# SYSTEMATIC APPROACH TO CYCLIC ORBIFOLDS 

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#### Abstract

We introduce an orbifold induction procedure which provides a systematic construction of cyclic orbifolds, including their twisted sectors. The procedure gives counterparts in the orbifold theory of all the current-algebraic constructions of conformal field theory and enables us to find the orbifold characters and their modular transformation properties.


## 1 Introduction

Twisted scalar fields [1,2] and twisted vertex operators [3,4] were introduced in 1971 and 1975 respectively, although their role in string theory was not fully understood until the theory of orbifolds [5-12] was developed in the mid 1980's.

In this paper, we focus on the class of cyclic orbifolds, which are those formed by modding out the $\mathbb{Z}_{\lambda}$ symmetry (cyclic permutations) of $\lambda$ copies of a mother conformal field theory. In particular, we give an orbifold induction procedure which generates the twisted sectors of these orbifolds directly from the mother theory,

$$
\left.\begin{array}{rl}
(\text { mother }) \text { CFT } & \rightarrow  \tag{1.1}\\
\lambda
\end{array} \text { (cyclic orbifold }\right)_{\lambda}
$$

without having to consider the tensor product theory. Thus, the orbifold induction procedure makes what is apparently the hardest part of the problem into the easiest part.

Starting on the sphere (Sections 2 and 3), we find first that the orbifold induction procedure generates a new class of infinite-dimensional algebras which we call the orbifold algebras. These algebras, which are operative in the twisted sectors of cyclic orbifolds, are the root-covering algebras of the infinite-dimensional algebras of conformal field theory. Some subalgebras of these orbifold algebras appear earlier in Refs. [13, 12, 14], and the role of these subalgebras in cyclic orbifolds was emphasized by Fuchs, Klemm and Schmidt [12]. With the help of the orbifold algebras, we find that all the familiar constructions of conformal field theory have their counterparts in the orbifolds, including affine-Sugawara constructions [15-18], coset constructions $[15,16,19,20]$ and affine-Virasoro constructions [21-23], as well as $\mathrm{SL}(2, \mathbb{R})$-Ward identities [24] and null-state differential equations of BPZ type. Higher-level counterparts of the vertex operator constructions [25-27,4,28,29] and conformal embeddings [15,25,30-32] are also found.

The partition functions of cyclic orbifolds have been given for prime $\lambda$ by Klemm and Schmidt [11]. The orbifold induction procedure on the torus (Section 4) allows us to go beyond the partition functions, giving a systematic description of the characters of the orbifold, including their modular properties and fusion rules.

## 2 Orbifold Algebras

The orbifold algebras , simple examples of which are displayed in mode form below, are the rootcovering algebras of the infinite-dimensional algebras of conformal field theory. The orbifold algebras operate in the twisted sectors of cyclic orbifolds and these algebras may be induced in the orbifold by the orbifold induction procedure (see Subsection 2.6) from the infinite-dimensional algebras of the mother conformal field theory. The identity of the orbifold algebras as rootcovering algebras will be clear in Section 3, where we express the induction in its local form. The origin of the orbifold algebras is discussed from a more conventional viewpoint of cyclic orbifolds in Subsection 3.8. In what follows, the positive integer $\lambda$ is the order of the cyclic orbifold.

### 2.1 Orbifold affine algebra

The simplest examples of orbifold algebras are the orbifold affine algebras, whose generators are

$$
\begin{equation*}
\hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right), \quad a=1 \ldots \operatorname{dim} \mathrm{~g}, \quad r=0 \ldots \lambda-1, \quad m \in \mathbb{Z} \tag{2.1}
\end{equation*}
$$

where g is any simple finite-dimensional Lie algebra. These algebras have the form

$$
\begin{align*}
{\left[\hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{J}_{b}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]=} & \mathrm{i} f_{a b}{ }^{c} \hat{J}_{c}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right)  \tag{2.2}\\
& +\hat{k} \eta_{a b}\left(m+\frac{r}{\lambda}\right) \delta_{m+n+\frac{r+s}{\lambda}, 0}
\end{align*}
$$

where $f_{a b}{ }^{c}$ and $\eta_{a b}$ are respectively the structure constants and Killing metric of g , and $\hat{k}$ is the level of the orbifold affine algebra. The algebra (2.2) is understood with the periodicity condition

$$
\begin{equation*}
\hat{J}_{a}^{(r \pm \lambda)}\left(m+\frac{r \pm \lambda}{\lambda}\right)=\hat{J}_{a}^{(r)}\left(m \pm 1+\frac{r}{\lambda}\right) . \tag{2.3}
\end{equation*}
$$

This is a special case (with $\epsilon=0$ ) of the general periodicity condition

$$
\begin{equation*}
\hat{A}^{(r \pm \lambda)}\left(m+\frac{r \pm \lambda+\epsilon}{\lambda}\right)=\hat{A}^{(r)}\left(m \pm 1+\frac{r+\epsilon}{\lambda}\right) \tag{2.4}
\end{equation*}
$$

which will be assumed to hold with $\epsilon=0$ or $\frac{1}{2}$ for the modes of all the orbifold algebras below. It will also be useful to have the relation

$$
\begin{equation*}
\hat{A}^{(-r)}\left(m+\frac{-r+\epsilon}{\lambda}\right)=\hat{A}^{(\lambda-r)}\left(m-1+\frac{\lambda-r+\epsilon}{\lambda}\right) \tag{2.5}
\end{equation*}
$$

which is a consequence of the periodicity (2.4).

### 2.2 Orbifold Virasoro algebra

The generators of the orbifold Virasoro algebras are

$$
\begin{equation*}
\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \quad r=0 \ldots \lambda-1, \quad m \in \mathbb{Z} \tag{2.6}
\end{equation*}
$$

which satisfy

$$
\begin{align*}
{\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{L}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]=} & \left(m-n+\frac{r-s}{\lambda}\right) \hat{L}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right) \\
& +\frac{\hat{c}}{12}\left(m+\frac{r}{\lambda}\right)\left(\left(m+\frac{r}{\lambda}\right)^{2}-1\right) \delta_{m+n+\frac{r+s}{\lambda}, 0} \tag{2.7}
\end{align*}
$$

and the periodicity condition (2.4) with $\epsilon=0$. The quantity $\hat{c}$ is the central charge of the orbifold Virasoro algebra.

### 2.3 Orbifold $\mathrm{N}=1$ superconformal algebra

## $\widehat{N S}$ sector

In the $\widehat{N S}$ sector of the $\mathrm{N}=1$ orbifold superconformal algebra, we have, in addition to the orbifold Virasoro generators (2.6), the set of $\lambda \widehat{N S}$ orbifold supercurrents

$$
\begin{equation*}
\hat{G}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right), \quad r=0 \ldots \lambda-1, \quad m \in \mathbb{Z} \tag{2.8}
\end{equation*}
$$

which satisfy the algebra

$$
\begin{align*}
{\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{G}^{(s)}\left(n+\frac{s+\frac{1}{2}}{\lambda}\right)\right]=} & \left(\frac{1}{2}\left(m+\frac{r}{\lambda}\right)-\left(n+\frac{s+\frac{1}{2}}{\lambda}\right)\right) \hat{G}^{(r+s)}\left(m+n+\frac{r+s+\frac{1}{2}}{\lambda}\right) \\
{\left[\hat{G}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right), \hat{G}^{(s)}\left(n+\frac{s+\frac{1}{2}}{\lambda}\right)\right]_{+}=} & 2 \hat{L}^{(r+s+1)}\left(m+n+\frac{r+s+1}{\lambda}\right)  \tag{2.9}\\
& +\frac{\hat{c}}{3}\left(\left(m+\frac{r+\frac{1}{2}}{\lambda}\right)^{2}-\frac{1}{4}\right) \delta_{m+n+\frac{r+s+1}{\lambda}, 0} .
\end{align*}
$$

The $\widehat{N S}$ orbifold supercurrents also satisfy the periodicity relation (2.4) with $\epsilon=\frac{1}{2}$.

## $\widehat{\widehat{R} \text { sector }}$

In the $\widehat{R}$ sector, the $\widehat{R}$ orbifold supercurrents $\hat{G}^{(r)}\left(m+\frac{r}{\lambda}\right)$ satisfy

$$
\begin{align*}
& {\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{G}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]=\left(\frac{1}{2}\left(m+\frac{r}{\lambda}\right)-\left(n+\frac{s}{\lambda}\right)\right) \hat{G}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right)} \\
& {\left[\hat{G}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{G}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]_{+}=2 \hat{L}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right)+\frac{\hat{c}}{3}\left(\left(m+\frac{r}{\lambda}\right)^{2}-\frac{1}{4}\right) \delta_{m+n+\frac{r+s}{\lambda}, 0}} \tag{2.10}
\end{align*}
$$

and the periodicity condition (2.4) with $\epsilon=0$.

### 2.4 Orbifold $N=2$ superconformal algebra

The orbifold $\mathrm{N}=2$ superconformal algebra has the counterparts $\widehat{N S}, \widehat{R}$, and $\widehat{T}$ of the usual sectors $N S, R$, and $T$ (twisted), all of which are included in the uniform notation for the generators

$$
\begin{gather*}
\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{G}^{(i, r)}\left(m+\frac{r+\delta^{i}}{\lambda}\right), \hat{J}^{(r)}\left(m+\frac{r+\epsilon}{\lambda}\right), \quad i=1,2, \quad r=0 \ldots \lambda-1  \tag{2.11}\\
\widehat{N S}: \quad \epsilon=0, \quad \delta^{1}=\delta^{2}=\frac{1}{2} \\
\widehat{R}: \epsilon=\delta^{1}=\delta^{2}=0  \tag{2.12}\\
\widehat{T}: \epsilon=\delta^{2}=\frac{1}{2}, \quad \delta^{1}=0
\end{gather*}
$$

where $i=1,2$ labels the two sets of orbifold supercurrents and $\hat{J}^{(r)}$ is the set of orbifold $U(1)$ currents. In addition to the orbifold Virasoro algebra (2.7), the $\mathrm{N}=2$ system satisfies

$$
\begin{array}{ll}
{\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{G}^{(i, s)}\left(n+\frac{s+\delta^{i}}{\lambda}\right)\right]} & =\left(\frac{1}{2}\left(m+\frac{r}{\lambda}\right)-\left(n+\frac{s+\delta^{i}}{\lambda}\right)\right) \hat{G}^{(i, r+s)}\left(m+n+\frac{r+s+\delta^{i}}{\lambda}\right) \\
{\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{J}^{(s)}\left(n+\frac{s+\epsilon}{\lambda}\right)\right]} & =-\left(n+\frac{s+\epsilon}{\lambda}\right) \hat{J}^{(r+s)}\left(m+n+\frac{r+s+\epsilon}{\lambda}\right) \\
{\left[\hat{J}^{(r)}\left(m+\frac{r+\epsilon}{\lambda}\right), \hat{J}^{(s)}\left(n+\frac{s+\epsilon}{\lambda}\right)\right]} & =\frac{\hat{c}}{3}\left(m+\frac{r+\epsilon}{\lambda}\right) \delta_{m+n+\frac{r+s+2 \epsilon}{}, 0}^{\lambda} \\
{\left[\hat{J}^{(r)}\left(m+\frac{r+\epsilon}{\lambda}\right), \hat{G}^{(i, s)}\left(n+\frac{s+\delta^{i}}{\lambda}\right)\right]} & =\mathrm{i}_{i j} \hat{G}^{\left(j, r+s+\epsilon+\delta^{i}-\delta^{j}\right)}\left(m+n+\frac{r+s+\epsilon+\delta^{i}}{\lambda}\right) \\
{\left[\hat{G}^{(i, r)}\left(m+\frac{r+\delta^{i}}{\lambda}\right), \hat{G}^{(j, s)}\left(n+\frac{s+\delta^{j}}{\lambda}\right)\right]_{+}} & =2 \delta_{i j} \hat{L}^{\left(r+s+\delta^{i}+\delta^{j}\right)}\left(m+n+\frac{r+s+\delta^{i}+\delta^{j}}{\lambda}\right) \\
& +\mathrm{i}_{i j}\left(m-n+\frac{r-s+\delta^{i}-\delta^{j}}{\lambda}\right) \hat{J}^{\left(r+s+\delta^{i}+\delta^{j}-\epsilon\right)}\left(m+n+\frac{r+s+\delta^{i}+\delta^{j}}{\lambda}\right) \\
& +\frac{\hat{c}}{3}\left(\left(m+\frac{r+\delta^{i}}{\lambda}\right)^{2}-\frac{1}{4}\right) \delta_{i j} \delta_{m+n+\frac{r+s+\delta^{i}+\delta^{j}}{\lambda}}^{\lambda}, 0 \tag{2.13}
\end{array}
$$

and the periodicity conditions (2.4).

