$ and let $\beta(\sigma)\colon \mathbb{Z}^\ell \to N(\sigma)$ be the map determined by the set $\{\tilde{b}_i : \rho_i \in \operatorname{link}(\sigma)\}$. The quotient stacky fan $\boldsymbol{\Sigma/\sigma}$ is the triple $\big(N(\sigma), \Sigma/\sigma, \beta(\sigma)\big)$.

Proposition 4.2. *If σ is a cone in the stacky fan $\boldsymbol{\Sigma}$ which satisfies Condition 4.1, then $\mathcal{X}(\boldsymbol{\Sigma/\sigma})$ defines a closed substack of $\mathcal{X}(\boldsymbol{\Sigma})$.*

Proof. By definition, $\mathcal{X}(\boldsymbol{\Sigma})$ is $[Z/G]$. Let $W(\sigma)$ be the closed subvariety of Z defined by the ideal $J(\sigma) := \langle z_i : \rho_i \subseteq \sigma\rangle$ in $\mathbb{C}[z_1, \ldots, z_n]$. The $\mathbb{C}$-valued points of $W(\sigma)$ are the $z \in \mathbb{C}^n$ such that the cone spanned by $\{\rho_i : z_i = 0\}$ contains σ and belongs to Σ. Hence, $\rho_i \not\subseteq \sigma \cup \operatorname{link}(\sigma)$ implies that $z_i \neq 0$. Since $J(\sigma)$ defines a coordinate subspace, $W(\sigma)$ is G-invariant and the groupoid $W(\sigma) \times G \rightrightarrows W(\sigma)$ defines a closed substack of $\mathcal{X}(\boldsymbol{\Sigma})$. It remains to show that $\mathcal{X}(\boldsymbol{\Sigma/\sigma})$ is the stack associated to $W(\sigma) \times G \rightrightarrows W(\sigma)$.

To begin, we construction a homomorphism from $W(\sigma) \times G \rightrightarrows W(\sigma)$ to the defining groupoid of $\mathcal{X}(\boldsymbol{\Sigma/\sigma})$. By renumbering the ρ_i, we may assume that $\tilde{\rho}_1, \ldots, \tilde{\rho}_\ell$ are the rays in $\operatorname{link}(\sigma)$. If $\mathbb{C}[\tilde{z}_1, \ldots, \tilde{z}_\ell]$ is the coordinate ring of $\mathbb{A}^\ell$, then

$$J_{\Sigma/\sigma} := \left\langle \textstyle\prod_{\rho_i \not\subseteq \tau} \tilde{z}_i : \sigma \subseteq \tau \text{ and } \tau \in \Sigma \right\rangle .$$

By definition, $\mathcal{X}(\boldsymbol{\Sigma/\sigma}) := [Z(\sigma)/G(\sigma)]$ where $Z(\sigma) := \mathbb{A}^\ell - \mathbb{V}(J_{\Sigma/\sigma})$ and $G(\sigma) := \operatorname{Hom}_{\mathbb{Z}}(\operatorname{DG}(\beta(\sigma)), \mathbb{C}^*)$. Let $m := \dim \sigma$. The description of the $\mathbb{C}$-values points of $W(\sigma)$ shows the projection $\mathbb{A}^n \to \mathbb{A}^\ell$ induces a surjection $\varphi_0\colon W(\sigma) \to Z(\sigma)$ with $\operatorname{Ker}(\varphi_0) = (\mathbb{C}^*)^{n-\ell-m}$. Applying Lemma 2.3 to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{n-\ell} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^\ell & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \tilde{\beta}} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \beta(\sigma)} & & \\
0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0
\end{array}$$

produces the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathbb{Z}^\ell)^\star & \longrightarrow & (\mathbb{Z}^n)^\star & \longrightarrow & (\mathbb{Z}^{n-\ell})^\star & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \beta(\sigma)^\vee} & & \big\downarrow{\scriptstyle \beta^\vee} & & \big\downarrow{\scriptstyle \tilde{\beta}^\vee} & & \\
0 & \longrightarrow & \operatorname{DG}(\beta(\sigma)) & \longrightarrow & \operatorname{DG}(\beta) & \longrightarrow & \operatorname{DG}(\tilde{\beta}) & \longrightarrow & 0
\end{array} \tag{4.2.5}$$

Since the cone σ is simplicial, $N_\sigma \cong \mathbb{Z}^m$ and $\mathrm{DG}(\tilde{\beta}) \cong \mathbb{Z}^{n-\ell-m}$. Applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to (4.2.5) gives the diagram with split exact rows

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{C}^*)^{n-\ell-m} \arrow[r] \arrow[d] & G \arrow[r, "\varphi_1"] \arrow[d, "\alpha"] & G(\sigma) \arrow[r] \arrow[d, "\alpha(\sigma)"] & 0 \\ 0 \arrow[r] & (\mathbb{C}^*)^{n-\ell} \arrow[r] & (\mathbb{C}^*)^n \arrow[r] & (\mathbb{C}^*)^\ell \arrow[r] & 0 \end{tikzcd} .$$

Hence, $\Phi := (\varphi_0 \times \varphi_1, \varphi_0)$ is a homomorphism of groupoids from $W(\sigma) \times G \rightrightarrows W(\sigma)$ to $Z(\sigma) \times G(\sigma) \rightrightarrows Z(\sigma)$.

To prove that $\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\sigma})$ is the stack associated to $W(\sigma) \times G \rightrightarrows W(\sigma)$, it suffices to show that Φ is a Morita equivalence. First, the commutative diagram

$$\begin{tikzcd} Z(\sigma) \times G(\sigma) \times (\mathbb{C}^*)^{2(n-\ell-m)} \arrow[d, "{(s,t,\mathrm{id})}"] & W(\sigma) \times G \arrow[l, "\cong"'] \arrow[r, "\varphi_0\times\varphi_1"] \arrow[d, "{(s,t)}"] & Z(\sigma) \times G(\sigma) \arrow[d, "{(s,t)}"] \\ Z(\sigma) \times Z(\sigma) \times (\mathbb{C}^*)^{2(n-\ell-m)} & W(\sigma) \times W(\sigma) \arrow[l, "\cong"'] \arrow[r, "\varphi_0\times\varphi_0"] & Z(\sigma) \times Z(\sigma) \end{tikzcd}$$

shows that $W(\sigma) \times G = \big(Z(\sigma) \times G(\sigma)\big) \times_{\varphi_0\times\varphi_0, Z(\sigma)\times Z(\sigma),(s,t)} \big(W(\sigma) \times W(\sigma)\big)$. Second, we have $\big(Z(\sigma) \times G(\sigma)\big) \times_{s,Z(\sigma),\varphi_0} W(\sigma) \cong Z(\sigma) \times G(\sigma) \times \mathbb{C}^{n-\ell-m}$ which implies that the map $t \circ \pi_1 \colon \big(Z(\sigma) \times G(\sigma)\big) \times_{s,Z(\sigma),\varphi_0} W(\sigma) \to Z(\sigma)$ splits. Therefore, Φ is a Morita equivalence and $\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\sigma})$ defines a closed substack of $\mathcal{X}(\boldsymbol{\Sigma})$. □

Each cone σ in Σ also corresponds to an open substack of $\mathcal{X}(\boldsymbol{\Sigma})$. This substack has a particularly nice description when $\dim \sigma = d := \operatorname{rank} N$ so that σ is of maximal dimension. In this case, let $\beta_\sigma \colon \mathbb{Z}^d \to N$ be the map determined by the set $\{b_i : \rho_i \subseteq \sigma\}$. The induced stacky fan $\boldsymbol{\sigma}$ is the triple $\big(N, \sigma, \beta_\sigma\big)$.

Proposition 4.3. *If σ is a d-dimensional cone in the stacky fan $\boldsymbol{\Sigma}$, then $\mathcal{X}(\boldsymbol{\sigma})$ defines an open substack of $\mathcal{X}(\boldsymbol{\Sigma})$. Moreover, $\mathcal{X}(\boldsymbol{\sigma})$ is isomorphic the quotient of $\mathbb{C}^d$ by the finite abelian group $N(\sigma)$.*

Proof. As in Lemma 3.2, let U_σ be the open subvariety of Z defined by the monomial $z_{\widehat{\sigma}} := \prod_{\rho_i \not\subseteq \sigma} z_i$. The $\mathbb{C}$-valued points of U_σ are the $z \in \mathbb{C}^n$ such that for each $z_i = 0$ the ray ρ_i is contained in σ. Since $\mathbb{V}(z_{\widehat{\sigma}})$ is a union of coordinate subspaces, U_σ is G-invariant and the groupoid $U_\sigma \times G \rightrightarrows U_\sigma$ defines an open substack of $\mathcal{X}(\boldsymbol{\Sigma})$. It remains to show that $\mathcal{X}(\boldsymbol{\sigma})$ is the stack associated to $U_\sigma \times G \rightrightarrows U_\sigma$.