### 2.5 Orbifold $\mathrm{W}_{3}$ algebra

This algebra has, in addition to the orbifold Virasoro generators, the $\hat{W}$ generators

$$
\begin{equation*}
\hat{W}^{(r)}\left(m+\frac{r}{\lambda}\right), \quad r=0 \ldots \lambda-1, \quad m \in \mathbb{Z} \tag{2.14}
\end{equation*}
$$

which satisfy the orbifold $W_{3}$ algebra

$$
\begin{align*}
& {\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{W}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]=\left(2\left(m+\frac{r}{\lambda}\right)-\left(n+\frac{s}{\lambda}\right)\right) \hat{W}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right)} \\
& {\left[\hat{W}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{W}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]=\frac{16 \lambda}{22 \lambda+5 \hat{c}}\left(m-n+\frac{r-s}{\lambda}\right) \hat{\Lambda}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right)} \\
& +\left(m-n+\frac{r-s}{\lambda}\right)\left(\frac{1}{15}\left(m+n+\frac{r+s+2}{\lambda}\right)\left(m+n+\frac{r+s+3}{\lambda}\right)-\frac{1}{6}\left(m+\frac{r+2}{\lambda}\right)\left(n+\frac{s+2}{\lambda}\right)\right) \hat{L}^{(r+s)}\left(m+n+\frac{r+s}{\lambda}\right) \\
& +\frac{\hat{c}}{360}\left(\left(m+\frac{r}{\lambda}\right)^{2}-4\right)\left(\left(m+\frac{r}{\lambda}\right)^{2}-1\right)\left(m+\frac{r}{\lambda}\right) \delta_{m+n+\frac{r+s}{\lambda}, 0} . \tag{2.15}
\end{align*}
$$

The composite operators $\hat{\Lambda}^{(r)}\left(m+\frac{r}{\lambda}\right), \quad r=0 \ldots \lambda-1$ are constructed with the normal-ordered products of two orbifold Virasoro operators

$$
\begin{align*}
& \hat{\Lambda}^{(r)}\left(m+\frac{r}{\lambda}\right)= \frac{1}{\lambda} \sum_{s=0}^{\lambda-1} \sum_{n}: \hat{L}^{(s)}\left(n+\frac{s}{\lambda}\right) \hat{L}^{(r-s)}\left(m-n+\frac{r-s}{\lambda}\right): \\
&-\left(\frac{3}{10}\left(m+\frac{r+2}{\lambda}\right)\left(m+\frac{r+3}{\lambda}\right)+\frac{\hat{c}}{12 \lambda}\left(1-\frac{1}{\lambda^{2}}\right)\right) \hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)  \tag{2.16}\\
&+\frac{\hat{c}}{1440}\left(1-\frac{1}{\lambda^{2}}\right)\left(\frac{119}{\lambda^{2}}-11\right) \delta_{m+\frac{r}{\lambda}, 0} \\
&: \hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right) \hat{L}^{(s)}\left(n+\frac{s}{\lambda}\right): \equiv \theta\left(m \geq-\frac{r+1}{\lambda}\right) \hat{L}^{(s)}\left(n+\frac{s}{\lambda}\right) \hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)  \tag{2.17}\\
&+\theta\left(m<-\frac{r+1}{\lambda}\right) \hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right) \hat{L}^{(s)}\left(n+\frac{s}{\lambda}\right)
\end{align*}
$$

and the periodicity of all the operators in the system is governed by (2.4) with $\epsilon=0$.

### 2.6 Orbifold induction procedure

The orbifold algebras above and other orbifold algebras, such as the orbifold $W_{n}$ algebras for $n>3$, can be obtained by an orbifold induction procedure from the infinite-dimensional algebras of the mother conformal field theory. We discuss here the mode form of the induction procedure, deferring its local form until Section 3.

In the induction procedure, every Virasoro primary field $A_{\Delta}$ of conformal weight $\Delta$ in the mother theory defines a set of $\lambda$ orbifold fields $\hat{A}_{\Delta}^{(r)}, r=0 \ldots \lambda-1$. We consider here only integer and half-integer moded primary fields with modes $A_{\Delta}(m+\epsilon)$ where $\epsilon=0, \frac{1}{2}$. (The induction procedure for general primary fields is discussed in Section 3.) Then the modes of the orbifold fields are given by the following definition:

$$
\begin{equation*}
\hat{A}_{\Delta}^{(r)}\left(m+\frac{r+\epsilon}{\lambda}\right) \equiv \lambda^{1-\Delta} A_{\Delta}(\lambda m+r+\epsilon) \tag{2.18}
\end{equation*}
$$

The simplest application of equation (2.18) is the induction of the orbifold affine algebra (2.2) via the relation

$$
\begin{equation*}
\hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right)=J_{a}(\lambda m+r) \tag{2.19}
\end{equation*}
$$

from the general affine algebra $[33,15]$

$$
\begin{equation*}
\left[J_{a}(m), J_{b}(n)\right]=\mathrm{i} f_{a b}^{c} J_{c}(m+n)+k \eta_{a b} m \delta_{m+n, 0} \tag{2.20}
\end{equation*}
$$

of the mother theory at level $k$. One finds that

$$
\begin{equation*}
\hat{k}=\lambda k \tag{2.21}
\end{equation*}
$$

where $\hat{k}$ is the level of the orbifold affine algebra.
For the quasi-primary fields $L$ and $\Lambda$, the induction relations read

$$
\begin{gather*}
\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right) \equiv \frac{1}{\lambda} L(\lambda m+r)+\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right)  \tag{2.22}\\
\hat{\Lambda}^{(r)}\left(m+\frac{r}{\lambda}\right) \equiv \lambda^{-3} \Lambda(\lambda m+r)-\frac{\hat{c}(22 \lambda+5 \hat{c})}{2880 \lambda}\left(1-\frac{1}{\lambda^{2}}\right)^{2} \delta_{m+\frac{r}{\lambda}, 0} \tag{2.23}
\end{gather*}
$$

and one finds that the central charge of the orbifold Virasoro algebra (2.7) is

$$
\begin{equation*}
\hat{c}=\lambda c \tag{2.24}
\end{equation*}
$$

where $c$ is the central charge of the Virasoro algebra in the mother theory. The same orbifold central charge $\hat{c}$ appears in the orbifold superconformal and $W_{3}$ algebras.

Special cases of the induction relations (2.18) and (2.22) are known in the literature [13,12, 14]. These special cases correspond to the induction of what we will call integral subalgebras (see Subsection 2.7) of the orbifold affine, orbifold Virasoro and orbifold superconformal algebras. It was also emphasized in Ref. [12] that these subalgebras of the orbifold algebras are operative in the twisted sectors of cyclic orbifolds.

## Unitarity

The orbifold induction procedure also implies that the states of the twisted sector of the orbifold are the states of the mother conformal field theory and the induced inner product of states is also the same as in the mother theory.

Although non-unitary conformal field theories can also be considered, we will generally assume in this paper that the mother conformal field theory is a unitary conformal field theory with a unique $\operatorname{SL}(2, \mathbb{R})$-invariant vacuum state $|0\rangle$ and corresponding vacuum field $\phi_{0}=\mathbf{1}$. In the orbifold the state $|0\rangle$ is the ground state of the twisted sector (twist field) with conformal weight

$$
\begin{equation*}
\hat{\Delta}=\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right) \tag{2.25}
\end{equation*}
$$

as measured by the orbifold Virasoro generator $\hat{L}^{(0)}(0)$.
The induction procedure from a unitary mother conformal field theory gives the twisted sectors of the unitary cyclic orbifolds, where the adjoint operations are defined by

$$
\begin{gather*}
\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)^{\dagger}=\hat{L}^{(\lambda-r)}\left(-(m+1)+\frac{\lambda-r}{\lambda}\right)  \tag{2.26}\\
\hat{A}_{\Delta}^{(r)}\left(m+\frac{r+\epsilon}{\lambda}\right)^{\dagger}=\hat{A}_{\Delta}^{(\lambda-2 \epsilon-r)}\left(-(m+1)+\frac{(\lambda-2 \epsilon-r)+\epsilon}{\lambda}\right) . \tag{2.27}
\end{gather*}
$$

These relations are nothing but the images of the adjoint operations in the unitary mother conformal field theory.

### 2.7 Subalgebras of the orbifold algebras

In this subsection, we discuss the integral subalgebras of the orbifold algebras and two $\mathrm{SL}(2, \mathbb{R})$ subalgebras of the orbifold Virasoro algebras. The integral subalgebras of the orbifold algebras are isomorphic to the corresponding infinite-dimensional algebras of the mother theory.

As a related remark, it is not difficult to see for $\lambda=N^{2}$ that the orbifold affine algebra generated by $\hat{J}_{a}^{(r)}, r=0 \ldots N^{2}-1$ contains a subalgebra (generated by $\hat{J}_{a}^{(r)}$ with $r$ divisible by $N$ ) which is a twisted affine Lie algebra [34]. This twisted affine algebra can be obtained by an outer-automorphic twist of $\lambda$ copies of a mother affine algebra at level $k$ by the $\mathbb{Z}_{\lambda}$ permutation symmetry of the copies.

## Integral affine subalgebra

In the orbifold affine algebra (2.2), the orbifold currents $\hat{J}_{a}^{(0)}(m), a=1 \ldots \operatorname{dim} \mathrm{~g}, \quad m \in \mathbb{Z}$ form an integral affine subalgebra which is an ordinary affine algebra at level $\hat{k}=\lambda k$. This integral subalgebra was studied in Refs. [13,14], and was called the winding subalgebra in Ref. [13].

## Integral Virasoro subalgebra

In the orbifold Virasoro algebra (2.7), the generators $\hat{L}^{(0)}(m), m \in \mathbb{Z}$ form an integral Virasoro subalgebra (which is an ordinary Virasoro algebra with central charge $\hat{c}=\lambda c$ ). This integral subalgebra was studied in Refs. [13,12, 14].

## $\underline{\text { Integral } N=1 \text { subalgebras }}$

The integral superconformal subalgebras of the orbifold $\mathrm{N}=1$ superconformal algebras (2.9) and
(2.10) are collected in the following table.

Orbifold sector Generators Integral $\mathrm{N}=1$ subalgebra

| $\lambda=2 l+1$ | $\widehat{N S}$ | $\hat{G}^{(l)}, \hat{L}^{(0)}$ | $N S$ |
| :---: | :---: | :---: | :---: |
| $\lambda=2 l$ | $\widehat{R}$ | $\hat{G}^{(0)}, \hat{L}^{(0)}$ | $R$ |
|  | $\widehat{N S}$ | - | - |
|  | $\widehat{R}$ | $\hat{G}^{(l)}, \hat{L}^{(0)}$ | $N S$ |
|  | $\widehat{R}$ | $\hat{G}^{(0)}, \hat{L}^{(0)}$ | $R$ |

This table shows in particular that, for even $\lambda$, the $\widehat{R}$ orbifold algebra contains both an ordinary NS and an ordinary R superconformal subalgebra, while the $\widehat{N S}$ orbifold algebra contains no superconformal subalgebras.

## $\underline{\text { Integral } N=2 \text { subalgebras }}$

The integral $\mathrm{N}=2$ superconformal subalgebras of the orbifold $N=2$ superconformal algebras (2.13) are given in the table below.

Orbifold sector Generators Integral $\mathrm{N}=2$ subalgebra

| $\lambda=2 l+1$ | $\widehat{N S}$ | $\hat{G}^{(i, l)}, \hat{J}^{(0)}, \hat{L}^{(0)}$ | $N S$ |
| :---: | :---: | :---: | :---: |
| $\lambda=2 l$ | $\widehat{R}$ | $\hat{G}^{(i, 0)}, \hat{J}^{(0)}, \hat{L}^{(0)}$ | $R$ |
|  | $\widehat{T}$ | $\hat{G}^{(1,0)}, \hat{G}^{(2, l)}, \hat{J}^{(l)}, \hat{L}^{(0)}$ | $T$ |
|  | $\widehat{N S}$ | - | - |
|  | $\widehat{R}$ | $\hat{G}^{(i, l)}, \hat{J}^{(0)}, \hat{L}^{(0)}$ | $N S$ |
|  | $\widehat{R}$ | $\hat{G}^{(i, 0)}, \hat{J}^{(0)}, \hat{L}^{(0)}$ | $R$ |
|  | $\widehat{R}$ | $\hat{G}^{(1,0)}, \hat{G}^{(2, l)}, \hat{J}^{(l)}, \hat{L}^{(0)}$ | $T$ |
|  | $\widehat{T}$ | - | - |

Except for the $\widehat{T}$ sector, these integral subalgebras have been studied in Ref. [12], where it was also noted for even $\lambda$ that the $\widehat{N S}$ sector of the orbifold is eliminated in string theory by the

GSO projection [35].

## $\underline{\mathrm{W}_{3} \text { integral subalgebras }}$

The orbifold $\mathrm{W}_{3}$ algebra (2.15) has no integral $\mathrm{W}_{3}$ subalgebras, so orbifoldization has, in this case, removed an infinite-dimensional symmetry of the mother theory.

## $\underline{S L(2, \mathbb{R}) \text { subalgebras }}$

Any integral Virasoro subalgebra contains an $\operatorname{SL}(2, \mathbb{R})$ subalgebra whose generators are

$$
\begin{equation*}
\hat{L}^{(0)}(1), \hat{L}^{(0)}(-1), \hat{L}^{(0)}(0) \tag{2.30}
\end{equation*}
$$

In addition, each orbifold Virasoro algebra contains a centrally-extended $\operatorname{SL}(2, \mathbb{R})$ subalgebra

$$
\begin{array}{ll}
{\left[\hat{L}^{(1)}\left(\frac{1}{\lambda}\right), \hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\right]} & =\frac{2}{\lambda} \hat{L}^{(0)}(0)-\frac{\hat{c}}{12 \lambda}\left(1-\frac{1}{\lambda^{2}}\right) \\
{\left[\hat{L}^{(0)}(0), \hat{L}^{(1)}\left(\frac{1}{\lambda}\right)\right]} & =-\frac{1}{\lambda} \hat{L}^{(1)}\left(\frac{1}{\lambda}\right)  \tag{2.31}\\
{\left[\hat{L}^{(0)}(0), \hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\right]} & =\frac{1}{\lambda} \hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)
\end{array}
$$

which is the image of the $\operatorname{SL}(2, \mathbb{R})$ algebra of the mother theory. The ground state $|0\rangle$ of the twisted sector of the orbifold is not $\mathrm{SL}(2, \mathbb{R})$-invariant under either of these $\mathrm{SL}(2, \mathbb{R})$ subalgebras, and we note in particular that

$$
\begin{gather*}
\hat{L}^{(1)}\left(\frac{1}{\lambda}\right)|0\rangle=\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)|0\rangle=0 \\
\hat{L}^{(0)}(0)|0\rangle=\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right)|0\rangle \tag{2.32}
\end{gather*}
$$

for the centrally-extended $\operatorname{SL}(2, \mathbb{R})$. Curiously however, we will see in Subsection 3.6 that the twisted sector of the orbifold contains $\operatorname{SL}(2, \mathbb{R})$ Ward identities associated to the centrallyextended SL $(2, \mathbb{R})$.

## 3 Twisted sectors on the sphere

### 3.1 Principal primary fields

Any Virasoro primary state $|\Delta\rangle$ in the mother theory is proportional to the state $|\hat{\Delta}\rangle$ in the orbifold, which is primary with respect to the integral Virasoro subalgebra

$$
\begin{equation*}
\hat{L}^{(0)}(m \geq 0)|\hat{\Delta}\rangle=\delta_{m, 0} \hat{\Delta}|\hat{\Delta}\rangle \tag{3.1}
\end{equation*}
$$

The conformal weight $\hat{\Delta}$ in (3.1) is

$$
\begin{equation*}
\hat{\Delta}=\frac{\Delta}{\lambda}+\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right) . \tag{3.2}
\end{equation*}
$$

There is also an infinite number of other primary states of the integral Virasoro subalgebra, which are not Virasoro primary in the mother theory. At induction order $\lambda$, examples of such states include all the descendants in the $\Delta$ module whose level in the module is less than or equal to $\lambda-1$. We focus in particular on the multiplet of $\lambda$ states

$$
\begin{gather*}
(L(-1))^{\lambda-1-r}|\Delta\rangle \sim\left(\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\right)^{\lambda-1-r}|\hat{\Delta}\rangle, \quad r=0 \ldots \lambda-1  \tag{3.3}\\
\hat{\Delta}_{r}=\frac{\lambda-1-r+\Delta}{\lambda}+\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right)
\end{gather*}
$$

which, up to normalization, we will call the principal primary states of the twisted sector. All these states are Virasoro primary with conformal weights $\hat{\Delta}_{r}$ under the integral Virasoro subalgebra, including the state $|\hat{\Delta}\rangle$ with $\hat{\Delta}=\hat{\Delta}_{\lambda-1}$.

The principal primary states have the distinction that all other primary states under the integral Virasoro subalgebra have conformal weights $\hat{h}=\left\{\hat{\Delta}_{r}\right\}+n$ which differ by a nonnegative integer $n$ from the conformal weights $\hat{\Delta}_{r}$ in (3.3). In what follows, we construct the principal primary fields $\hat{\varphi}_{\Delta}^{(r)}(z), r=0 \ldots \lambda-1$ of the twisted sector of the orbifold, which, as we shall see, are interpolating fields for the principal primary states (3.3).

We begin with a primary field $\varphi_{\Delta}$ of conformal weight $\Delta$ in the mother theory, which satisfies

$$
\begin{equation*}
\left[L(m), \varphi_{\Delta}(z)\right]=z^{m}\left(z \partial_{z}+\Delta(m+1)\right) \varphi_{\Delta}(z) . \tag{3.4}
\end{equation*}
$$

We will assume that the primary field has a diagonal monodromy such that

$$
\begin{gather*}
\varphi_{\Delta}\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=\varphi_{\Delta}(z) \mathrm{e}^{2 \pi \mathrm{i} \theta} \\
\varphi_{\Delta}(z)[L(m), \theta]=0  \tag{3.5}\\
\theta|0\rangle=0
\end{gather*}
$$

examples of which are provided by the familiar abelian vertex operators and by all simple currents [36-40].

The principal primary fields $\hat{\varphi}_{\Delta}^{(r)}(z), r=0 \ldots \lambda-1$ of the twisted sector of the orbifold are defined as

$$
\begin{align*}
\hat{\varphi}_{\Delta}^{(r)}(z) & \equiv \sum_{s=0}^{\lambda-1} \hat{\varphi}_{\Delta}\left(z \mathrm{e}^{2 \pi \mathrm{i} s}\right) \mathrm{e}^{\frac{2 \pi \mathrm{is} s}{\lambda}(-\theta+r+1+\Delta(\lambda-1))}  \tag{3.6}\\
& =\rho(z)^{\Delta} \sum_{s=0}^{-1} \varphi_{\Delta}\left(z^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{is} s}{\lambda}}\right) \mathrm{e}^{\frac{2 \pi \mathrm{is} s}{\lambda}(-\theta+r+1)}
\end{align*}
$$

where

$$
\begin{gather*}
\rho(z) \equiv \frac{1}{\lambda} z^{\frac{1}{\lambda}-1}  \tag{3.7}\\
\hat{\varphi}_{\Delta}(z) \equiv \rho(z)^{\Delta} \varphi_{\Delta}\left(z^{\frac{1}{\lambda}}\right) \tag{3.8}
\end{gather*}
$$

$$
\begin{equation*}
\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{\varphi}_{\Delta}(z)\right]=z^{m+\frac{r}{\lambda}}\left(z \partial_{z}+\Delta\left(m+\frac{r}{\lambda}+1\right)\right) \hat{\varphi}_{\Delta}(z) \tag{3.9}
\end{equation*}
$$

The commutation relation (3.9) follows from (3.4), (3.7) and (3.8).
The principal primary fields have the following properties,

$$
\begin{gather*}
\hat{\varphi}_{\Delta}^{(r \pm \lambda)}(z)=\hat{\varphi}_{\Delta}^{(r)}(z)  \tag{3.10}\\
\hat{\varphi}_{\Delta}^{(r)}\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=\hat{\varphi}_{\Delta}^{(r)}(z) \mathrm{e}^{\frac{2 \pi \mathrm{i}}{\lambda}(\theta-r-1-\Delta(\lambda-1))}  \tag{3.11}\\
\sum_{r=0}^{\lambda} \hat{\varphi}_{\Delta}^{(r)}(z)=\lambda \hat{\varphi}_{\Delta}(z)  \tag{3.12}\\
{\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{\varphi}_{\Delta}^{(s)}(z)\right]=z^{m+\frac{r}{\lambda}}\left(z \partial_{z}+\Delta\left(m+\frac{r}{\lambda}+1\right)\right) \hat{\varphi}_{\Delta}^{(s+r)}(z)} \tag{3.13}
\end{gather*}
$$

which follow from eqs. (3.4-9).
The reader should bear in mind that the field $\hat{\varphi}_{\Delta}(z)$ in (3.8) is a locally-conformal transformation, by $z \rightarrow z^{\frac{1}{\lambda}}$, of the Virasoro primary field $\varphi_{\Delta}(z)$, where the quantity $\rho(z)$ in (3.7)

$$
\begin{equation*}
\rho(z)=\frac{\partial\left(z^{\frac{1}{\lambda}}\right)}{\partial z} \tag{3.14}
\end{equation*}
$$

is the Jacobian of the transformation. The field $\hat{\varphi}_{\Delta}(z)$ therefore lives on the root coverings of the sphere. Moreover, equations (3.6) and (3.12) show that the principal primary fields are the fields with definite monodromy which result from the monodromy decomposition of the conformally-transformed field $\hat{\varphi}_{\Delta}(z)$. This conformal transformation has appeared in free-field examples in Ref. [9].

The principal primary fields $\hat{\varphi}_{\Delta}^{(r)}$ create the principal primary states $\left|\hat{\Delta}_{r}\right\rangle, r=0 \ldots \lambda-1$ as follows,

$$
\begin{align*}
\left|\hat{\Delta}_{r}\right\rangle \quad & \equiv \lim _{z \rightarrow 0} z^{\Delta\left(1-\frac{1}{\lambda}\right)-1+\frac{r+1}{\lambda}} \hat{\varphi}_{\Delta}^{(r)}(z)|0\rangle \\
& =\lambda^{1-\Delta} \frac{\left(\partial^{\lambda-r-1} \varphi \Delta\right)(0)}{(\lambda-r-1)!}|0\rangle=\frac{\lambda^{\left.1-\Delta_{L(-1)}\right)^{\lambda-r-1}}}{(\lambda-r-1)!}|\Delta\rangle \\
& =\frac{\lambda^{\lambda-r-1}\left\{\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\right\}^{\lambda-r-1}}{(\lambda-r-1)!}\left|\hat{\Delta}_{\lambda-1}\right\rangle  \tag{3.15}\\
\left|\hat{\Delta}_{\lambda-1}\right\rangle & =\lambda^{1-\Delta} \lim _{z \rightarrow 0} \varphi_{\Delta}(z)|0\rangle=\lambda^{1-\Delta}|\Delta\rangle .
\end{align*}
$$

The relations in (3.15) identify the principal primary states as those shown in eq. (3.3). Using these results, the following algebraic properties of the principal primary states $\left|\hat{\Delta}_{r}\right\rangle$ are easily
verified:

$$
\begin{gather*}
\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)\left|\hat{\Delta}_{s}\right\rangle=0, m>0 \\
\hat{L}^{(0)}(0)\left|\hat{\Delta}_{r}\right\rangle=\hat{\Delta}_{r}\left|\hat{\Delta}_{r}\right\rangle \\
\hat{\Delta}_{r}=\left(\frac{\lambda-r-1+\Delta}{\lambda}\right)+\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right) \\
\hat{L}^{(r)}\left(\frac{r}{\lambda}\right)\left|\hat{\Delta}_{s}\right\rangle=0, \quad r+s \geq \lambda  \tag{3.16}\\
\hat{L}^{(r)}\left(\frac{r}{\lambda}\right)\left|\hat{\Delta}_{s}\right\rangle=\left(\frac{\Delta(r+1)+\lambda-s-r-1}{\lambda}\right)\left|\hat{\Delta}_{s+r}\right\rangle, \quad r=1 \ldots \lambda-1 \\
\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\left|\hat{\Delta}_{r}\right\rangle=\left(1-\frac{r}{\lambda}\right)\left|\hat{\Delta}_{r-1}\right\rangle .
\end{gather*}
$$

The first three relations in (3.16) include the fact that the principal primary states are primary under the integral Virasoro subalgebra, with conformal weights $\hat{\Delta}_{r}$, as they should be. The state $\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\left|\hat{\Delta}_{0}\right\rangle$ is not in general Virasoro primary under the integral Virasoro subalgebra.

## Example: ground state of the twisted sector

Some of the principal primary fields and states may be null. This is seen in the simplest example, which is the ground state of the twisted sector. The vacuum field of the mother theory is $\varphi_{0}(z)=1$, so that we obtain for the corresponding set of principal primary fields,

$$
\begin{gather*}
\hat{\varphi}_{0}^{(r)}(z)=\lambda \delta_{r+1,0 \bmod \lambda}=\lambda \delta_{r, \lambda-1} \\
\lim _{z \rightarrow 0} \hat{\varphi}_{0}^{(\lambda-1)}(z)|0\rangle=\lambda|0\rangle \tag{3.17}
\end{gather*}
$$

These fields and states are null for $r \neq \lambda-1$ because the state $\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)|0\rangle$ is null, which corresponds to the fact that the ground state of this twisted sector is non-degenerate.

## Modes

For those Virasoro primary fields of the mother theory which have integer or half-integer conformal weight $\Delta$ and trivial monodromy, the discussion above can be expressed in modes. In the mother theory, the mode resolution of $\varphi_{\Delta}(z)$ can be written as

$$
\begin{gather*}
\varphi_{\Delta}(z)=\sum_{m \in \mathbb{Z}} \varphi_{\Delta}(m+1-\Delta) z^{-(m+1-\Delta)} z^{-\Delta} \\
\lim _{z \rightarrow 0} \varphi_{\Delta}(z)|0\rangle=\varphi_{\Delta}(-\Delta)|0\rangle  \tag{3.18}\\
\varphi_{\Delta}(-\Delta+m)|0\rangle=0, \quad m>0
\end{gather*}
$$

Then one obtains the mode form of the principal primary fields and states,

$$
\begin{gather*}
\hat{\varphi}_{\Delta}^{(r)}(z)=\sum_{m \in \mathbb{Z}} \hat{\varphi}_{\Delta}^{(r)}\left(m+\frac{r+1-\Delta}{\lambda}\right) z^{-\left(m+\frac{r+1-\Delta}{\lambda}\right)} z^{-\Delta}  \tag{3.19}\\
\hat{\varphi}_{\Delta}^{(r)}\left(m+\frac{r+1-\Delta}{\lambda}\right)=\lambda^{1-\Delta} \varphi_{\Delta}(\lambda m+r+1-\Delta)  \tag{3.20}\\
{\left[\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{\varphi}_{\Delta}^{(s)}\left(n+\frac{s+1-\Delta}{\lambda}\right)\right]=\left(\left(m+\frac{r}{\lambda}\right)(\Delta-1)-\left(n+\frac{s+1-\Delta}{\lambda}\right)\right) \hat{\varphi}_{\Delta}^{(s+r)}\left(n+m+\frac{s+r+1-\Delta}{\lambda}\right)}  \tag{3.22}\\
{\left[\hat{L}^{(0)}(0), \hat{\varphi}_{\Delta}^{(r)}\left(-1+\frac{r+1-\Delta}{\lambda}\right)\right]=\left(\frac{\lambda-r-1+\Delta}{\lambda}\right) \hat{\varphi}_{\Delta}^{(r)}\left(-1+\frac{r+1-\Delta}{\lambda}\right)}  \tag{3.21}\\
\left|\hat{\Delta}_{r}\right\rangle=\hat{\varphi}_{\Delta}^{(r)}\left(-1+\frac{r+1-\Delta}{\lambda}\right)|0\rangle=\lambda^{1-\Delta} \varphi_{\Delta}(-\Delta+r+1-\lambda)|0\rangle . \tag{3.23}
\end{gather*}
$$

The coefficient on the right side of eq. (3.22) is the non-shift part of the conformal weight $\hat{\Delta}_{r}$ in (3.16). We also note that (except for the orbifold Virasoro generators) the generators of the orbifold algebras are the modes of principal primary fields of the orbifolds, although the mode-labeling convention (3.20) of the principal primary fields can differ by a cyclic relabeling from the mode-labeling convention (2.18) of the orbifold algebras.

### 3.2 OPE's and correlators of principal primary fields

In this subsection, we will use the orbifold induction procedure to compute the operator product expansion (OPE) of two principal primary fields.

We begin with the OPE of two Virasoro primary fields in the mother theory,

$$
\begin{equation*}
\varphi_{\Delta_{1}}(z) \varphi_{\Delta_{2}}(w)=\sum_{i} \frac{F_{12}{ }^{i}}{(z-w)^{\Delta_{1}+\Delta_{2}-\Delta_{i}}}\left(1+\sum_{p=1}^{\infty} a_{p}^{i}(z-w)^{p} \partial_{w}^{p}\right) \chi_{\Delta_{i}}(w) \tag{3.24}
\end{equation*}
$$

where $\chi_{\Delta_{i}}$ are Virasoro quasi-primary fields. The constants $a_{p}^{i}$ in (3.24) are fixed by $\operatorname{SL}(2, \mathbb{R})$ invariance

$$
\begin{equation*}
a_{p}^{i}=\frac{1}{p!} \prod_{q=0}^{p-1} \frac{\left(\Delta_{i}+\Delta_{1}-\Delta_{2}+q\right)}{\left(2 \Delta_{i}+q\right)} \tag{3.25}
\end{equation*}
$$

and further relations between $F$ 's follow from conformal invariance when sets of quasi-primary fields are in the same Virasoro module.