We construct a homomorphism from the defining groupoid of $\mathcal{X}(\boldsymbol{\sigma})$ to $U_\sigma \times G \rightrightarrows U_\sigma$. Since σ is a d-dimensional simplicial cone, $J_\sigma = \langle 1 \rangle$ and $Z_\sigma := \mathbb{A}^d$. By definition, $\mathcal{X}(\boldsymbol{\sigma}) := [Z_\sigma/G_\sigma]$ where $G_\sigma := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta_\sigma), \mathbb{C}^*)$. The description of the $\mathbb{C}$-values points of U_σ yields a closed embedding $\psi_0 \colon Z_\sigma \to U_\sigma$ where

$$\psi_0(Z_\sigma) = \mathbb{C}^d \times \mathbf{1} \subset \mathbb{C}^d \times (\mathbb{C}^*)^{n-d} \cong U_\sigma \,.$$

Applying Lemma 2.3 and the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^d & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{n-d} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta_\sigma} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow[\mathrm{id}]{} & N & \longrightarrow & 0 & & \end{array}$$

produces the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_\sigma & \longrightarrow & G & \xrightarrow{\psi_1} & (\mathbb{C}^*)^{n-d} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \alpha_\sigma} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & (\mathbb{C}^*)^d & \longrightarrow & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^{n-d} & \longrightarrow & 0 \end{array} . \tag{4.3.6}$$

Hence, $\Psi := (\psi_0 \times \psi_1, \psi_0)$ is a homorphism of groupoids from $Z_\sigma \times G_\sigma \rightrightarrows Z_\sigma$ to $U_\sigma \times G \rightrightarrows U_\sigma$ and an element $g \in G$ belongs to G_σ if and only if $(Z_\sigma \cdot g) \cap Z_\sigma \neq \emptyset$.

Next, we establish that $G_\sigma \cong N(\sigma)$. The definition of $N(\sigma)$ gives the exact sequence

$$0 \longrightarrow \mathbb{Z}^{d+r} \xrightarrow{[B_\sigma\ Q]} \mathbb{Z}^{d+r} \longrightarrow N(\sigma) \longrightarrow 0$$

where B_σ is the submatrix of B whose columns correspond to the $\rho_i \subseteq \sigma$. Since $N(\sigma)^\star = 0$, we obtain the exact sequence

$$0 \longrightarrow (\mathbb{Z}^{d+r})^\star \xrightarrow{[B_\sigma\ Q]^\star} (\mathbb{Z}^{d+r})^\star \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\big(N(\sigma),\mathbb{Z}\big) \longrightarrow 0$$

which implies that $\mathrm{DG}(\beta) = \mathrm{Ext}^1_{\mathbb{Z}}\big(N(\sigma),\mathbb{Z}\big) = \mathrm{Hom}_{\mathbb{Z}}\big(N(\sigma),\mathbb{Q}/\mathbb{Z}\big)$. Hence, the group G_σ is $\mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_{\mathbb{Z}}\big(N(\sigma),\mathbb{Q}/\mathbb{Z}\big),\mathbb{C}^*\big)$. We identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^*$ via the map $p \mapsto \exp(2\pi\sqrt{-1}p)$ to obtain a natural homomorphism from $N(\sigma)$ to G_σ. By expressing $N(\sigma)$ as a direct sum of cyclic groups, one verifies that this map is an isomorphism.

Finally, to prove that $\mathcal{X}(\boldsymbol{\sigma})$ is the stack associated to $U_\sigma \times G \rightrightarrows U_\sigma$, it suffices to show that Ψ is a Morita equivalence. First, because an element $g \in G$ belongs to G_σ if and only if $(Z_\sigma \cdot g) \cap Z_\sigma \neq \emptyset$, the commutative diagram

$$\begin{array}{ccc} Z_\sigma \times G_\sigma & \xrightarrow{\psi_0 \times \psi_1} & U_\sigma \times G \\ \Big\downarrow{\scriptstyle (s,t)} & & \Big\downarrow{\scriptstyle (s,t)} \\ Z_\sigma \times Z_\sigma & \xrightarrow{\psi_0 \times \psi_0} & U_\sigma \times U_\sigma \end{array}$$

establishes that $Z_\sigma \times G_\sigma = \big(Z_\sigma \times Z_\sigma\big) \times_{\psi_0\times\psi_0, Z_\sigma \times Z_\sigma, (s,t)} \big(U_\sigma \times G\big)$. Secondly, we have $\big(U_\sigma \times G\big) \times_{s,U_\sigma,\psi_0} Z_\sigma \cong Z_\sigma \times G$ which implies that $\pi_1\colon \big(U_\sigma \times G\big) \times_{s,U_\sigma,\psi_0} Z_\sigma \to U_\sigma \times G$ corresponds to the closed immersion $\psi_0 \times \mathrm{id}\colon Z_\sigma \times G \to U_\sigma \times G$. Lemma 3.2 implies that $t\colon U_\sigma \times G \to U_\sigma$ is finite. Since the action of $\mathrm{Coker}(\psi_1)$ on $\psi_0(Z_\sigma)$ surjects onto U_σ, we deduce that $t \circ \pi_1\colon \big(U_\sigma \times G\big) \times_{s,U_\sigma,\psi_0} Z_\sigma \to U_\sigma$ is a finite surjective morphism of nonsingular varieties and hence flat. Because the geometric fibers of $t \circ \pi_1$ correspond to G_σ, a finite set of reduced points, the map $t \circ \pi_1$ is also étale and therefore locally surjective. We conclude that Ψ is a Morita equivalence and $\mathcal{X}(\boldsymbol{\sigma})$ defines an open substack of $\mathcal{X}(\boldsymbol{\Sigma})$. $\square$

Remark 4.4. Assuming that every cone in Σ is contained in a d-dimensional cone, Proposition 4.3 produces an étale atlas of $\mathcal{X}(\boldsymbol{\Sigma})$.

Remark 4.5. More generally, if $\boldsymbol{\Sigma}' := (N', \Sigma', \beta')$ and $\boldsymbol{\Sigma} := (N, \Sigma, \beta)$ are two stacky fans, then a *morphism of stacky fans* is a homomorphism $\phi\colon N' \to N$ satisfying:

- for each cone $\sigma' \in \Sigma'$, there exists a $\sigma \in \Sigma$ such that $\phi_{\mathbb{Q}}(\sigma') \subseteq \sigma$ where $\phi_{\mathbb{Q}}\colon N' \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q}$;
- for each $\bar{b}'_i \in \sigma'$, the element $\phi_{\mathbb{Q}}(\bar{b}'_i)$ is an integer combination of the $\bar{b}_i \in \sigma$ where $\sigma \in \Sigma$ is any cone that contains $\phi_{\mathbb{Q}}(\sigma')$.

For each morphism $\phi\colon \boldsymbol{\Sigma}' \to \boldsymbol{\Sigma}$, there is a morphism $\mathcal{X}(\boldsymbol{\Sigma}') \to \mathcal{X}(\boldsymbol{\Sigma})$. Since we do not make use of this construction, the proof is left to the reader.

For each d-dimensional cone σ in the stacky fan $\boldsymbol{\Sigma}$, we define $\operatorname{Box}(\sigma)$ to be the set of elements $v \in N$ such that $\bar{v} = \sum_{\rho_i \subseteq \sigma} q_i \bar{b}_i$ for some $0 \leq q_i < 1$. Hence, the set $\operatorname{Box}(\sigma)$ is in one-to-one correspondence with the elements in the finite group $N(\sigma)$. Let $\operatorname{Box}(\boldsymbol{\Sigma})$ be the union of $\operatorname{Box}(\sigma)$ for all d-dimensional cones $\sigma \in \Sigma$. For each $v \in N$, we write $\sigma(\bar{v})$ for the unique minimal cone containing $\bar{v}$.