Because OPE's are form invariant under conformal transformations [24] and $\hat{\varphi}_{\Delta}$ is a conformal transformation of $\varphi_{\Delta}$ (see Subsection 3.1), it follows that

$$
\begin{equation*}
\hat{\varphi}_{\Delta_{1}}(z) \hat{\varphi}_{\Delta_{2}}(w)=\sum_{i} \frac{F_{12}{ }^{i}}{(z-w)^{\Delta_{1}+\Delta_{2}-\Delta_{i}}}\left(1+\sum_{p=1}^{\infty} a_{p}^{i}(z-w)^{p} \partial_{w}^{p}\right) \hat{\chi}_{\Delta_{i}}(w) \tag{3.26}
\end{equation*}
$$

where $\hat{\chi}_{\Delta_{i}}(w)$ is the corresponding conformal transformation (by $z \rightarrow z^{\frac{1}{\lambda}}$ ) of $\chi_{\Delta_{i}}$.

For simplicity, we now assume that the mother field $\varphi_{\Delta_{1}}$ has trivial monodromy, which implies that

$$
\begin{equation*}
\theta_{i}=\theta_{2}, \quad \Delta_{1}+\Delta_{2}-\Delta_{i} \in \mathbb{Z} \tag{3.27}
\end{equation*}
$$

where $\theta_{i}$ is the monodromy of $\hat{\chi}_{\Delta_{i}}$. Then one computes from (3.6) and (3.26) the OPE's of two principal primary fields,

$$
\begin{equation*}
\hat{\varphi}_{\Delta_{1}}^{(r)}(z) \hat{\varphi}_{\Delta_{2}}^{(s)}(w)=\sum_{i} \frac{F_{12}{ }^{i}}{(z-w)^{\Delta_{1}+\Delta_{2}-\Delta_{i}}}\left(1+\sum_{p=1}^{\infty} a_{p}^{i}(z-w)^{p} \partial_{w}^{p}\right) \hat{\chi}_{\Delta_{i}}^{\left(r+s+1+\Delta_{i}-\Delta_{1}-\Delta_{2}\right)}(w)+\text { reg. } \tag{3.28}
\end{equation*}
$$

where

$$
\begin{gather*}
\hat{\chi}_{\Delta_{i}}^{(r)}(w)=\sum_{s=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{is} s}{\lambda}\left(r+1-\theta_{i}+\Delta_{i}(\lambda-1)\right)} \hat{\chi}_{\Delta_{i}}\left(w \mathrm{e}^{2 \pi \mathrm{i} s}\right) \\
\hat{\chi}_{\Delta_{i}}^{(r+\lambda)}(w)=\hat{\chi}_{\Delta_{i}}^{(r)}(w) \tag{3.29}
\end{gather*}
$$

is obtained for the monodromy decomposition of the conformal transformation of the quasiprimary fields. Given the monodromy algebra of $\theta_{1} \neq 0$ with $\varphi_{\Delta_{2}}$ in the mother theory, these OPE's can be obtained in full generality.

Using the monodromy decomposition (3.6), we may also express the correlators of any number of principal primary fields in terms of vacuum averages in the mother theory,

$$
\begin{align*}
& \langle 0| \hat{\varphi}_{\Delta_{1}}^{\left(r_{1}\right)}\left(z_{1}\right) \ldots \hat{\varphi}_{\Delta_{n}}^{\left(r_{n}\right)}\left(z_{n}\right)|0\rangle=\left(\prod_{i=1}^{n} \rho^{\Delta_{i}}\left(z_{i}\right)\right) \sum_{s_{1}=0}^{\lambda-1} \ldots \sum_{s_{n}=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{i}}{\lambda} \sum_{j=1}^{\lambda-1} s_{j}\left(r_{j}+1\right)} \Phi_{(\Delta)}^{(s)}(z) \\
& \Phi_{(\Delta)}^{(s)}(z)=\left\langle\varphi_{\Delta_{1}}\left(z_{1}^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{i} s_{1}}{\lambda}}\right) \mathrm{e}^{-\frac{2 \pi \mathrm{i} s_{1} \theta_{1}}{\lambda}} \ldots \varphi_{\Delta_{n}-1}\left(z_{n-1}^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{i} s_{n-1}}{\lambda}}\right) \mathrm{e}^{-\frac{2 \pi \mathrm{i} s_{n-1} \theta_{n-1}}{\lambda}} \varphi_{\Delta_{n}}\left(z_{n}^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{i} s_{n}}{\lambda}}\right)\right\rangle . \tag{3.30}
\end{align*}
$$

Although the monodromy algebra of $\theta$ 's and $\varphi$ 's is needed to evaluate these quantities in the general case, the evaluation is straightforward for fields in the mother theory with integer and half-integer conformal weight $\Delta$ and trivial monodromy. As an example, we consider the two-point correlators, which satisfy

$$
\begin{equation*}
\left\langle\varphi_{\Delta}(z) \varphi_{\Delta}(w)\right\rangle=\frac{B}{(z-w)^{2 \Delta}} \tag{3.31}
\end{equation*}
$$

in the mother theory. Then eq. (3.30) gives

$$
\begin{gather*}
\langle 0| \hat{\varphi}_{\Delta}^{(r)}(z) \hat{\varphi}_{\Delta}^{(s)}(w)|0\rangle=\frac{\lambda B}{(2 \Delta-1)!}\left(\frac{w}{z}\right) \frac{\Delta(\lambda+1)-(s+1)}{\lambda}\left(w^{1-2 \Delta} \delta_{r+s+2-2 \Delta, 0 \bmod \lambda}\right) D_{\Delta}(w) \frac{1}{z-w}  \tag{3.32}\\
D_{\Delta}(w)=\prod_{k=0}^{2 \Delta-2}\left(w \partial_{w}+\frac{\lambda-s+k}{\lambda}\right)
\end{gather*}
$$

for the two-point correlators in the twisted sector of the orbifold.

### 3.3 Local form of the orbifold stress tensor

The stress tensor of the mother theory is moded as

$$
\begin{equation*}
T(z) \equiv \sum_{m} L(m) z^{-m-2} \tag{3.33}
\end{equation*}
$$

We define the local form of the twist classes of the orbifold stress tensor as

$$
\begin{gather*}
\hat{T}^{(r)}(z) \equiv \sum_{m \in \mathbb{Z}} \hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right) z^{-\left(m+\frac{r}{\lambda}\right)-2}, r=0 \ldots \lambda-1  \tag{3.34}\\
\hat{T}(z) \equiv \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} \hat{T}^{(r)}(z) \tag{3.35}
\end{gather*}
$$

where the modes $\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)$ are given in (2.22), and the relation (3.35) parallels the relation (3.12) for the principal primary fields. It follows that

$$
\begin{equation*}
\hat{T}(z)=\rho(z)^{2} T\left(z^{\frac{1}{\lambda}}\right)+\frac{\hat{c}}{24 \lambda z^{2}}\left(1-\frac{1}{\lambda^{2}}\right) \tag{3.36}
\end{equation*}
$$

where $\hat{c}=\lambda c$ is the central charge of the orbifold Virasoro algebra (2.7). As expected, $\hat{T}(z)$ in (3.36) is a conformal transformation of $T(z)$ by the same base-space transformation $\left(z \rightarrow z^{\frac{1}{\lambda}}\right)$ that gave $\hat{\varphi}_{\Delta}(z)$ for the principal primary fields (see Subsection 3.1). The shift term in (3.36) is the Schwarzian derivative for this transformation, which can appear because $T(z)$ is only quasiprimary under itself. This confirms the general prescription for the induction of quasi-primary fields seen on the right of eqs. (3.26) and (3.28).

The twist classes $\hat{T}^{(r)}$ can also be viewed as the fields with definite monodromy which result from the monodromy decomposition of $\hat{T}$,

$$
\begin{align*}
& \hat{T}^{(r)}(z)= \sum_{s=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{i} s}{\lambda} r} \hat{T}\left(z \mathrm{e}^{2 \pi \mathrm{i} s}\right) \\
&= \rho(z)^{2} \sum_{s=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{is}}{\lambda}(r+2)} T\left(z^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{i} s}{\lambda}}\right)+\frac{\hat{c}}{24 z^{2}}\left(1-\frac{1}{\lambda^{2}}\right) \delta_{r, 0}  \tag{3.37}\\
& \hat{T}^{(r)}\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=\hat{T}^{(r)}(z) \mathrm{e}^{-\frac{2 \pi \mathrm{i} i}{\lambda}}  \tag{3.38}\\
& \hat{T}^{(r \pm \lambda)}(z)=\hat{T}^{(r)}(z) \tag{3.39}
\end{align*}
$$

and the monodromy relations (3.38) identify the field

$$
\begin{equation*}
\hat{T}^{(0)}(z)=\sum_{m \in \mathbb{Z}} \hat{L}^{(0)}(m) z^{-m-2} \tag{3.40}
\end{equation*}
$$

as the chiral stress tensor of the twisted sector of the orbifold.
Moreover, the OPE's among $\hat{T}^{(r)}$ and the principal primary fields $\hat{\varphi}_{\Delta}^{(s)}$ may be computed from their monodromy decompositions and their OPE's in the mother theory. The results are

$$
\begin{gather*}
\hat{T}^{(r)}(z) \hat{\varphi}_{\Delta}^{(s)}(w)=\left(\frac{\Delta}{(z-w)^{2}}+\frac{\partial_{w}}{(z-w)}\right) \hat{\varphi}_{\Delta}^{(s+r)}(w)+\text { reg. }  \tag{3.41}\\
\hat{T}^{(r)}(z) \hat{T}^{(s)}(w)=\frac{(\hat{c} / 2) \delta_{r+s, 0 \bmod \lambda}}{(z-w)^{4}}+\left(\frac{2}{(z-w)^{2}}+\frac{\partial_{w}}{(z-w)}\right) \hat{T}^{(r+s)}(w)+\text { reg. } \tag{3.42}
\end{gather*}
$$

where eq. (3.42) is the local form of the orbifold Virasoro algebra (2.7). In obtaining (3.42), one recognizes that the coefficient of the central term is

$$
\begin{equation*}
\frac{c}{2} \hat{\varphi}_{0}^{(r+s-1)}(z)=\frac{\hat{c}}{2} \delta_{r+s, 0 \bmod \lambda} \tag{3.43}
\end{equation*}
$$

where $\hat{\varphi}_{0}^{(r)}$ is the ground state field (3.17) of the twisted sector.

### 3.4 Orbifold currents and applications

In this subsection, we study the orbifold currents $\hat{J}_{a}^{(r)}, r=0 \ldots \lambda-1$, which satisfy the orbifold affine algebra (2.2) at level $\hat{k}=\lambda k$, and discuss various applications of these currents.

The local form of the orbifold currents is

$$
\begin{gather*}
\hat{J}_{a}^{(r)}(z)=\sum_{m \in \mathbb{Z}} \hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right) z^{-\left(m+\frac{r}{\lambda}\right)-1}=\sum_{s=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{i}}{\lambda} r s} \hat{J}_{a}\left(z \mathrm{e}^{2 \pi \mathrm{i} \mathrm{i}}\right)  \tag{3.44}\\
\hat{J}_{a}(z)=\rho(z) J_{a}\left(z^{\frac{1}{\lambda}}\right)  \tag{3.45}\\
\hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right)|0\rangle=0, \quad m \geq 0  \tag{3.46}\\
\left|\hat{\Delta}_{r}\right\rangle_{a}=\lim _{z \rightarrow 0} z^{\frac{r}{\lambda}} \hat{J}_{a}^{(r)}(z)|0\rangle=\hat{J}_{a}^{(r)}\left(-1+\frac{r}{\lambda}\right)|0\rangle=J_{a}(-\lambda+r)|0\rangle  \tag{3.47}\\
\hat{J}_{a}^{(r)}(z) \hat{J}_{b}^{(s)}(w)=\frac{\hat{k} \eta_{a b} \delta_{r+s, 0 \bmod \lambda}}{(z-w)^{2}}+\frac{\mathrm{i} f_{a b}{ }^{c} \hat{J}_{c}^{(r+s)}(w)}{(z-w)}+\mathrm{reg} .  \tag{3.48}\\
\langle 0| \hat{J}_{a}^{(r)}(z) \hat{J}_{b}^{(s)}(w)|0\rangle=\left(\hat{k} \eta_{a b} \delta_{r+s, 0 \bmod \lambda}\right)\left(\frac{w}{z}\right)^{1-\frac{s}{\lambda}}\left(\frac{1}{(z-w)^{2}}+\frac{\lambda-s}{\lambda w} \frac{1}{z-w}\right) \tag{3.49}
\end{gather*}
$$

where (3.48) is the local form of the orbifold affine algebra (2.2) and (3.49) is a special case of (3.32).

## Affine-Sugawara construction on the orbifold

We consider next the cyclic orbifolds formed from tensor products of affine-Sugawara constructions [15-18] on simple affine algebras. Beginning with a general affine-Sugawara construction at level $k$ of g in the mother theory, the induction procedure gives the orbifold affine-Sugawara construction

$$
\begin{align*}
& \hat{L}_{\mathrm{g}}^{(r)}\left(m+\frac{r}{\lambda}\right)= \frac{1}{\lambda} L_{\mathrm{g}}^{a b}(k) \sum_{s=0}^{\lambda-1}: \hat{J}_{a}^{(s)}\left(n+\frac{s}{\lambda}\right) \hat{J}_{b}^{(r-s)}\left(m-n+\frac{r-s}{\lambda}\right):  \tag{3.50}\\
&+\frac{\hat{c}_{\mathbf{g}}}{24}\left(1-\frac{1}{\lambda^{2}}\right) \delta_{m+\frac{r}{\lambda}, 0, \quad r=0 \ldots \lambda-1} \\
& \frac{1}{\lambda} L_{\mathrm{g}}^{a b}(k)=\frac{1}{\lambda} \frac{\eta^{a b}}{2 k+Q_{\mathrm{g}}}=\frac{\eta^{a b}}{2(\lambda k)+\left(\lambda Q_{\mathrm{g}}\right)}, \quad \hat{c}_{\mathrm{g}}=\lambda c_{\mathrm{g}}(k)=\frac{2 \lambda k \operatorname{dimg}}{2 k+Q_{\mathrm{g}}}  \tag{3.51}\\
&: \hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right) \hat{J}_{b}^{(s)}\left(n+\frac{s}{\lambda}\right): \equiv \theta\left(m+\frac{r}{\lambda} \geq 0\right) \hat{J}_{b}^{(s)}\left(n+\frac{s}{\lambda}\right) \hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right) \\
&+\theta\left(m+\frac{r}{\lambda}<0\right) \hat{J}_{a}^{(r)}\left(m+\frac{r}{\lambda}\right) \hat{J}_{b}^{(s)}\left(n+\frac{s}{\lambda}\right)  \tag{3.52}\\
& \hat{L}_{\mathrm{g}}^{(r)}\left(m+\frac{r}{\lambda}\right)|0\rangle= \delta_{m+\frac{r}{\lambda}, 0} \frac{\hat{c}_{\mathrm{g}}}{24}\left(1-\frac{1}{\lambda^{2}}\right)|0\rangle, \quad m \geq 0 \tag{3.53}
\end{align*}
$$

in the twisted sectors of the corresponding cyclic orbifold, where $\eta^{a b}$ is the inverse Killing metric of g . The orbifold currents in (3.50-3.53) have level $\hat{k}=\lambda k$.

The orbifold currents are principal primary fields under the orbifold affine-Sugawara construction, and we obtain the relations

$$
\begin{gather*}
{\left[\hat{L}_{\mathrm{g}}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{J}_{a}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]=-\left(n+\frac{s}{\lambda}\right) \hat{J}_{a}^{(s+r)}\left(n+m+\frac{s+r}{\lambda}\right)}  \tag{3.54}\\
{\left[\hat{L}_{\mathrm{g}}^{(r)}\left(m+\frac{r}{\lambda}\right), \hat{J}_{a}^{(s)}(z)\right]=\partial_{z}\left(z^{m+\frac{r}{\lambda}+1} \hat{J}_{a}^{(s+r)}(z)\right)}  \tag{3.55}\\
\hat{T}_{\mathrm{g}}^{(r)}(z) \hat{J}_{a}^{(s)}(w)=\left(\frac{1}{(z-w)^{2}}+\frac{1}{(z-w)} \partial_{w}\right) \hat{J}_{a}^{(s+r)}(w)+\mathrm{reg} . \tag{3.56}
\end{gather*}
$$

which express in the orbifold the fact that the currents have conformal weight $\Delta=1$ in the mother theory.

One also finds that the principal primary states (3.47) of the orbifold currents have conformal weights

$$
\begin{equation*}
\hat{\Delta}_{r}^{\mathrm{g}}=\left(1-\frac{r}{\lambda}\right)+\frac{\hat{c}_{\mathbf{g}}}{24}\left(1-\frac{1}{\lambda^{2}}\right), \quad r=0 \ldots \lambda-1 \tag{3.57}
\end{equation*}
$$

under the integral Virasoro subalgebra of the orbifold affine-Sugawara construction (3.50). Similarly, the principal primary states associated to the affine primary states of the mother theory have orbifold conformal weights

$$
\begin{equation*}
\hat{\Delta}_{r}^{\mathrm{g}}(T)=\frac{\lambda-r-1+\Delta_{\mathrm{g}}(T)}{\lambda}+\frac{\hat{c}_{\mathrm{g}}}{24}\left(1-\frac{1}{\lambda^{2}}\right), \quad r=0 \ldots \lambda-1 \tag{3.58}
\end{equation*}
$$

where $\Delta_{\mathrm{g}}(T)$ is the conformal weight of irrep $T$ in the mother theory.

## Higher-level vertex operator constructions

The vertex operator constructions [25-27,4,28,29] of the affine algebras are well known at level one: In the mother theory, one has for level one of the untwisted simply-laced algebras,

$$
\begin{align*}
E_{\alpha}(z) & =c_{\alpha}: \mathrm{e}^{\mathrm{i} \alpha \cdot Q}:  \tag{3.59}\\
J_{A}(z) & =\mathrm{i} \partial_{z} Q_{A}(z)
\end{align*}
$$

where $\alpha \in \Delta$ with $\alpha^{2}=2, A=1 \ldots$ rankg and $c_{\alpha}$ is the Klein transformation. The string coordinate $Q$ in (3.59) involves the familiar zero-mode operators $q$ and $p$, which satisfy canonical commutation relations.

The orbifold induction relation (2.18) then gives the vertex operator construction of the orbifold affine algebra at level $\hat{k}=\lambda$,

$$
\begin{align*}
\hat{E}_{\alpha}^{(r)}\left(m+\frac{r}{\lambda}\right) & =E_{\alpha}(\lambda m+r)=c_{\alpha} \oint \frac{d z}{2 \pi \mathrm{i}} z^{\lambda m+r}: \mathrm{e}^{\mathrm{i} \alpha \cdot Q(z)}:  \tag{3.60}\\
& =c_{\alpha} \oint_{\lambda} \frac{d w}{2 \pi \mathrm{i}} w^{m+\frac{r}{\lambda}} \rho(w): \mathrm{e}^{\mathrm{i} \alpha \cdot \hat{Q}(w)}:, \quad r=0 \ldots \lambda-1
\end{align*}
$$

$$
\begin{gather*}
\hat{E}_{\alpha}^{(r)}(w)=c_{\alpha} \sum_{s=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{i} s r}{\lambda}} \rho\left(w \mathrm{e}^{2 \pi \mathrm{i} s}\right): \mathrm{e}^{\mathrm{i} \alpha \cdot \hat{Q}\left(w \mathrm{e}^{2 \pi \mathrm{i} s}\right)}:  \tag{3.61}\\
\hat{J}_{A}^{(r)}(w)=\sum_{s=0}^{\lambda-1} \mathrm{e}^{\frac{2 \pi \mathrm{i}}{\lambda} r s} \hat{J}_{A}\left(w \mathrm{e}^{2 \pi \mathrm{i} s}\right) \tag{3.62}
\end{gather*}
$$

where $\hat{\varphi}_{\Delta}(z)=\rho^{\Delta}(z) \varphi_{\Delta}\left(z^{\frac{1}{\lambda}}\right)$ and $\Delta=0$ for $Q$. In (3.60), the symbol $\oint_{\lambda}$ means that the contour goes counterclockwise around the origin $\lambda$ times. The normal ordering in (3.60) and (3.61) is

$$
\begin{gather*}
: \mathrm{e}^{\mathrm{i} \alpha \cdot \hat{Q}(w)}: \equiv \mathrm{e}^{\mathrm{i} \alpha \cdot q} w^{\frac{\alpha \cdot p}{\lambda}} \mathrm{e}^{\mathrm{i} \alpha \cdot \hat{Q}^{-}(w)} \mathrm{e}^{\mathrm{i} \alpha \cdot \hat{Q}^{+}(w)}  \tag{3.63}\\
\hat{Q}_{A}^{ \pm}(w) \equiv \pm \frac{\mathrm{i}}{\lambda}\left\{\sum_{r=1}^{\lambda-1} \frac{\lambda}{r} \hat{J}_{A}^{( \pm r)}\left( \pm \frac{r}{\lambda}\right) w^{\mp \frac{r}{\lambda}}+\sum_{r=0}^{\lambda-1} \sum_{m=1}^{\infty}\left(m+\frac{r}{\lambda}\right)^{-1} \hat{J}_{A}^{( \pm r)}\left( \pm\left(m+\frac{r}{\lambda}\right)\right) w^{\mp\left(m+\frac{r}{\lambda}\right)}\right\} \tag{3.64}
\end{gather*}
$$

and the orbifold Cartan currents $\hat{J}_{A}^{(-r)}$ in (3.64) are defined as

$$
\begin{equation*}
\hat{J}_{A}^{(-r)}\left(m-\frac{r}{\lambda}\right)=\hat{J}_{A}^{(\lambda-r)}\left(m-1+\frac{\lambda-r}{\lambda}\right), \quad A=1 \ldots \operatorname{rankg} \tag{3.65}
\end{equation*}
$$

in agreement with the general relation (2.5).
According to the induction procedure, the construction (3.61-62) for $\hat{J}_{a}^{(r)}=\left(\hat{E}_{\alpha}^{(r)}, \hat{J}_{A}^{(r)}\right)$ satisfies the orbifold affine algebra (3.48) at level $\hat{k}=\lambda$. This can also be checked directly from the OPE's

$$
\begin{equation*}
: \mathrm{e}^{\mathrm{i} \alpha \cdot \hat{Q}(z)}:: \mathrm{e}^{\mathrm{i} \beta \cdot \hat{Q}(w)}:=\left(z^{\frac{1}{\lambda}}-w^{\frac{1}{\lambda}}\right)^{\alpha \cdot \beta}: \mathrm{e}^{\mathrm{i}(\alpha \cdot \hat{Q}(z)+\beta \cdot \hat{Q}(w))}: . \tag{3.66}
\end{equation*}
$$

We emphasize in particular that the integral affine subalgebra generated by $\hat{J}_{a}^{(0)}=\left(\hat{E}_{\alpha}^{(0)}, \hat{J}_{A}^{(0)}\right)$, which is an ordinary untwisted simply-laced affine algebra, is represented by this construction at level $\hat{k}=\lambda$.

## Cosets, conformal embeddings at level $\lambda$ and so on

The $\mathrm{g} / \mathrm{h}$ coset construction $[15,16,19,20]$ has the stress tensor $T_{\mathrm{g} / \mathrm{h}}=T_{\mathrm{g}}-T_{\mathrm{h}}$ and Virasoro central charge $c_{\mathrm{g} / \mathrm{h}}=c_{\mathrm{g}}-c_{\mathrm{h}}$, where h is a simple subalgebra of the simple ${ }^{1}$ Lie algebra g at level $k$. The image of this coset construction in the twisted sector of the corresponding cyclic orbifold is the set of orbifold stress tensors

$$
\begin{align*}
& \hat{T}_{\mathrm{g} / \mathrm{h}}^{(r)}=\hat{T}_{\mathrm{g}}^{(r)}-\hat{T}_{\mathrm{h}}^{(r)}, \quad r=0 \ldots \lambda-1  \tag{3.67}\\
& \hat{c}_{\mathrm{g} / \mathrm{h}}=\hat{c}_{\mathrm{g}}-\hat{c}_{\mathrm{h}}=\lambda c_{\mathrm{g} / \mathrm{h}},
\end{align*}
$$

which satisfy the local form (3.42) of the orbifold Virasoro algebra with central charge $\hat{c}_{\mathrm{g} / \mathrm{h}}$. Here, $\hat{T}_{\mathrm{g}}^{(r)}$ (at level $\hat{k}=\lambda k$ of the orbifold affine algebra) is given in (3.50), and $\hat{T}_{\mathrm{h}}^{(r)}$ has the same form with $L_{\mathrm{g}}^{a b} \rightarrow L_{\mathrm{h}}^{a b}$.

[^0]The conformal embeddings [15, 25, 30-32] are the unitary coset constructions with $c_{\mathrm{g} / \mathrm{h}}=0$ and hence $T_{\mathrm{g}}=T_{\mathrm{h}}$. In the mother theory, these embeddings are known to occur only at level one of the ambient affine Lie algebra. It follows that each of these conformal embeddings has its image in the corresponding cyclic orbifold

$$
\begin{equation*}
\hat{T}_{\mathrm{g}}^{(r)}=\hat{T}_{\mathrm{h}}^{(r)}, \quad r=0 \ldots \lambda-1 \tag{3.68}
\end{equation*}
$$

at $\hat{c}_{\mathrm{g} / \mathrm{h}}=0$ and level $\hat{k}=\lambda$ of the ambient orbifold affine algebra (and its integral affine subalgebra). These higher level conformal embeddings include the chiral stress tensor of the twisted sector when $r=0$. We mention in particular the orbifold image of a familiar conformal embedding, whose form

$$
\begin{equation*}
\hat{T}^{(r)}\left(\text { simply-laced } \mathrm{g}_{\lambda}\right)=\hat{T}^{(r)}(\text { Cartan }(\text { simply-laced } \mathrm{g})), \quad r=0 \ldots \lambda-1 \tag{3.69}
\end{equation*}
$$

can also be verified directly from the higher-level orbifold vertex operator construction (3.6162).

Beyond the coset constructions, one has the general affine-Virasoro construction [21-23]

$$
\begin{equation*}
T(z)=L^{a b}: J_{a}(z) J_{b}(z):, \quad c=2 k \eta_{a b} L^{a b} \tag{3.70}
\end{equation*}
$$

where $L^{a b}$ is any solution of the Virasoro master equation. The image of the general affineVirasoro construction in the twisted sector of the corresponding cyclic orbifold is the set of orbifold stress tensors

$$
\begin{equation*}
\hat{T}^{(r)}(z)=\frac{1}{\lambda} L^{a b} \sum_{s=0}^{\lambda-1}: \hat{J}_{a}^{(s)}(z) \hat{J}_{b}^{(r-s)}(z):+\frac{\hat{c}}{24 \lambda z^{2}}\left(1-\frac{1}{\lambda^{2}}\right) \tag{3.71}
\end{equation*}
$$

which satisfy the local form (3.42) of the orbifold Virasoro algebra with $\hat{c}=\lambda c$. The normal ordering of the orbifold currents is given in eq. (3.52).

### 3.5 Ising orbifolds at $\hat{c}=\frac{\lambda}{2}$

As another example of the induction procedure, we construct the twisted sectors of Ising orbifolds at $\hat{c}=\frac{\lambda}{2}$, starting from the mother theory at $c=\frac{1}{2}$. This example has the induction of a Ramond sector as an additional feature.