Lemma 4.6. *If Σ is a complete fan, then the elements $v \in \operatorname{Box}(\boldsymbol{\Sigma})$ are in one-to-one correspondence with elements $g \in G$ which fix a point of Z, and $[Z^g/G] \cong \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$.*

Proof. By definition, an element $v \in \operatorname{Box}(\boldsymbol{\Sigma})$ corresponds to an element in $N(\tau)$ for some d-dimensional cone $\tau \in \Sigma$. In the proof of Proposition 4.3, we give an isomorphism between $N(\tau)$ and and G_τ. Hence, there is a bijection sending v to an element g in the subgroup $G_\tau \subseteq G$. In addition, (4.3.6) implies that g act trivially on points $z \in Z$ with $z_i = 0$ for all $\rho_i \subseteq \tau$ which shows that g fixes a point in Z.

Conversely, suppose $g \in G$ fixes a point $z \in Z$. Since the action of G on Z is defined via the map $\alpha\colon G \to (\mathbb{C}^*)^n$ where $g \mapsto \big(\alpha_1(g), \dots, \alpha_n(g)\big)$, we see that either $\alpha_i(g) = 1$ or $z_i = 0$ for all $1 \leq i \leq n$. The definition of Z guarantees that there exists a cone in Σ containing all the rays ρ_i for which $z_i = 0$. Let σ be the minimal cone with this property. Because Σ is a simplicial fan, the ray ρ_i is contained in σ if and only if $\alpha_i(g) \neq 1$. Thus, the closed subvariety $W(\sigma)$ defined in Proposition 4.2 is equal to the invariant subvariety Z^g. Moreover, our choice of σ implies that the element g stabilizes $\psi_0(Z_\tau)$ for every d-dimensional cone τ which contains σ. It follows that g corresponds to an element $v \in \operatorname{Box}(\boldsymbol{\Sigma})$. Finally, σ is clearly the intersection of all maximal cones τ for which v corresponds to an element in $N(\tau)$. Therefore, $\sigma = \sigma(\bar{v})$ and Proposition 4.2 establishes that $[Z^g/G] = [W(\sigma)/G] \cong \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$. $\square$

For a Deligne-Mumford stack $\mathcal{X}$, its *inertia stack* $\mathcal{I}(\mathcal{X})$ is defined to be the fibered product $\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, \Delta} \mathcal{X}$ where Δ denotes the diagonal map. For a scheme S, an object in $\mathcal{I}\big(\mathcal{X}\big)(S)$ can be identified with pair (x, ϕ) where x is an object in $\mathcal{X}(S)$ and ϕ is an automorphism of x. A morphism from $(x, \phi) \to (x', \phi')$ is a morphism $\gamma\colon x \to x'$ in $\mathcal{X}(S)$ such that $\gamma \circ \phi = \phi' \circ \gamma$. Since we are working over $\mathbb{C}$, the inertia stack $\mathcal{I}(\mathcal{X})$ is naturally isomorphic to the stack of representable morphisms from constant cyclotomic gerbes to $\mathcal{X}$; see Section 4.4 in [AGV].

Proposition 4.7. *If Σ is a complete fan, then $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big) = \coprod_{v\in \mathrm{Box}(\mathbf{\Sigma})} \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$ where $\sigma(\bar{v})$ is the minimal cone in Σ containing $\bar{v}$.*

Proof. Let S be a connected scheme. An object x of $\mathcal{X}(\mathbf{\Sigma})(S)$ is a principal G-bundle $E \to S$ with a G-equivariant morphism $f\colon E \to Z$. An automorphism ϕ is an automorphism of the principal G-bundle $E \to S$ that is compatible with $E \to Z$. Since S is connected, ϕ corresponds to multiplication by an element $g \in G$. Moreover, because f is G-equivariant and $f = f \circ \phi$, the map f factors through Z^g. Hence, the principal G-bundle $E \to S$ with $E \to Z^g$ is an object in $[Z^g/G](S)$.

For an arbitrary scheme S and an object in $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)(S)$, we can assign an object in $\coprod_{g\in G}[Z^g/G](S)$ by considering the connected components of S. Finally, Lemma 4.6 shows that $Z^g \neq \emptyset$ if and only if g corresponds to an element $v \in \mathrm{Box}(\mathbf{\Sigma})$ and that $[Z^g/G] \cong \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$. □

Remark 4.8. By combining Proposition 3.7 and Proposition 4.7, we see that the coarse moduli space of $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is isomorphic to the disjoint union of $X\big(\Sigma/\sigma(\bar{v})\big)$ for all $v \in \mathrm{Box}(\Sigma)$. In particular, we recover the description of the twisted sectors in Section 6 of [Pod].

5. Module Structure on $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$

The goal of this section to describe the orbifold Chow ring of a complete toric Deligne-Mumford stack as an abelian group. Throughout this section, we assume all fans are complete and simplicial and all Chow rings have rational coefficients.

We first introduce the *deformed group ring* $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ associated to the stacky fan $\mathbf{\Sigma} = \big(N, \Sigma, \beta\big)$. As a vector space, $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is simply the group ring $\mathbb{Q}[N]$; in other words, $\mathbb{Q}[N]^{\mathbf{\Sigma}} = \bigoplus_{c\in N} \mathbb{Q}\cdot y^c$ where y is a formal variable. Multiplication in $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is defined as follows:

$$y^{c_1}\cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there exists } \sigma\in\Sigma \text{ such that } \bar{c}_1\in\sigma \text{ and } \bar{c}_2\in\sigma;\\ 0 & \text{otherwise.}\end{cases} \tag{5.0.7}$$

We endow $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ with a $\mathbb{Q}$-grading as follows: if $\bar{c} = \sum_{\rho_i\subseteq\sigma(\bar{c})} m_i\bar{b}_i$ where $\sigma(\bar{c})$ is the minimal cone in Σ containing $\bar{c}$, then $\deg(y^c) := \sum m_i \in \mathbb{Q}$.

Given a stacky fan $\mathbf{\Sigma}$, we denote by $S_{\mathbf{\Sigma}}$ the subring of $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ generated over $\mathbb{Q}$ by the monomials y^{b_i}. Since Σ is simplicial, the ring $S_{\mathbf{\Sigma}}$ is isomorphic to the quotient $\mathbb{Q}[x_1,\ldots,x_n]/I_\Sigma$ where the ideal I_Σ is generated by the square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_s}$ with $\rho_{i_1}+\cdots+\rho_{i_s}\notin\Sigma$. In particular, $S_{\mathbf{\Sigma}}$ is a $\mathbb{Z}$-graded ring and I_Σ is the Stanley-Reisner ideal associated to Σ.

To describe the Chow ring of $\mathcal{X}(\mathbf{\Sigma})$, we need certain line bundles corresponding to the rays $\rho_1,\ldots,\rho_n$. Since the category of coherent sheaves on $\mathcal{X}(\mathbf{\Sigma})$ is equivalent to the category of G-equivariant sheaves on Z (Example 7.21 in [Vis]), we can define L_i

for $1 \leq i \leq n$ to be the line bundle on $\mathcal{X}(\mathbf{\Sigma})$ corresponding to the trivial line bundle $\mathbb{C} \times Z$ on Z with the G-action on $\mathbb{C}$ is given by the ith component α_i of $\alpha \colon G \to (\mathbb{C}^*)^n$.

We first calculate the non-orbifold Chow ring of $\mathcal{X}(\mathbf{\Sigma})$.

Lemma 5.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then there is an isomorphism of $\mathbb{Z}$-graded rings*

$$\frac{S_{\mathbf{\Sigma}}}{\left\langle \sum_{i=1}^n \theta(\bar{b}_i) \cdot y^{b_i} : \theta \in N^\star \right\rangle} \longrightarrow A^*\big(\mathcal{X}(\mathbf{\Sigma})\big)$$

defined by $y^{b_i} \mapsto c_1(L_i)$.