## $\widehat{\widehat{B H / N} S}$ sector

The induction of this sector of the Ising orbifolds begins with the $B H / N S$ sector [15,41] of the critical Ising model

$$
\begin{gather*}
T_{B H / N S}=-\frac{1}{2}: H \partial H:, \quad c=\frac{1}{2}  \tag{3.72}\\
H(z)=H\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=\sum_{m \in \mathbb{Z}} b\left(m+\frac{1}{2}\right) z^{-\left(m+\frac{1}{2}\right)-\frac{1}{2}} . \tag{3.73}
\end{gather*}
$$

The orbifold induction procedure maps the $B H / N S$ sector of the mother theory into the $\widehat{B H / N} S$ sector of the Ising orbifold, where the principal primary fields corresponding to the $B H / N S$ fermion have the form

$$
\begin{gather*}
\hat{b}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right)=\sqrt{\lambda} b\left(\lambda m+r+\frac{1}{2}\right), \quad r=0 \ldots \lambda-1  \tag{3.74}\\
\hat{H}^{(r)}(z)=\rho^{\frac{1}{2}}(z) \sum_{s=0}^{\lambda-1} H\left(z^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{is}}{\lambda}}\right) \mathrm{e}^{\frac{2 \pi \mathrm{is}}{\lambda}(r+1)}=\sum_{m} \hat{b}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right) z^{-\left(m+\frac{r+\frac{1}{2}}{\lambda}\right)-\frac{1}{2}}  \tag{3.75}\\
\hat{H}^{(r)}\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=\hat{H}^{(r)}(z) \mathrm{e}^{-2 \pi \mathrm{i}\left(\frac{r+\frac{1}{2}}{\lambda}+\frac{1}{2}\right)}  \tag{3.76}\\
\hat{H}^{(r)}(z) \hat{H}^{(s)}(w)=\frac{\lambda \delta_{r+s+1,0 \bmod \lambda}^{z-w}}{z-\mathrm{reg} .}  \tag{3.77}\\
\langle 0| \hat{H}^{(r)}(z) \hat{H}^{(s)}(w)|0\rangle=\left(\lambda \delta_{r+s+1,0 \bmod \lambda}\right)\left(\frac{w}{z}\right)^{\frac{1}{2}\left(\frac{1}{\lambda}-1\right)+\frac{r}{\lambda}} \frac{1}{z-w} . \tag{3.78}
\end{gather*}
$$

The orbifold two-point correlators in (3.78) are examples of the more general result (3.32). The generators of the orbifold Virasoro algebra in the $\widehat{B H / N} S$ sector are

$$
\begin{align*}
& \hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)= \frac{1}{2 \lambda} \sum_{s=0}^{\lambda-1} \sum_{n}\left(m-n-\frac{r-s-\frac{1}{2}}{\lambda}\right): \hat{b}^{(s)}\left(n+\frac{s+\frac{1}{2}}{\lambda}\right) \hat{b}^{(r-s-1)}\left(m-n+\frac{r-s-1+\frac{1}{2}}{\lambda}\right): \\
&+\delta_{r, 0} \frac{1}{48}\left(\lambda-\frac{1}{\lambda}\right), \quad r=0 \ldots \lambda-1 \\
&: \hat{b}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right) \hat{b}^{(s)}\left(n+\frac{s+\frac{1}{2}}{\lambda}\right): \equiv-\theta\left(m+\frac{r+\frac{1}{2}}{\lambda}>0\right) \hat{b}^{(s)}\left(n+\frac{s+\frac{1}{2}}{\lambda}\right) \hat{b}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right)  \tag{3.79}\\
&+\theta\left(m+\frac{r+\frac{1}{2}}{\lambda}<0\right) \hat{b}^{(r)}\left(m+\frac{r+\frac{1}{2}}{\lambda}\right) \hat{b}^{(s)}\left(n+\frac{s+\frac{1}{2}}{\lambda}\right) \tag{3.80}
\end{align*}
$$

with central charge $\hat{c}=\frac{\lambda}{2}$. The $B H / N S$ vacuum $|0\rangle$ of the mother theory is now the ground state of the twisted sector, which is the twist field of the orbifold. It is a primary field under the integral Virasoro subalgebra $\hat{L}^{(0)}(m)$ with conformal weight

$$
\begin{equation*}
\hat{\Delta}=\frac{1}{48}\left(\lambda-\frac{1}{\lambda}\right) . \tag{3.81}
\end{equation*}
$$

The principal primary states created by the orbifold fermion fields $\hat{H}^{(r)}(z)$ are primary with conformal weights

$$
\begin{equation*}
\hat{\Delta}_{r}=1+\frac{\lambda}{48}-\frac{1}{\lambda}\left(r+\frac{25}{48}\right), \quad r=0 \ldots \lambda-1 \tag{3.82}
\end{equation*}
$$

under the integral Virasoro subalgebra of the orbifold.

## $\widehat{R}$ sector

To induce the $\widehat{R}$ sector of the Ising orbifolds, we begin with the $R$ sector [42] of the critical Ising model

$$
\begin{gather*}
T_{R}=-\frac{1}{2}: H \partial H:+\frac{1}{16 z^{2}}, \quad c=\frac{1}{2}  \tag{3.83}\\
H(z)=\sum_{m \in \mathbb{Z}} b(m) z^{-m-\frac{1}{2}}  \tag{3.84}\\
H\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=-H(z), \quad \theta=\frac{1}{2} \tag{3.85}
\end{gather*}
$$

In this case, the monodromy $\theta$ of the $R$ fermion in (3.85) is a constant. So long as we do not assume that $\theta$ annihilates the ground state, the operator formalism of Subsection 3.1 is still applicable. We obtain the orbifold fermion fields of the $\widehat{R}$ sector

$$
\begin{gather*}
\hat{b}^{(r)}\left(m+\frac{r}{\lambda}\right)=\sqrt{\lambda} b(\lambda m+r), r=0 \ldots \lambda-1  \tag{3.86}\\
\hat{H}^{(r)}(z)=\rho^{\frac{1}{2}}(z) \sum_{s=0}^{\lambda-1} H\left(z^{\frac{1}{\lambda}} \mathrm{e}^{\frac{2 \pi \mathrm{is}}{\lambda}}\right) \mathrm{e}^{\frac{2 \pi \mathrm{is}}{\lambda}\left(r+\frac{1}{2}\right)}=\sum_{m} \hat{b}^{(r)}\left(m+\frac{r}{\lambda}\right) z^{-\left(m+\frac{r}{\lambda}\right)-\frac{1}{2}}  \tag{3.87}\\
\hat{H}^{(r)}\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=\hat{H}^{(r)}(z) \mathrm{e}^{-2 \pi \mathrm{i}\left(\frac{r}{\lambda}+\frac{1}{2}\right)}  \tag{3.88}\\
\hat{H}^{(r)}(z) \hat{H}^{(s)}(w)=\frac{\lambda \delta_{r+s, 0 \bmod \lambda}}{z-w}+\text { reg. } \tag{3.89}
\end{gather*}
$$

The generators of the orbifold Virasoro algebra in the $\widehat{R}$ sector are

$$
\begin{align*}
\hat{L}^{(r)}\left(m+\frac{r}{\lambda}\right)= & \frac{1}{2 \lambda} \sum_{s=0}^{\lambda-1} \sum_{n}\left(m-n-\frac{r-s}{\lambda}\right): \hat{b}^{(s)}\left(n+\frac{s}{\lambda}\right) \hat{b}^{(r-s)}\left(m-n+\frac{r-s}{\lambda}\right): \\
& +\delta_{m+\frac{r}{\lambda}, 0}\left(\frac{1}{16 \lambda}+\frac{1}{48}\left(\lambda-\frac{1}{\lambda}\right)\right), \quad r=0 \ldots \lambda-1  \tag{3.90}\\
: \hat{b}^{(r)}\left(m+\frac{r}{\lambda}\right) \hat{b}^{(s)}\left(n+\frac{s}{\lambda}\right): \equiv & -\theta\left(m+\frac{r}{\lambda}>0\right) \hat{b}^{(s)}\left(n+\frac{s}{\lambda}\right) \hat{b}^{(r)}\left(m+\frac{r}{\lambda}\right) \\
& +\frac{1}{2} \delta_{m+\frac{r}{\lambda}, 0}\left[\hat{b}^{(0)}(0), \hat{b}^{(s)}\left(n+\frac{s}{\lambda}\right)\right]  \tag{3.91}\\
& +\theta\left(m+\frac{r}{\lambda}<0\right) \hat{b}^{(r)}\left(m+\frac{r}{\lambda}\right) \hat{b}^{(s)}\left(n+\frac{s}{\lambda}\right)
\end{align*}
$$

with central charge $\hat{c}=\frac{\lambda}{2}$. The ground state $|r\rangle$ of the mother Ramond sector becomes the ground state of the $\widehat{R}$ sector of the orbifold. It is a primary state with conformal weight

$$
\begin{equation*}
\hat{\Delta}=\frac{1}{24}\left(\frac{\lambda}{2}+\frac{1}{\lambda}\right) \tag{3.92}
\end{equation*}
$$

as measured by the integral Virasoro subalgebra $\hat{L}^{(0)}(m)$.

### 3.6 Orbifold $\operatorname{SL}(2, \mathbb{R})$ Ward identities

As discussed in Subsection 2.7, the orbifold Virasoro algebra has a centrally-extended SL(2,R) subalgebra (see 2.31) which is generated by

$$
\begin{equation*}
\hat{L}^{(1)}\left(\frac{1}{\lambda}\right), \hat{L}^{(0)}(0), \quad \hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right) . \tag{3.93}
\end{equation*}
$$

These generators do not form a subalgebra of the integral Virasoro subalgebra of the orbifold, and moreover, eq. (2.32) shows that the ground state $|0\rangle$ of the twisted sector is not annihilated by these generators. Nevertheless, we find the orbifold $\operatorname{SL}(2, \mathbb{R})$ Ward identities

$$
\begin{gather*}
\langle 0|\left[\hat{L}^{(1)}\left(\frac{1}{\lambda}\right), A\right]|0\rangle=\langle 0|\left[\hat{L}^{(0)}(0), A\right]|0\rangle=0 \\
\langle 0|\left[\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right), A\right]|0\rangle=0 \tag{3.94}
\end{gather*}
$$

in the twisted sector, where $A$ is any operator. These Ward identities are the images in the orbifold of the $\mathrm{SL}(2, \mathbb{R})$ Ward identities [24] of the mother theory.

As an application, we will work out the explicit form of the orbifold Ward identities (3.94) in the case of the principal primary fields. For this, we need the commutators

$$
\begin{align*}
{\left[\hat{L}^{(1)}\left(\frac{1}{\lambda}\right), \hat{\varphi}_{\Delta}^{(s)}(z)\right] } & =z^{\frac{1}{\lambda}}\left(z \partial_{z}+\Delta\left(1+\frac{1}{\lambda}\right)\right) \hat{\varphi}_{\Delta}^{(s+1)}(z) \\
{\left[\hat{L}^{(0)}(0), \hat{\varphi}_{\Delta}^{(s)}(z)\right] } & =\left(z \partial_{z}+\Delta\right) \hat{\varphi}_{\Delta}^{(s)}(z)  \tag{3.95}\\
{\left[\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right), \hat{\varphi}_{\Delta}^{(s)}(z)\right] } & =z^{-\frac{1}{\lambda}}\left(z \partial_{z}+\Delta\left(1-\frac{1}{\lambda}\right)\right) \hat{\varphi}_{\Delta}^{(s-1)}(z)
\end{align*}
$$

which follow from the more general relation (3.13). Then we find the orbifold $\mathrm{SL}(2, \mathbb{R})$ Ward identities

$$
\begin{align*}
& A_{\Delta_{1} \ldots \Delta_{n}}^{s_{1} \ldots s_{n}}(z) \equiv\langle 0| \hat{\varphi}_{\Delta_{1}}^{\left(s_{1}\right)}\left(z_{1}\right) \ldots \hat{\varphi}_{\Delta_{n}}^{\left(s_{n}\right)}\left(z_{n}\right)|0\rangle  \tag{3.96}\\
0= & \sum_{i=1}^{n} z_{i}^{\frac{1}{\lambda}}\left(z_{i} \partial_{i}+\Delta_{i}\left(1+\frac{1}{\lambda}\right)\right) A_{\Delta_{1} \ldots \Delta_{i} \ldots \Delta_{n}}^{s_{1} \ldots s_{i}+1 \ldots s_{n}}(z) \\
0= & \sum_{i=1}^{n}\left(z_{i} \partial_{i}+\Delta_{i}\right) A_{\Delta_{1} \ldots \Delta_{i} \ldots \Delta_{n}}^{s_{1} \ldots s_{i} \ldots s_{n}}(z)  \tag{3.97}\\
0= & \sum_{i=1}^{n} z_{i}^{-\frac{1}{\lambda}}\left(z_{i} \partial_{i}+\Delta_{i}\left(1-\frac{1}{\lambda}\right)\right) A_{\Delta_{1} \ldots \Delta_{i} \ldots \Delta_{n}}^{s_{1} \ldots s_{i}-1 \ldots s_{n}}(z)
\end{align*}
$$

for the matrix correlators defined in (3.96). Since $\hat{\varphi}^{(s)}$ is cyclic in $s$, these correlators are cyclic in each index $s_{i}$.

The matrix differential equations (3.97) are difficult to solve in general, although there are some simple observations about the solutions which are easily made:

- Because $\hat{L}^{(0)}(0)$ is the scale operator, the system (3.97) implies that the singularities of
correlators and OPE's in the twisted sectors of the orbifold are controlled by the conformal weights of the fields in the mother theory. We have already seen this phenomenon in many examples (c.f. equations 3.28, 3.32, 3.41, 3.42, 3.48, 3.49, 3.56, 3.77, 3.78 and 3.89).
- Orbifold correlators are not translation invariant in the twisted sector (c.f. 3.32): In the first place, the operator $\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)$, which is the image of the translation operator in the mother theory, is no longer the generator of translations (c.f. 3.95). There is, on the other hand, a translation operator $\hat{L}^{(0)}(-1)$ in the integral Virasoro subalgebra, but this operator fails to annihilate the ground state of the twisted sector.
- The solution for the one-point correlators

$$
\begin{equation*}
\langle 0| \hat{\varphi}_{\Delta}^{(r)}(z)|0\rangle=0, \quad r=0 \ldots \lambda-1 \tag{3.98}
\end{equation*}
$$

follows easily from the system (3.97), and the result (3.98) also follows directly from eq. (3.6) because $\left\langle\varphi_{\Delta}\right\rangle=0$ in the mother theory. The orbifold two-point correlators in (3.32), (3.49) and (3.78) are also solutions of the system (3.97), but these are only particular solutions corresponding to trivial monodromy in the mother theory. Beyond this, the solutions of the system (3.97) must be quite complex: One expects the analogues of the usual ambiguities for $n \geq 4$, and, moreover, the general solutions for $n \geq 2$ must also allow for the structure induced in the orbifold by the monodromy algebra of the mother theory (c.f. eq. 3.30).