Proof. For $1 \leq i \leq n$, let a_i denote the unique minimal lattice generator of ρ_i in Σ and let ℓ_i be the positive integer satisfying the relation $\bar{b}_i = \ell_i a_i$. The Jurkiewicz-Danilov Theorem (see Page 134 in [Oda]) states that there is a surjective homomorphism of graded rings from $\mathbb{Q}[x_1, \ldots, x_n]$ to $A^*\big(X(\Sigma)\big)$ given by $x_i \mapsto D_i$ where D_i is the torus invariant Weil divisor on $X(\Sigma)$ associated with ρ_i. The kernel of this map is the ideal I_Σ plus the ideal generated by the linear relations $\sum_{i=1}^n \theta(a_i) \cdot x_i$ for all $\theta \in N^\star$. Example 6.7 in [Vis] establishes a natural isomorphism $A^*\big(\mathcal{X}(\mathbf{\Sigma})\big) \cong A^*\big(X(\Sigma)\big)$ defined by $c_1(L_i) \mapsto \ell_i^{-1} \cdot D_i$. Since we have $\sum_{i=1}^n \theta(a_i) \cdot \ell_i \cdot x_i = \sum_{i=1}^n \theta(\bar{b}_i) \cdot x_i$ for all $\theta \in N^\star$, the composition of these two isomorphism establishes the claim. $\square$

This lemma allow us to establish Theorem 1.1 at the level of $\mathbb{Q}$-graded $\mathbb{Q}$-vector spaces. More precisely, we prove the following the result. If M is a $\mathbb{Q}$-graded module and c is a rational number, then we write $M[c]$ for the cth shift of M; its defined by the formula $M[c]_{c'} = M_{c'+c}$.

Proposition 5.2. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then there is an isomorphism of $\mathbb{Q}$-graded $\mathbb{Q}$-vector spaces:*

$$\frac{\mathbb{Q}[N]^{\mathbf{\Sigma}}}{\langle \sum_{i=1}^n \theta(b_i) \cdot y^{b_i} : \theta \in N^\star \rangle} \cong \bigoplus_{v \in \operatorname{Box}(\mathbf{\Sigma})} A^*\big(\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}))\big)\big[\deg(y^v)\big] \,.$$

Proof. The definition of $S_{\mathbf{\Sigma}}$ and $\operatorname{Box}(\mathbf{\Sigma})$ implies that $\mathbb{Q}[N]^{\mathbf{\Sigma}} = \bigoplus_{v \in \operatorname{Box}(\mathbf{\Sigma})} y^v \cdot S_{\mathbf{\Sigma}}$. We first analyze the individual summands. Fix an element $v \in \operatorname{Box}(\mathbf{\Sigma})$ and let $\tau := \sigma(\bar{v})$ be the minimal cone in Σ containing $\bar{v}$. It follows from the definition of multiplication in the deformed group ring that $y^v \cdot S_{\mathbf{\Sigma}}$ is isomorphic to the quotient of $S_{\mathbf{\Sigma}}$ by the ideal generated by the elements y^c where c lies outside the cones in Σ containing τ.

Let $S_{\mathbf{\Sigma}/\boldsymbol{\tau}}$ denote the subring of $\mathbb{Q}[N(\tau)]^{\mathbf{\Sigma}/\boldsymbol{\tau}}$ generated by $y^{\tilde{b}_i}$ for $\rho_i \in \operatorname{link}(\tau)$. By renumbering the rays in Σ, we may assume that $\tilde{\rho}_1, \ldots, \tilde{\rho}_\ell$ are the rays in $\operatorname{link}(\tau)$. Recall that $\tilde{b}_i$ is the image of b_i in $N(\tau)$. For each ray $\rho_i \in \tau$, choose an element $\theta_i \in N^\star$ such that $\theta_i(b_i) = 1$ and $\theta_i(b_j) = 0$ for all $\bar{b}_i \neq \bar{b}_j \in \tau$. Consider the map defined by

$$y^{b_i} \mapsto \begin{cases} y^{\tilde{b}_i} & \text{for } \rho_i \subseteq \operatorname{link}(\tau); \\ -\sum_{j=1}^{\ell} \theta_i(b_j) \cdot y^{\tilde{b}_j} & \text{for } \rho_i \subseteq \tau; \\ 0 & \text{for } \rho_i \not\subseteq \tau \cup \operatorname{link}(\tau). \end{cases}$$

Since this map is compatible with the multiplicative structures on $S_{\boldsymbol{\Sigma}}$ and $S_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}$, it induces a surjective homomorphism from $S_{\boldsymbol{\Sigma}}$ to $S_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}$. Clearly, the kernel contains the elements $\theta_i(b_i) \cdot y^{b_i} + \sum_{j=1}^{\ell} \theta_i(b_j) y^{b_j}$ for all $\rho_i \in \tau$ and the elements y^c where c lies outside the cones in Σ containing τ. Given any other element of the kernel, we can use these relations to obtain a linear combination of monomials y^w with $\bar{w} \in \operatorname{link}(\tau)$ which also belongs to the kernel. However, this is only possible if all the coefficients of y^w are zero which implies that the given elements generate the kernel.

Since Lemma 5.1 establishes that

$$\frac{S_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}}{\langle \sum_{i=1}^{\ell} \tilde{\theta}(\tilde{b}_i) \cdot y^{\tilde{b}_i} : \tilde{\theta} \in N(\tau)^\star \rangle} \cong A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\tau})\big)\,,$$

we have a surjective $\mathbb{Q}$-graded $\mathbb{Q}$-linear map from $y^v \cdot S_{\boldsymbol{\Sigma}}$ to $A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\tau})\big)[\deg(y^v)]$ whose kernel is generated by the elements $\theta_i(b_i) \cdot y^{b_i} + \sum_{j=1}^{\ell} \theta_i(b_j) y^{b_j}$ for all $\rho_i \in \tau$ and the pullbacks of the linear relations $\sum_{i=1}^{\ell} \tilde{\theta}(\tilde{b}_i) \cdot y^{\tilde{b}_i}$ where $\tilde{\theta} \in N(\tau)^\star$. Finally, taking the direct sum over all $v \in \operatorname{Box}(\boldsymbol{\Sigma})$ produces a surjective $\mathbb{Q}$-graded $\mathbb{Q}$-linear map from $\mathbb{Q}[N]^{\boldsymbol{\Sigma}}$ to $\bigoplus_{v \in \operatorname{Box}(\boldsymbol{\Sigma})} A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}))\big)\big[\deg(y^v)\big]$ whose kernel is generated by the elements $\sum_{i=1}^{n} \theta(b_i) \cdot y^{b_i}$ where $\theta \in N^\star$. $\square$

Remark 5.3. Although the elements θ_i in the proof of Proposition 5.2 are not uniquely determined, the possible choices differ by elements in $N(\tau)^\star$. It follows that the surjection from $y^v \cdot S_{\boldsymbol{\Sigma}}$ to $S_{\boldsymbol{\Sigma}/\boldsymbol{\tau}}[\deg(y^v)]$ is not canonically defined, but surjection from $y^v \cdot S_{\boldsymbol{\Sigma}}$ to $A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\tau})\big)\big[\deg(y^v)\big]$ is.

Remark 5.4. The degree shift in Proposition 5.2 is also called the *age* of the component of the inertia stack.

6. The Product Structure on $A^*_{orb}\big(\mathcal{X}(\boldsymbol{\Sigma})\big)$

In this section, we study multiplication in $A^*_{orb}\big(\mathcal{X}(\boldsymbol{\Sigma})\big)$. Specifically, we complete the proof of Theorem 1.1 by showing that multiplication in the deformed group ring coincides with the orbifold product.

To compare the two products, we first give a combinatorial description of the moduli space $\mathcal{K} := \mathcal{K}_{0,3}\big(\mathcal{X}(\boldsymbol{\Sigma}), 0\big)$ of 3-pointed twisted stable maps of genus zero and degree zero to $\mathcal{X}(\boldsymbol{\Sigma})$. The moduli space $\mathcal{K}$ is a proper Deligne-Mumford stack with a smooth projective coarse moduli space; see Theorem 3.6.2 in [AGV]. In addition, Lemma 6.2.1 in [AGV] gives three evaluation maps denoted $\operatorname{ev}_i \colon \mathcal{K} \to \mathcal{I}\big(\mathcal{X}(\boldsymbol{\Sigma})\big)$ for $1 \leq i \leq 3$. Proposition 4.7 shows that $\mathcal{I}\big(\mathcal{X}(\boldsymbol{\Sigma})\big) = \coprod_{v \in \operatorname{Box}(\boldsymbol{\Sigma})} \mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v})\big)$, so we can index the components of $\mathcal{K}$ by the images of the evaluation maps. Let $\mathcal{K}_{v_1,v_2,v_3}$ be the component of $\mathcal{K}$ such that ev_i maps to $\mathcal{X}\big(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$ for $1 \leq i \leq 3$.

For brevity, we write $v_1 + v_2 + v_3 \equiv 0$ to indicate that there exists a cone $\sigma \in \Sigma$ containing $\bar{v}_i$ for $1 \leq i \leq 3$ such that the sum $v_1 + v_2 + v_3$ belongs to the subgroup N_σ in N.

Proposition 6.1. *If $\mathcal{X}(\mathbf{\Sigma})$ is a complete toric Deligne-Mumford stack, then*

$$\mathcal{K} = \coprod_{\substack{(v_1,v_2,v_3)\in \mathrm{Box}(\mathbf{\Sigma})^3 \\ v_1+v_2+v_3\equiv 0}} \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1,\bar{v}_2,\bar{v}_3)\big)\,,$$

where $\sigma(\bar{v}_1,\bar{v}_2,\bar{v}_3)$ is the minimal cone in Σ containing $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$.

Proof. We begin by examining the geometric points of $\mathcal{K}$. A $\mathbb{C}$-valued point of $\mathcal{K}$ is a representable morphism f from a twisted curve $\mathcal{C}$ to $\mathcal{X}(\mathbf{\Sigma})$ such that the induced map on coarse moduli spaces sends $\mathbb{P}^1$ to a point $x \in X(\Sigma)$. Hence, the map f factors through a closed substack $\mathcal{B}G'$ in $\mathcal{X}(\mathbf{\Sigma})$ where $G' \subseteq G$ is the isotropy group of $x \in \mathcal{X}(\mathbf{\Sigma})$ and $\mathcal{B}G'$ is the classifying stack $[x/G']$. Corollary 1.6.2 in [LM] shows that the morphism from $\mathcal{C}$ to $\mathcal{B}G'$ is also representable which implies that the fibered product $\widehat{C} := \mathcal{C} \times_{\mathcal{B}G'} x$ is a scheme. Since $\mathcal{C}$ is smooth, we see that $\widehat{C}$ is a smooth curve, although it is typically disconnected. Let H be the subgroup of G' that acts trivially on the set of connected components of $\widehat{C}$. Since G' is abelian, the group H is the stabilizer of each connected component of $\widehat{C}$. By choosing a connected component C of $\widehat{C}$, we obtain $\mathcal{C} \cong [C/H]$. Assuming the points $\{0,1,\infty\}$ in $\mathbb{P}^1$ correspond to the markings on $\mathcal{C}$, the properties of a twisted curve imply that the map $\mathcal{C} \to \mathbb{P}^1$ is an isomorphism over $\mathbb{P}^1 - \{0,1,\infty\}$. It follows that C is a proper smooth Galois cover of $\mathbb{P}^1$ with Galois group H branched over 0, 1 and ∞. Specifically, if $\gamma_1, \gamma_2, \gamma_3$ are the generators of the fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}$ corresponding to counterclockwise loops around 0, 1, ∞ respectively, then C is induced by a homomorphism $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}) \to G$ sending γ_1 to g_i such that $g_1 \cdot g_2 \cdot g_3 = 1$ and g_i generate H as a subgroup of G.

By definition, the map ev_i is induced by the representable morphism from the cyclotomic gerbe in $\mathcal{C}$ lying over the corresponding point in $\mathbb{P}^1$ to $\mathcal{X}(\mathbf{\Sigma})$; recall that over $\mathbb{C}$ the inertia stack $\mathcal{I}(\mathcal{X}(\mathbf{\Sigma})$ is canonically isomorphic to the stack of representable morphisms from a constant cyclotomic gerbe to $\mathcal{X}(\mathbf{\Sigma})$. Hence, the evaluation map ev_i sends f to the geometric point (x, g_i) in the inertia stack. Because g_i belongs to the isotropy group of x, it fixes a point in Z. Thus, Lemma 4.6 shows that g_i corresponds to an element $v_i \in \mathrm{Box}(\mathbf{\Sigma})$ and ev_i maps to the component $[Z^{g_i}/G] = \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$ of the inertia stack. Moreover, the condition that $g_1 \cdot g_2 \cdot g_3 = 1$ means that there exists a cone $\sigma \in \Sigma$ containing $\bar{v}_1, \bar{v}_2, \bar{v}_3$ and the sum $v_1 + v_2 + v_3$ belongs to the subgroup N_σ in N. Therefore, the component $\mathcal{K}_{v_1,v_2,v_3}$ is nonempty if and only if $v_1 + v_2 + v_3 \equiv 0$.

The morphisms $\mathrm{ev}_i \colon \mathcal{K}_{v_1,v_2,v_3} \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$ are compatible with the inclusion maps into $\mathcal{X}(\mathbf{\Sigma})$ for $1 \leq i \leq 2$ which yields a morphism

$$e \colon \mathcal{K}_{v_1,v_2,v_3} \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1)\big) \times_{\mathcal{X}(\mathbf{\Sigma})} \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_2)\big) = [Z^{g_1}/G] \times_{[Z/G]} [Z^{g_2}/G] \quad .$$

Because H is the subgroup of G generated by g_1 and g_2 (note: $g_3 = g_1^{-1}g_2^{-1}$), we have $Z^{g_1} \times_Z Z^{g_2} = Z^{\langle g_1, g_2\rangle} = Z^H$. It follows that $[Z^{g_1}/G] \times_{[Z/G]} [Z^{g_2}/G] = [Z^H/G]$. Our analysis of the geometric points of $\mathcal{K}$ shows that e induces a bijection between the $\mathbb{C}$-valued points of the coarse moduli spaces of $\mathcal{K}_{v_1,v_2,v_3}$ and $[Z^H/G]$. Since both $\mathcal{K}_{v_1,v_2,v_3}$ and $[Z^H/G]$ are smooth Deligne-Mumford stacks, their coarse moduli spaces have at worst quotient singularities. Applying Theorem VI.1.5 in [Kol], we deduce that, in fact, e produces an isomorphism between the coarse moduli spaces.

To prove that e is an isomorphism of stacks, it remains to show that e gives an isomorphism between the isotropy groups of $\mathbb{C}$-valued points. Indeed, since $\mathcal{K}$ is smooth (see page 18 in [AGV]) and e is representable, the isomorphism follows from a similar statement for the lifting of e to the atlases. Proposition 7.1.1 in [ACV] indicates that the automorphism group of a twisted stable curve is the direct product of the automorphism groups of the nodes which implies that our curve $\mathcal{C}$ has only the trivial automorphism. Hence, an isotropy of the twisted stable map $f\colon \mathcal{C} \to \mathcal{B}G' \subseteq \mathcal{X}(\mathbf{\Sigma})$ corresponds to a diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ \downarrow & & \downarrow \\ \mathcal{C} & = & \mathcal{C} \end{array}$$

where ϕ is a G'-equivariant map of principal G'-bundles over $\mathcal{C}$. Since $\mathcal{C}$ is connected, the map ϕ is multiplication by an element of G'. Therefore, the isotropy group of the map f is precisely G' which completes the proof. □

Proposition 6.1 also provides a presentation for the universal twisted stable curve over $\mathcal{K}$. To describe the universal curve, we focus on the component $\mathcal{K}_{v_1,v_2,v_3}$. As above, we write H for the subgroup of G corresponding to $\{v_1, v_2, v_3\}$ and $C \to \mathbb{P}^1$ for the associated Galois cover. Consider the quotient stack

$$\mathcal{U}_{v_1,v_2,v_3} := [(Z^H \times C)/(G \times H)] = [Z^H/G] \times [C/H] \quad .$$

If S is a scheme, then the objects in $\mathcal{U}_{v_1,v_2,v_3}(S)$ are principal $(G \times H)$-bundles $E \to S$ with a $(G \times H)$-equivariant map $E \to Z^H \times C$. The *twisted projection map* π from $\mathcal{U}_{v_1,v_2,v_3}$ to $\mathcal{K}_{v_1,v_2,v_3} = [Z^H/G]$ is defined as follows: If H acts on E via the map $h \mapsto (h^{-1}, h) \in G \times H$, then E/H is a principal G-bundle over S. To obtain an object in $\mathcal{K}_{v_1,v_2,v_3}(S)$, observe that the $(G \times H)$-equivariant map $E \to Z^H \times C$ induces a G-equivariant map from E/H to Z^H. By verifying that π is compatible with morphisms in $\mathcal{U}_{v_1,v_2,v_3}(S)$ and $\mathcal{K}_{v_1,v_2,v_3}(S)$, we conclude that π is a morphism of stacks. With these definitions, we have

Corollary 6.2. *The universal twisted stable curve over* $\mathcal{K}_{v_1,v_2,v_3} \cong [Z^H/G]$ *is given by the twisted projection map* $\pi\colon \mathcal{U}_{v_1,v_2,v_3} = [(Z^H \times C)/(G \times H)] \to [Z^H/G]$.