### 3.7 Orbifold null-state differential equations

Under the orbifold induction procedure, any null state of the mother theory maps to a null state in the twisted sector of the cyclic orbifold, and, as a consequence, every BPZ null-state differential equation has its image in the corresponding orbifold.

As an example, we will focus on the cyclic orbifolds formed from tensor products of the Virasoro minimal models [24], and in particular on the images of the simplest null states of the mother theories which now read in the orbifold

$$
\begin{gather*}
0=\left(\lambda\left(\hat{L}^{(\lambda-1)}\left(-1+\frac{\lambda-1}{\lambda}\right)\right)^{2}-\frac{2}{3}(1+2 \Delta) \hat{L}^{(\lambda-2)}\left(-1+\frac{\lambda-2}{\lambda}\right)\right)\left|\hat{\Delta}_{\lambda-1}\right\rangle  \tag{3.99}\\
\hat{\Delta}_{\lambda-1}=\frac{\Delta}{\lambda}+\frac{\hat{c}}{24}\left(1-\frac{1}{\lambda^{2}}\right), \quad \Delta=\left\{\begin{array}{l}
\frac{m}{4(m+3)} \\
\text { or } \\
\frac{m+5}{4(m+2)}
\end{array}\right.  \tag{3.100}\\
\hat{c}=\lambda\left(1-\frac{6}{(m+2)(m+3)}\right), m \geq 0 . \tag{3.101}
\end{gather*}
$$

Following the usual procedure, one then obtains the following null-state matrix differential equations

$$
\begin{equation*}
B_{\Delta}^{r_{1} \ldots r_{n-1}}(z) \equiv\langle 0| \hat{\varphi}_{\Delta_{1}}^{\left(r_{1}\right)}\left(z_{1}\right) \ldots \hat{\varphi}_{\Delta_{n-1}}^{\left(r_{n-1}\right)}\left(z_{n-1}\right)\left|\hat{\Delta}_{\lambda-1}\right\rangle \tag{3.102}
\end{equation*}
$$

$$
\begin{align*}
0= & \sum_{i=1}^{n-1}\left\{\lambda\left(z_{i}^{1-\frac{1}{\lambda}} \partial_{i}+\frac{\Delta_{i}\left(1-\frac{1}{\lambda}\right)}{z_{i}^{\frac{1}{\lambda}}}\right)^{2}+\frac{2}{3}(1+2 \Delta)\left(z_{i}^{1-\frac{2}{\lambda}} \partial_{i}+\frac{\Delta_{i}\left(2-\frac{1}{\lambda}\right)}{z_{i}^{\frac{2}{\lambda}}}\right)\right\} B_{\Delta}^{r_{1} \ldots r_{i}-2 \ldots r_{n-1}}(z) \\
& +2 \lambda \sum_{i<j}\left(z_{i}^{1-\frac{1}{\lambda}} \partial_{i}+\frac{\Delta_{i}\left(1-\frac{1}{\lambda}\right)}{z_{i}^{\frac{1}{\lambda}}}\right)\left(z_{j}^{1-\frac{1}{\lambda}} \partial_{j}+\frac{\Delta_{j}\left(1-\frac{1}{\lambda}\right)}{z_{j}^{\frac{1}{\lambda}}}\right) B_{\Delta}^{r_{1} \ldots r_{i}-1 \ldots r_{j}-1 \ldots r_{n-1}}(z) \tag{3.103}
\end{align*}
$$

for the correlators defined in (3.102). Here $\hat{\varphi}_{\Delta_{i}}^{\left(r_{i}\right)}$ are the principal primary fields corresponding to the Virasoro primary fields $\varphi_{\Delta_{i}}$ of conformal weight $\Delta_{i}$ in the minimal model. Because there is no translation invariance in the twisted sector of the orbifold, we have been unable to express this system in a more familiar form.

### 3.8 The conventional view of orbifolds

In this subsection we want to explain in further detail why the twisted sectors discussed above are in fact the twisted sectors of cyclic orbifolds. The discussion will also shed further light on the higher-twist components of the orbifold algebras of Section 2. To this end, we first review the more conventional viewpoint of orbifolds.

## General orbifolds

At this point it is helpful to recall a few standard facts about the algebraic description of orbifolds (see e.g. Ref. [10]): Given a conformal field theory with a chiral algebra $\mathcal{A}$ and a symmetry of $\mathcal{A}$, the idea of the orbifold construction is to consider the subalgebra $\mathcal{A}^{(0)}$ of $\mathcal{A}$ that is fixed under the symmetry. The subalgebra $\mathcal{A}^{(0)}$ is the chiral algebra of a new conformal field theory, called the orbifold theory. It is a well-known result [43] in conformal field theory that, in order to obtain a modular invariant partition function, we should find all irreducible representations of $\mathcal{A}^{(0)}$ and retain each inequivalent representation once.

Some representations of $\mathcal{A}^{(0)}$ are easy to obtain: Any irreducible representation of the original chiral algebra $\mathcal{A}$ can be decomposed into irreducible representations of $\mathcal{A}^{(0)}$. The states in these $\mathcal{A}^{(0)}$-modules form what is called the untwisted sector of the orbifold theory. An important feature of the untwisted sector is that the representation theory of the original chiral algebra $\mathcal{A}$ provides us with a good handle on this sector. In the process of this decomposition, two effects can occur, and we will encounter both of them:

- Irreducible modules of $\mathcal{A}$ can be reducible under $\mathcal{A}^{(0)}$.
- Irreducible modules which are inequivalent as modules of $\mathcal{A}$ can be isomorphic as modules of the subalgebra $\mathcal{A}^{(0)}$.

Generically, however, there are modules of $\mathcal{A}^{(0)}$ which are not contained in any $\mathcal{A}$ module. The states in these modules form what is called the twisted sector of the orbifold. This sector usually causes problems in the description of orbifolds, since the representation of $\mathcal{A}$ does not provide a handle on it. However, as we will emphasize shortly, in the case of cyclic permutation orbifolds, the orbifold induction procedure of Sections 2 and 3 has given us in fact an explicit construction of the states in the twisted sector.

To further understand the twisted sector, we briefly review a few more facts about orbifolds in their geometric description. For simplicity, we will assume that the symmetry group $G$ of $\mathcal{A}$ by which we mod out is abelian. Let us further imagine that the theory in question can be constructed from string coordinates $X(\tau, \sigma)$, on which $G$ acts non-trivially. The worldsheet is parametrized by $\tau$ and $\sigma$, where $\sigma+2 \pi \equiv \sigma$. If one mods out by $G$, one must also introduce twisted sectors $[6,7]$, in which the string closes only up to the action of $G$,

$$
\begin{equation*}
X(\tau, \sigma+2 \pi)=g X(\tau, \sigma) \tag{3.104}
\end{equation*}
$$

where $g$ is an element of $G$. This relation tells us that the string coordinate in the twisted sectors is a multi-valued function on the world-sheet. This is the origin of root coverings of the worldsheet in the calculation of orbifold $N$-point functions in examples [9] and it is also the origin of the root-covering algebras in Section 2.

To translate this prescription into a more algebraic form, we introduce the complex variable $z=\exp (2 \pi \mathrm{i}(\sigma+\mathrm{i} \tau))$ which maps the worldsheet cylinder to the complex plane, where time flows radially. In these coordinates, the requirement (3.104) translates into a monodromy property

$$
\begin{equation*}
X\left(z \mathrm{e}^{2 \pi \mathrm{i}}\right)=g X(z) \tag{3.105}
\end{equation*}
$$

of the string coordinate. After quantization, the string coordinate becomes an operator and the modes of $\partial X$ contain both twisted and untwisted Cartan currents; The untwisted Cartan currents form the representation of the orbifold chiral algebra in the twisted sector, while the twisted modes are the twisted sector representatives of the other generators of the original chiral algebra $\mathcal{A}$, which now play the role of intertwiners in the twisted sector.

Many of the principles discussed in this subsection will be seen again in our discussion of orbifolds on the torus (cf. Section 4).

## Cyclic orbifolds

We now specialize this discussion to the case of cyclic permutation orbifolds. These orbifolds are constructed by modding out the symmetry $\mathbb{Z}_{\lambda}$ which acts by cyclic permutations on the tensor product of $\lambda$ copies of the mother theory, so that they have Virasoro central charge $\hat{c}=\lambda c$, where $c$ is the central charge of the mother theory.

As is true in general orbifolds, the orbifold chiral algebra $\mathcal{A}^{(0)}$ of the cyclic orbifold is a subalgebra of the original chiral algebra of the tensor product theory. The subalgebra $\mathcal{A}^{(0)}$ is represented in the twisted sector by the $r=0$ component of the orbifold affine and Virasoro algebras of Section 2. The rest of the original chiral algebra still operates (as an abstract algebra) in the twisted sectors of the orbifold, but it is represented in the twisted sectors by operators with non-trivial monodromy. These operators are the higher-twist components $r=1 \ldots \lambda-1$ of the orbifold affine and Virasoro algebras of Section 2.

Let us illustrate this in the simplest case, when $\lambda=2$ and the conformal field theory is a tensor product of two affine-Sugawara constructions at level $k$. The symmetry by which we
want to mod out is the permutation of the two factors in the tensor product. In the untwisted sector, we have a set of operators in $\mathcal{A}$ that are even under the $\mathbb{Z}_{2}$ permutation

$$
\begin{equation*}
J_{a}(z) \otimes 1+\mathbf{1} \otimes J_{a}(z) \tag{3.106}
\end{equation*}
$$

which span an affine Lie algebra at level $2 k$. Since the symmetry operates trivially on operators of the form (3.106), then, according to (3.105), the abstract algebra generated by these operators should be represented in the twisted sector of the orbifold by integer moded operators,

$$
\begin{equation*}
J_{a}(z) \otimes \mathbf{1}+\mathbf{1} \otimes J_{a}(z) \quad \rightarrow \quad \hat{J}_{a}^{(0)}(z) \tag{3.107}
\end{equation*}
$$

where $\hat{J}_{a}^{(0)}(z)$ generate the integral affine subalgebra of the $\lambda=2$ orbifold affine algebra (2.2). The odd part of $\mathcal{A}$ in the untwisted sector is

$$
\begin{equation*}
J_{a}(z) \otimes \mathbf{1}-\mathbf{1} \otimes J_{a}(z) \tag{3.108}
\end{equation*}
$$

and it transforms in the adjoint representation with respect to the even part $\mathcal{A}^{(0)}$. According to (3.105), the odd part of $\mathcal{A}$ must be represented in the twisted sector by half-integer moded operators

$$
\begin{equation*}
J_{a}(z) \otimes \mathbf{1}-\mathbf{1} \otimes J_{a}(z) \quad \rightarrow \quad \hat{J}_{a}^{(1)}(z) \tag{3.109}
\end{equation*}
$$

where $\hat{J}_{a}^{(1)}(z)$ is the set of $r=1$ generators of the $\lambda=2$ orbifold affine algebra (2.2).
In summary, the orbifold algebras are simply the representations in the twisted sectors of the chiral algebra of the mother theory; moreover, the $r=0$ subalgebra of the orbifold algebra is the representative in the twisted sector of the chiral algebra of the orbifold, while the higher-twist components of the orbifold algebra have become intertwiners in the twisted sector.

## 4 Characters and modular transformations

So far our discussion has focussed on the orbifold conformal field theory for a worldsheet of genus zero. We will now discuss some aspects of genus one, that is, the characters and their modular transformation properties. In what follows we will, for the sake of simplicity, first discuss the case of $\lambda=2$.

### 4.1 The untwisted sector for $\lambda=2$

We begin with the untwisted sector. According to the discussion in the previous section, the states of the untwisted sector of the cyclic orbifold are obtained from states in the tensor product theory, which consists of $\lambda$ copies of the mother conformal field theory $\mathcal{C}$. We assume here that the mother theory is a rational conformal field theory with Virasoro central charge $c$. To each primary state with conformal weight $\Delta_{i}$ in this theory we associate the irreducible module $\mathcal{H}_{i}$ spanned by the primary field and its descendants with respect to the chiral algebra of $\mathcal{C}$. We will use characters to describe these modules:

$$
\begin{equation*}
\chi_{i}(\tau, z) \equiv \operatorname{Tr}_{\mathcal{H}_{i}} \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-c / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)} \tag{4.1}
\end{equation*}
$$

In case the theory contains commuting spin one currents $J_{a}(z)$, we have also included in the definition of the character a set of Cartan angles, collected in the vector $z$. The primary fields of the tensor product theory are labeled by $\lambda$ indices, each of which labels a primary field of the mother theory. The corresponding module is the tensor product of the modules of the mother theory and, correspondingly, the characters of the tensor product theory

$$
\begin{equation*}
\chi_{i j}(\tau, z)=\chi_{i}(\tau, z) \chi_{j}(\tau, z) \tag{4.2}
\end{equation*}
$$

are just the products of the characters of the mother theory.

## Off-diagonal fields

Let us now describe the characters of the primary fields in the untwisted sector of the orbifold theory. If $i$ and $j$ are different, the direct $\operatorname{sum}\left(\mathcal{H}_{i} \otimes \mathcal{H}_{j}\right) \oplus\left(\mathcal{H}_{j} \otimes \mathcal{H}_{i}\right)$ carries a representation of the $\mathbb{Z}_{2}$ permutation symmetry, which we use to decompose the direct sum into two submodules of the orbifold chiral algebra: One of these submodules contains the states which are symmetric under the permutation and the other contains the antisymmetric states. The symmetric and antisymmetric states are of the form

$$
\begin{equation*}
|x\rangle \otimes|y\rangle+|y\rangle \otimes|x\rangle, \quad|x\rangle \otimes|y\rangle-|y\rangle \otimes|x\rangle \tag{4.3}
\end{equation*}
$$

respectively, where $|x\rangle \in \mathcal{H}_{i}$ and $|y\rangle \in \mathcal{H}_{j}$. However, the two modules are isomorphic: Symmetric and antisymmetric states come always in pairs, and the mapping

$$
\begin{equation*}
|x\rangle \otimes|y\rangle+|y\rangle \otimes|x\rangle \rightarrow|x\rangle \otimes|y\rangle-|y\rangle \otimes|x\rangle \tag{4.4}
\end{equation*}
$$

is an intertwiner of the representations. In the orbifold theory we should mod out by this permutation and we should keep only one of these two modules. Hence we retain only a single primary field called $(i j)$ in the untwisted sector of the orbifold theory with character

$$
\begin{equation*}
\mathcal{X}_{(i j)}(\tau, z)=\chi_{i}(\tau, z) \chi_{j}(\tau, z), \quad i<j . \tag{4.5}
\end{equation*}
$$

Characters of the orbifold theory will be denoted by the symbol $\mathcal{X}$.