Proof. Fix a map $S \to [Z^H/G]$ where S is a scheme and consider the fibered product $\mathcal{D} := \mathcal{U}_{v_1,v_2,v_3} \times_{[Z^H/G]} S$. Assuming that $S \to [Z^H/G]$ corresponds to the principal G-bundle $E \to S$ with a G-equivariant map $E \to Z^H$, we have $\mathcal{D} = [(E \times C)/(G \times H)]$ where the $(G \times H)$-action is given by $(e, c, g, h) \mapsto (e \cdot gh^{-1}, c \cdot h)$. The twisted projection map π induces a map $\mathcal{D} \to [E/G] = S$. Because the *anti-diagonal* action of H on $E \times C$ is free, the quotient $Y := (E \times C)/H$ is a scheme. Hence, we have $\mathcal{D} = [Y/G]$ where the G-action on Y is induced by the action on $E \times C$. Since H acts trivially on Z^H, the G-equivariant map $E \to Z^H$ induces a G-equivariant map $Y \to Z^H$ which shows that

$\mathcal{D}$ maps to $[Z^H/G] \subseteq [Z/G] = \mathcal{X}(\mathbf{\Sigma})$. Moreover, if $R = R_1 + R_2 + R_3$ is the ramification divisor of the Galois cover $C \to \mathbb{P}^1$, then the image of the open set $E \times (C - R)$ gives an open substack of $[Y/G]$ which is isomorphic to $S \times \big(\mathbb{P}^1 - \{0, 1, \infty\}\big)$. By definition, the evaluation map ev_i from $\mathcal{D}$ to the inertia stack $\mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ arises from the representable morphism from $[(E \times R_i)/(G \times H)]$ to $\mathcal{X}(\mathbf{\Sigma})$. In particular, ev_i is induced by the closed embedding $[Z^H/G] \to [Z^{g_i}/G] \cong \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$. We conclude that $\mathcal{U}_{v_1,v_2,v_3}$ is a family of twisted stable curves over $[Z^H/G]$ with a map $f\colon \mathcal{U}_{v_1,v_2,v_3} \to \mathcal{X}(\mathbf{\Sigma})$ and evaluation maps $\mathrm{ev}_i\colon \mathcal{U}_{v_1,v_2,v_3} \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big) \subseteq \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ for $1 \leq i \leq 3$.

Let $\mathcal{U}'$ denote the universal family of twisted stable curves over $\mathcal{K}_{v_1,v_2,v_3}$. By the universal mapping property of $\mathcal{U}'$, there exists a map $\mu\colon [Z^H/G] \to \mathcal{K}_{v_1,v_2,v_3}$ such that

$$\begin{array}{ccc} \mu^*(\mathcal{U}') & \longrightarrow & \mathcal{U}' \\ \downarrow & & \downarrow \\ [Z^H/G] & \xrightarrow{\mu} & \mathcal{K}_{v_1,v_2,v_3} \end{array}$$

is a Cartesian diagram. Combining definition of e with the first paragraph, we see that $e \circ \mu = \mathrm{id}$. Since Proposition 6.1 shows that e is an isomorphism, we conclude that μ is also an isomorphism and $\mathcal{U}_{v_1,v_2,v_3}$ is isomorphic to $\mathcal{U}'$. □

Next, we describe the virtual fundamental class on $\mathcal{K}$. Let L_k denote the line bundle on $\mathcal{X}(\mathbf{\Sigma})$ corresponding to the line bundle $\mathbb{C} \times Z$ on Z where the G-action on $\mathbb{C}$ given by the kth component α_k of $\alpha\colon G \to (\mathbb{C}^*)^n$.

Proposition 6.3. *Let $\mathcal{K}_{v_1,v_2,v_3}$ be a component of the moduli space $\mathcal{K}$. If the integers $m_k \in \{1, 2\}$ are defined by the relation $v_1 + v_2 + v_3 = \sum_{\rho_i \in \sigma(\bar{v}_1,\bar{v}_2,\bar{v}_3)} m_k b_k$ in N, then the virtual fundamental class of the component $\mathcal{K}_{v_1,v_2,v_3}$ is*

$$\prod_{m_k = 2} c_1(L_k)\big|_{\mathcal{X}(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1,\bar{v}_2,\bar{v}_3))} \quad .$$

Proof. Let f be the natural map from $\mathcal{U}_{v_1,v_2,v_3}$ to $\mathcal{X}(\mathbf{\Sigma})$ and let $\pi\colon \mathcal{U}_{v_1,v_2,v_3} \to [Z^H/G]$ be the twisted projection map. Since $\mathcal{K}_{v_1,v_2,v_3}$ is smooth, the virtual fundamental class of $\mathcal{K}$ is given by the top Chern class of the bundle $R^1\pi_* f^*(T_{\mathcal{X}(\mathbf{\Sigma})})$; see Section 6.2 in [AGV]. To calculate this Chern class, observe that the pullback of the tangent bundle $f^*(T_{\mathcal{X}(\mathbf{\Sigma})})$ corresponds to a $(G \times H)$-equivariant bundle $\mathcal{V}$ on $Z^H \times C$; $\mathcal{V}$ is a trivial vector bundle of rank n where the $(G \times H)$-action is induced by the map $\alpha\colon G \to (\mathbb{C}^*)^n$ on it basis. Let $p\colon Z^H \times C \to Z^H$ be the projection map and let p_*^H be the invariant pushforward (pushing forward and taking invariant sections). Since the associated derived functor $R^1 p_*^H$ sends $(G \times H)$-equivariant sheaves on $Z^H \times C$ to G-equivariant sheaves on Z^H, it suffices to compute $R^1 p_*^H(\mathcal{V})$.

Let $\mathcal{W}_k$ be the trivial line bundle on $Z^H \times C$ with $(G \times H)$-action induced by the kth component α_k of $\alpha\colon G \to (\mathbb{C}^*)^n$ and consider the following exact sequence of vector bundles on $Z^H \times C$:

$$0 \longrightarrow p^*(T_{Z^H}) \longrightarrow \mathcal{V} \longrightarrow \bigoplus_{\rho_k \in \sigma(\bar{v}_1,\bar{v}_2,\bar{v}_3)} \mathcal{W}_k \longrightarrow 0 \quad .$$

Since the H-invariant part of $R^1p_*p^*(T_{Z^H}) = R^1p_*(\mathcal{O}_{Z^H\times C})\otimes T_{Z^H}$ is trivial, it suffices to calculate $R^1p_*^H(\mathcal{W}_k)$. Given a point $z \in Z^H$, the restriction of $\mathcal{W}_k$ to $z\times C$ is isomorphic to the trivial line bundle $\mathcal{L}_k$ on C with the the H-action induced by α_k. Since the Leray spectral sequence degenerates, we have $H^1(C,\mathcal{L}_k) \cong H^1(\mathbb{P}^1, p'_*\mathcal{L}_k)$ where $p'\colon C \to \mathbb{P}^1$ is the Galois cover. Because $v_j \in \operatorname{Box}(\boldsymbol{\Sigma})$ for $1 \leq j \leq 3$, there are $a_{j,k} \in \mathbb{Q}$ such that $0 \leq a_{j,k} \leq 1$ and $\bar{v}_j = \sum a_{j,k}\bar{b}_k$ where $\rho_k \in \sigma(\bar{v}_1,\bar{v}_2,\bar{v}_3)$. By hypothesis, we have $v_1+v_2+v_3 \equiv 0$ which means that $a_{1,k}+a_{2,k}+a_{3,k}$ is an integer between 0 and 2. Lemma 4.6 establishes that v_j corresponds to an element $g_j \in G$ and the proof of Proposition 4.3 shows that $\alpha_k(g_i) = \exp(2\pi\sqrt{-1}a_{j,k})$. It follows that $p_*^H\big(\mathcal{W}_k\big|_{z\times C}\big)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-a_{k,1}-a_{k,2}-a_{k,3})$. Since

$$\dim H^1\big(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(-a_{k,1}-a_{k,2}-a_{k,3})\big) = 1 \qquad \text{when } a_{1,k}+a_{2,k}+a_{3,k} = 2,$$

we deduce that, in this case, $R^1p_*^H(\mathcal{W}_k)$ is the line bundle $\mathbb{C}\times Z$ on Z where the G-action on $\mathbb{C}$ given by the kth component α_k. When $a_{1,k}+a_{2,k}+a_{3,k} \neq 2$, the cohomology group vanishes and $R^1p_*^H(\mathcal{W}_k)$ is zero. Therefore, we have

$$R^1\pi_*f^*(T_{\mathcal{X}(\boldsymbol{\Sigma})}) \cong \bigoplus_{m_k=2} L_k\big|_{[Z^H/G]}$$

and taking the top Chern class completes the proof. □

Remark 6.4. The calculation of the virtual class in Proposition 6.3 is analogous to the factors $c(g,h)$ used by Fantechi and Göttsche for the orbifold cohomology of global quotients [FG].

We end this section with a proof of Theorem 1.1. Let $\iota\colon \mathcal{I}\big(\mathcal{X}(\boldsymbol{\Sigma})\big) \to \mathcal{I}\big(\mathcal{X}(\boldsymbol{\Sigma})\big)$ denote the natural involution on the inertia stack defined by $(x,\phi)\mapsto(x,\phi^{-1})$ and let $\check{\mathrm{ev}}_3 := \iota\circ\mathrm{ev}_3$ be the twisted evaluation map; see Section 4.5 in [AGV]. If $\gamma_1,\gamma_2 \in A^*\big(\mathcal{I}\big(\mathcal{X}(\boldsymbol{\Sigma})\big)\big)$, then the orbifold product (Definition 6.2.2 in [AGV]) is

$$\gamma_1 * \gamma_2 := (\check{\mathrm{ev}}_3)_*\big(\mathrm{ev}_1^*(\gamma_1)\cup\mathrm{ev}_2^*(\gamma_2)\cup[\mathcal{K}]^{vir}\big)$$

where $[\mathcal{K}]^{vir}$ denotes the virtual fundamental class on $\mathcal{K}$. This definition agrees with the definition of the quantum product in degree zero.

Remark 6.5. Proposition 6.1 shows that the component $\mathcal{K}_{v_1,v_2,v_3}$ of the moduli stack is nonempty if and only if $v_1+v_2+v_3\equiv 0$. Hence, if $\gamma_1 \in A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1)\big)$ and $\gamma_2 \in A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}_2)\big)$, then the orbifold $\gamma_1*\gamma_2$ is nonzero only if there is a cone in Σ containing $\bar{v}_1$ and $\bar{v}_2$.

Proof of Theorem 1.1. By combining Proposition 4.7 and Proposition 5.2, we obtain the following isomorphism of $\mathbb{Q}$-graded $\mathbb{Q}$-vector spaces:

$$A^*_{orb}\big(\mathcal{X}(\boldsymbol{\Sigma})\big) = \bigoplus_{v\in\operatorname{Box}(\boldsymbol{\Sigma})} A^*\big(\mathcal{X}(\boldsymbol{\Sigma}/\boldsymbol{\sigma}(\bar{v}))\big)\big[\deg(y^v)\big] \cong \frac{\mathbb{Q}[N]^{\boldsymbol{\Sigma}}}{\langle\sum_{i=1}^n \theta(b_i)\cdot y^{b_i} : \theta\in N^\star\rangle}\,.$$

It remains to show that the orbifold product agrees with the product structure on the deformed group ring. Since the elements of $\operatorname{Box}(\boldsymbol{\Sigma})$ generates $A^*_{orb}\big(\mathcal{X}(\boldsymbol{\Sigma})\big)$ as a module over the y^{b_i}, it suffices to show that $y^c * y^{b_i} = y^c \cdot y^{b_i}$ and $y^{v_1} * y^{v_2} = y^{v_1} \cdot y^{v_2}$ where $c \in N$ and $v_1, v_2 \in \operatorname{Box}(\mathbf{\Sigma})$.

We first consider the product $y^c * y^{b_i}$ where $c \in N$. By taking advantage of the linear relations $\sum_{i=1}^n \theta(b_i) \cdot y^{b_i}$ for $\theta \in N^\star$, we reduce to the case that b_i does not lie in the minimal cone $\sigma(\bar{c})$ containing $\bar{c}$. Let v be the representative of c in $\operatorname{Box}(\mathbf{\Sigma})$. By Remark 6.5, the only contribution to the product $y^c * y^{b_i}$ comes from the component $\mathcal{K}_{v,0,v'}$ where $v' \in \operatorname{Box}(\mathbf{\Sigma})$ is defined by the equation $v + v' = \sum_{\rho_i \in \sigma(\bar{c})} b_i$. Hence, $\mathcal{K}_{v,0,v'}$ is isomorphic to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$, both $\mathrm{ev}_1, \check{\mathrm{ev}}_3 \colon \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big) \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$ are the identity map and $\mathrm{ev}_2 \colon \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big) \to \mathcal{X}(\mathbf{\Sigma})$ is the closed embedding. The restriction of y^{b_i} from $\mathcal{X}(\mathbf{\Sigma})$ to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$ is equal to $y^{\tilde{b}_i}$ if $\bar{b}_i$ and $\sigma(\bar{c})$ lie in a cone of Σ and is equal to zero otherwise. Since Proposition 6.3 shows the the virtual fundamental class is 1, if $\mathrm{ev}_2^*(y^{b_i}) \neq 0$ then $y^c * y^{b_i}$ is simply multiplication in $A^*(\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{c})\big)$ and Proposition 5.2 shows that this agrees with multiplication in the deformed group ring. Moreover, when $\mathrm{ev}_2^*(y^{b_i}) = 0$, we have $y^c * y^{b_i} = 0 = y^c \cdot y^{b_i}$.

Next, consider the product $y^{v_1} * y^{v_2}$ where $v_1, v_2 \in \operatorname{Box}(\mathbf{\Sigma})$. If $\bar{v}_1$ and $\bar{v}_2$ are not contained in a cone, then Remark 6.5 implies that $y^{v_1} * y^{v_2} = 0$ and (5.0.7) implies that $y^{v_1} \cdot y^{v_2} = 0$. On the other hand, suppose the cone $\sigma \in \Sigma$ contains $\bar{v}_1$ and $\bar{v}_2$. Let $v_3 \in \operatorname{Box}(\mathbf{\Sigma})$ be the element such that $\bar{v}_3 \in \sigma(\bar{v}_1, \bar{v}_2)$ and $v_1 + v_2 + v_3 \equiv 0$; in other words, there exists integers m_i such that $v_1 + v_2 + v_3 = \sum_{\rho_i \in \sigma(\bar{v}_1, \bar{v}_2, \bar{v}_3)} m_i b_i$ and $1 \leq m_i \leq 2$. Proposition 6.1 shows that the component $\mathcal{K}_{v_1,v_2,v_3}$ is isomorphic to $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big)$ and the evaluation map ev_i correspond to the closed embedding $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big) \to \mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)$. If I is the set of indices i such that $m_i = 2$, then Proposition 6.3 shows that the virtual fundamental class on $\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_1, \bar{v}_2, \bar{v}_3)\big)$ is the product of the pullbacks of the divisor classes y^{b_i} where $i \in I$. Because of the degree shift, the class $y^{v_i} \in A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ is identified with the class $1 \in A^*\big(\mathcal{X}\big(\mathbf{\Sigma}/\boldsymbol{\sigma}(\bar{v}_i)\big)\big)$ and $y^{v_1} * y^{v_2}$ is the image of the virtual fundamental class under the twisted evaluation map $\check{\mathrm{ev}}_3$. In particular, if J denotes the set of indices i such that $\bar{b}_i \in \sigma(\bar{v}_1, \bar{v}_2)$ but $b_i \notin \sigma(\bar{v}_3)$, then unravelling the identification maps shows that $y^{v_1} * y^{v_2} = y^{\check{v}_3} \cdot \prod_{i \in I} y^{b_i} \cdot \prod_{j \in J} y^{b_j}$ where $\check{v}_3$ is the representation of $-v_3$ in $\in \operatorname{Box}(\mathbf{\Sigma})$. The factor $y^{\check{v}_3}$ arises from the involution $\iota \colon \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma}) \to \mathcal{I}\big(\mathcal{X}(\mathbf{\Sigma})\big)$. Since $\check{v}_3 + \sum_{i \in I} b_i + \sum_{j \in J} b_j = v_1 + v_2$, we conclude that $y^{v_1} * y^{v_2} = y^{v_1} \cdot y^{v_2}$. □

7. Applications to Crepant Resolutions

In this section, we relate the orbifold Chow ring to the Chow ring of a crepant resolution by showing that both rings are fibres of a flat family. This provide a new proof that the graded components of these Chow rings have the same dimension. On the other hand, we also establish that these Chow rings are not generally isomorphic.

If $X(\Sigma)$ is a complete simplicial toric variety, then there is an associated toric Deligne-Mumford stack. Specifically, the fan Σ gives rise to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$ where N is the distinguished lattice in the vector space containing Σ and $\beta \colon \mathbb{Z}^n \to N$ is the map defined by the minimal lattice points on the rays in Σ. Proposition 3.7 shows that $X(\Sigma)$ is the coarse moduli space of $\mathcal{X}(\mathbf{\Sigma})$. The toric variety $X(\Sigma)$ is Gorenstein and

$X(\Sigma') \to X(\Sigma)$ is a crepant resolution if and only if there is Σ-linear support function $h' \colon \mathbb{Q}^d \to \mathbb{Q}$ such that $h'(0) = 0$ and $h'(b_i) = -1$ for $1 \leq i \leq m$.

Theorem 7.1. *Let $X(\Sigma)$ be a complete simplicial Gorenstein toric variety and let $\mathcal{X}(\mathbf{\Sigma})$ be the associated toric Deligne-Mumford stack. If Σ' is a regular subdivison of Σ such that $X(\Sigma')$ is a crepant resolution of $X(\Sigma)$, then there is a flat family $T \to \mathbb{P}^1$ of schemes such that $T_0 \cong \operatorname{Spec} A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ and $T_\infty \cong \operatorname{Spec} A^*\big(X(\Sigma')\big)$.*

Proof. Set $d := \dim X(\Sigma)$. Let $b_1, \dots, b_n$ be minimal lattice points on the rays in Σ and let $b_{n+1}, \dots, b_m$ be the minimal lattice points on the additional rays in Σ'. Since Σ' is a regular subdivision of Σ, there is a Σ'-linear support function $h \colon N \to \mathbb{Z}$ such that $h(b_i) = 0$ for $1 \leq i \leq n$, $h(b_i) > 0$ for $n+1 \leq i \leq m$ and $h(c_1 + c_2) \geq h(c_1) + h(c_2)$ for all lattice points c_1, c_2 lying in the same cone of Σ. Moreover, the inequality is strict unless c_1 and c_2 lie in the same cone of Σ'.

Consider the quotient of $(\mathbb{Q}[t,t^{-1}])[N]^{\mathbf{\Sigma}} := \mathbb{Q}[t,t^{-1}] \otimes_{\mathbb{Q}} \mathbb{Q}[N]^{\mathbf{\Sigma}}$ by the relations

$$\sum_{i=1}^{m} \theta(b_i)\, y^{b_i}\, t^{h(b_i)} = 0 \qquad \text{for all } \theta \in N^\star, \tag{7.1.8}$$

as a family parametrized by t. Since $h(b_i) = 0$ if and only if $1 \leq i \leq n$, the limit as $t \to 0$ is the quotient $\mathbb{Q}[N]^{\mathbf{\Sigma}} / \big\langle \sum_{i=1}^{n} \theta(b_i) y^{b_i} : \theta \in N^\star \big\rangle$ which is isomorphic to $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ by Theorem 1.1. To calculate the limit as $t \to \infty$, we consider a different basis for $(\mathbb{Q}[t])[N]^{\mathbf{\Sigma}}$. Let $\breve{y}^c := y^c\, t^{h(c)}$ for $c \in N$. Under this change of basis, the relations (7.1.8) become $\sum_{i=1}^{m} \theta(b_i) \breve{y}^{b_i}$ and the product becomes:

$$\breve{y}^{c_1} \cdot \breve{y}^{c_2} = \begin{cases} \breve{y}^{c_1+c_2}\, t^{h(c_1)+h(c_2)-h(c_1+c_2)} & \text{if there exists } \sigma \in \Sigma \text{ such that } c_1, c_2 \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Since $h(c_1 + c_2) \geq h(c_1) + h(c_2)$ and equality holds if and only if c_1 and c_2 lie in the same cone of Σ', we obtain the following as t tends to ∞:

$$\breve{y}^{c_1} \cdot \breve{y}^{c_2} = \begin{cases} \breve{y}^{c_1+c_2} & \text{if there exists } \sigma' \in \Sigma' \text{ such that } c_1, c_2 \in \sigma', \\ 0 & \text{otherwise;} \end{cases}$$

which is isomorphic to $A^*\big(X(\Sigma')\big)$ by Lemma 5.1.

Finally, the generic freeness lemma (Theorem 14.4 in [Eis]) implies that there exists an element $p(t) \in \mathbb{Q}[t]$ such that $(\mathbb{Q}[t,t^{-1},p(t)^{-1}])[N]^{\mathbf{\Sigma}} / \big\langle \sum_{i=1}^{m} \theta(b_i)\, y^{b_i}\, t^{h(b_i)} : \theta \in N^\star \big\rangle$ is a free $\mathbb{Q}[t,t^{-1},p(t)^{-1}]$-module. Thus, we obtain the required flat family $T \to \mathbb{P}^1$ by extending this family over the roots of $p(t)$. □

We end with an example in which $A^*\big(X(\Sigma')\big)$ is not isomorphic to $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$.

Example 7.2. Let $N = \mathbb{Z}^2$ and let $\Sigma \subseteq \mathbb{R}^2$ be the complete fan in which the rays are generated by the lattice points $b_1 := (1,0)$, $b_2 := (0,-1)$ and $b_3 := (-1,2)$. Hence, the toric variety $X(\Sigma)$ is the weighted projective space $\mathbb{P}(1,2,1)$ and the associated toric Deligne-Mumford stack is the quotient $[(\mathbb{C}^3 - \{0\})/\mathbb{C}^*]$ where the action is given by

$(z_1, z_2, z_3) \cdot \lambda = (\lambda z_1, \lambda^2 z_2, \lambda z_3)$. If we simply write x_i for the element $y^{b_i} \in \mathbb{Q}[N]$, then Theorem 1.1 implies that

$$A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big) \cong \frac{\mathbb{Q}[x_1,x_2,x_3,x_4]}{\big\langle x_1x_3 - x_4^2, x_2x_4, x_1 - x_3, -x_2 + 2x_3 \big\rangle} \cong \frac{\mathbb{Q}[x_3,x_4]}{\big\langle x_3^2 - x_4^2, x_3x_4 \big\rangle}\,.$$

Let Σ' be the fan obtained from Σ by inserting the ray generated by $b_4 := (0,1)$. It follows that $X(\Sigma')$ is the Hirzeburch surface $\mathbb{F}_2$, $X(\Sigma') \to X(\Sigma)$ is a crepant resolution (it blows down the (-2)-curve in $\mathbb{F}_2$), and Lemma 5.1 gives:

$$\begin{aligned} A^*\big(X(\Sigma')\big) &\cong \frac{\mathbb{Q}[x_1,x_2,x_3,x_4]}{\big\langle x_1x_3, x_2x_4, x_1 - x_3, -x_2 + 2x_3 + x_4 \big\rangle} \\ &\cong \frac{\mathbb{Q}[x_3,x_4]}{\big\langle x_3^2, 2x_3x_4 + x_4^2 \big\rangle} = \frac{\mathbb{Q}[x_3,x_4]}{\big\langle x_3^2, (x_3+x_4)^2 \big\rangle}\,. \end{aligned}$$

Since there is a degree one element $x \in A^*\big(X(\Sigma')\big)$ such that $x^2 = 0$ and $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big)$ does not contain such an element, we conclude that $A^*_{orb}\big(\mathcal{X}(\mathbf{\Sigma})\big) \ncong A^*\big(X(\Sigma')\big)$.

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