## Diagonal fields

The diagonal fields for which $i=j$ also split into two representations, symmetric and antisymmetric under the orbifold chiral algebra. We will present strong evidence ${ }^{2}$ that these representations are indeed irreducible representations of the orbifold chiral algebra: We will compute the characters of the corresponding modules and show that they span a module of the

[^1]modular group $S L(2, \mathbb{Z})$. We denote the diagonal primary fields by $(i, \psi)$, where $\psi$ takes the values 0 for the symmetric and 1 for the antisymmetric representation.

To compute the characters of these two primary fields, we remark that the permutation $\pi$ gives rise to an involution $T_{\pi}$ on $\mathcal{H}_{i} \otimes \mathcal{H}_{i}$ which maps the state $|x\rangle \otimes|y\rangle$ to the state $|y\rangle \otimes|x\rangle$. We introduce the character-valued index

$$
\begin{equation*}
\mathcal{X}_{i}^{\pi}(\tau, z) \equiv \operatorname{Tr}_{\mathcal{H}_{i} \otimes \mathcal{H}_{i}} T_{\pi} \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-\hat{c} / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)} \tag{4.6}
\end{equation*}
$$

where $L(0)$ is the zero mode of the Virasoro algebra of the tensor product theory and $\hat{c}=\lambda c$ is the central charge of the tensor product theory. The index $\mathcal{X}_{i}^{\pi}$ is a close relative of the twining characters introduced in [45]: It encodes the action of an outer automorphism of the chiral algebra on the space of states, in this case the action of the permutation symmetry. In the case at hand the index is easy to compute: Only states of the form $|x\rangle \otimes|x\rangle$ contribute to $\mathcal{X}_{i}^{\pi}$, and evaluation of their contribution gives

$$
\begin{equation*}
\mathcal{X}_{i}^{\pi}(\tau, z)=\chi_{i}(2 \tau, 2 z) \tag{4.7}
\end{equation*}
$$

since the eigenvalues of $L(0)$ and $J(0)$ are additive.
Because $\frac{1}{2}\left(1 \pm T_{\pi}\right)$ is the projection operator on symmetric and antisymmetric states respectively in $\mathcal{H}_{i} \otimes \mathcal{H}_{i}$, we may then evaluate the characters of the symmetric and antisymmetric parts separately:

$$
\begin{align*}
\mathcal{X}_{(i, \psi)}(\tau, z) & =\operatorname{Tr}_{\mathcal{H}_{i} \otimes \mathcal{H}_{i}} \frac{1}{2}\left(\mathbf{1}+\mathrm{e}^{2 \pi \mathrm{i} \psi / 2} T_{\pi}\right) \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-\hat{c} / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)}=\frac{1}{2} \chi_{i}(\tau, z)^{2}+\mathrm{e}^{\pi \mathrm{i} \psi} \frac{1}{2} \mathcal{X}_{i}^{\pi}(\tau, z) \\
& =\frac{1}{2} \chi_{i}(\tau, z)^{2}+\mathrm{e}^{\pi \mathrm{i} \psi} \frac{1}{2} \chi_{i}(2 \tau, 2 z) . \tag{4.8}
\end{align*}
$$

It is easy to see that this field content reproduces the partition function of the untwisted sector derived in [11]:

$$
\begin{align*}
Z_{\text {untwisted }}(\tau, z) & \equiv \sum_{i<j}\left|\mathcal{X}_{(i j)}(\tau, z)\right|^{2}+\sum_{i}\left|\mathcal{X}_{(i, 0)}(\tau, z)\right|^{2}+\left|\mathcal{X}_{(i, 1)}(\tau, z)\right|^{2} \\
& =\frac{1}{2} \sum_{i, j}\left|\chi_{i}(\tau, z)\right|^{2}\left|\chi_{j}(\tau, z)\right|^{2}+\frac{1}{2} \sum_{i}\left|\chi_{i}(2 \tau, 2 z)\right|^{2} \tag{4.9}
\end{align*}
$$

where we have used equations (4.5) and (4.8) to obtain the final form in (4.9).
For later use we also read off the modular matrix $T$ describing the modular transformation $\tau \rightarrow \tau+1$ in the untwisted sector: It is a diagonal matrix with entries

$$
\begin{equation*}
T_{(i j)}=T_{i} T_{j}, \quad T_{(i, \psi)}=T_{i}^{2} . \tag{4.10}
\end{equation*}
$$

Here $T_{i}=\exp \left(2 \pi \mathrm{i}\left(\Delta_{i}-\frac{c}{24}\right)\right)$ are the diagonal elements of the modular matrix $T_{i j}=\delta_{i j} T_{i}$ of the mother conformal field theory.

### 4.2 The untwisted sector for general $\lambda$

The untwisted sector of cyclic permutation orbifolds for general $\lambda$ can be described quite similarly, except that the notation becomes somewhat more complicated. We describe fields by
multiindices $\vec{i}=\left(i_{1}, i_{2}, \ldots, i_{\lambda}\right)$, which generalizes the notation $(i j)$ in the previous subsection. Correspondingly, we introduce the following notation for the modules of the tensor product theory:

$$
\begin{equation*}
\mathcal{H}_{\vec{i}} \equiv \mathcal{H}_{i_{1}} \otimes \mathcal{H}_{i_{2}} \otimes \ldots \otimes \mathcal{H}_{i_{\lambda}} . \tag{4.11}
\end{equation*}
$$

The action of the group of cyclic permutations of indices organizes the multiindices into orbits; to each orbit $\vec{i}$ we associate its stabilizer $\mathcal{S}_{\vec{i}}$, the subgroup of elements of $\mathbb{Z}_{\lambda}$ that leaves $\vec{i}$ fixed. Note that the stabilizer subgroup does not depend on the choice of representative of the orbit, since $\mathbb{Z}_{\lambda}$ is abelian.

Suppose now that $\pi \in \mathcal{S}_{\vec{i}}$. Define $b(\pi)$ to be the order of $\pi$ and $a(\pi)=\lambda / b(\pi)$. As we saw for the diagonal fields in the case $\lambda=2$, we can implement the action of $\pi$ on the corresponding module and introduce analogous character-valued indices. By the same reasoning, we obtain

$$
\begin{align*}
\mathcal{X}_{\vec{i}}^{\pi}(\tau, z) & \equiv \operatorname{Tr}_{\mathcal{H}_{\vec{i}}} T_{\pi} \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-\hat{c} / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)}  \tag{4.12}\\
& =\prod_{l=1}^{a(\pi)} \chi_{i_{l}}(b(\pi) \tau, b(\pi) z)
\end{align*}
$$

Again, we have to decompose the irreducible module of the tensor product theory into submodules of the orbifold chiral algebra. These submodules are obtained by decomposing $\mathcal{H}_{\vec{i}}$ into subspaces on which all $T_{\pi}$ act diagonally, when $\pi$ is in the stabilizer. The decomposition

$$
\begin{equation*}
\mathcal{H}_{\vec{i}}=\oplus_{\psi} \mathcal{H}_{(\vec{i}, \psi)} \tag{4.13}
\end{equation*}
$$

should be such that on each subspace $\mathcal{H}_{(\vec{i}, \psi)}$ any $\rho \in \mathcal{S}_{\vec{i}}$ acts as a phase $\mathrm{e}^{2 \pi \mathrm{i} \psi m(\rho) / \lambda} \mathbf{1}$. Here we identify $\rho \in \mathcal{S}_{\vec{i}}$ with an integer $m(\rho) \in\{0, \ldots, \lambda-1\}$, and $\psi$ is an integer ranging from 0 to $\left|\mathcal{S}_{\vec{i}}\right|-1$, where $\left|\mathcal{S}_{\vec{i}}\right|$ is the number of elements in the stabilizer $\mathcal{S}_{\vec{i}}$. In other words, the true characters are labeled by a multiindex $\vec{i}$ and an element $\psi$ of the character group of the stabilizer.

To compute the characters of the spaces $\mathcal{H}_{(\vec{i}, \psi)}$ separately, we remark that

$$
\begin{equation*}
\mathcal{X}_{\vec{i}}^{\rho}(\tau, z)=\sum_{\psi=0}^{\left|\mathcal{S}_{\hat{i}}\right|-1} \operatorname{Tr}_{\mathcal{H}_{(\vec{i}, \psi)}} T_{\rho} \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-\hat{c} / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)}=\sum_{\psi=0}^{\left|\mathcal{S}_{\hat{i}}\right|-1} \mathrm{e}^{2 \pi \mathrm{i} \psi m(\rho) / \lambda} \operatorname{Tr}_{\mathcal{H}_{(\vec{i}, \psi)}} \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-\hat{c} / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)} \tag{4.14}
\end{equation*}
$$

Inverting this relation, we find

$$
\begin{equation*}
\mathcal{X}_{(\vec{i}, \psi)}(\tau, z)=\operatorname{Tr}_{\mathcal{H}_{(\vec{i}, \psi)}} \mathrm{e}^{2 \pi \mathrm{i} \tau(L(0)-\hat{c} / 24)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)}=\frac{1}{\left|\mathcal{S}_{\vec{i}}\right|} \sum_{\rho \in S_{\vec{i}}} \mathrm{e}^{-2 \pi \mathrm{i} \psi m(\rho) / \lambda} \mathcal{X}_{\vec{i}}^{\rho}(\tau, z) \tag{4.15}
\end{equation*}
$$

for the characters of the primary fields in the untwisted sector.

## Prime $\lambda$

Let us briefly discuss the special case when $\lambda$ is prime. In this case the stabilizer is either
trivial or $\mathbb{Z}_{\lambda}$. In the first case, we have $\mathcal{S}_{\vec{i}}=\{e\}$ and the character is simply a product of the characters of the mother theory:

$$
\begin{equation*}
\mathcal{X}_{(\vec{i}, 0)}(\tau, z)=\mathcal{X}_{\vec{i}}^{e}(\tau, z)=\prod_{l=1}^{\lambda} \chi_{i_{l}}(\tau, z) \tag{4.16}
\end{equation*}
$$

In case the stabilizer is non-trivial (which corresponds to the diagonal fields in the previous subsection), we find that the multiindex has to be of the form $\vec{i}=(i, i, \ldots, i)$. Moreover, we have

$$
\begin{equation*}
\mathcal{X}_{\vec{i}}^{\pi}(\tau, z)=\chi_{i}(\lambda \tau, \lambda z) \tag{4.17}
\end{equation*}
$$

when $\pi \neq e$.
The general formula (4.15) gives in the case $\psi=0$

$$
\begin{equation*}
\mathcal{X}_{(\vec{i}, 0)}(\tau, z)=\frac{1}{\lambda}\left(\chi_{i}(\tau, z)\right)^{\lambda}+\frac{\lambda-1}{\lambda} \chi_{i}(\lambda \tau, \lambda z) \tag{4.18}
\end{equation*}
$$

while for $\psi \neq 0$ we get

$$
\begin{equation*}
\mathcal{X}_{(\vec{i}, \psi)}(\tau, z)=\frac{1}{\lambda}\left(\chi_{i}(\tau, z)\right)^{\lambda}-\frac{1}{\lambda} \chi_{i}(\lambda \tau, \lambda z) . \tag{4.19}
\end{equation*}
$$

Note that this result is independent of $\psi$ so that $\lambda-1$ characters of the orbifold theory coincide. As in equation (4.9), one can compute the partition function of the untwisted sector and find full agreement with [11].

### 4.3 The twisted sector for $\lambda=2$

Let us now turn to the twisted sectors. We claim that the characters of the primary fields in the twisted sector can be obtained by considering the action of the orbifold chiral algebra on the modules of the original theory. As a first step let us calculate (for arbitrary $\lambda$ ):

$$
\begin{equation*}
\chi_{k}\left(\frac{\tau+n}{\lambda}, z\right)=\operatorname{Tr}_{\mathcal{H}_{k}} \mathrm{e}^{2 \pi \mathrm{i} \frac{\tau+n}{\lambda}\left(L(0)-\frac{c}{24}\right)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot J(0)}=\operatorname{Tr}_{\mathcal{H}_{k}} \mathrm{e}^{2 \pi \mathrm{i} \tau\left(\hat{L}^{(0)}(0)-\frac{\hat{c}}{24}\right)} \mathrm{e}^{2 \pi \mathrm{i} \cdot \cdot \hat{J}(0)} \mathrm{e}^{2 \pi \mathrm{in}\left(\hat{L}^{(0)}(0)-\frac{\hat{c}}{24}\right)} . \tag{4.20}
\end{equation*}
$$

Here $\hat{L}^{(0)}(0)$ is the generator of the orbifold Virasoro algebra introduced in (2.7) and $\hat{c}=\lambda c$ is the Virasoro central charge of the orbifold theory. The right side of equation (4.20) shows that, up to an extra $n$-dependent insertion, this is just the character of the module of the appropriate orbifold algebra.

The conformal weights of descendants of the same primary field differ only by an integer, and we should therefore decompose the vector space $\mathcal{H}_{k}$ into subspaces consisting of states that differ in conformal weights (with respect to $\left.\hat{L}^{(0)}(0)\right)$ by integers. We denote these subspaces by $\mathcal{H}_{\widehat{(k, \psi)}}$ and their characters by $\mathcal{X}_{\widehat{(k, \psi)}}$. The fractional part of the $\hat{L}^{(0)}(0)$-conformal weight of states in $\mathcal{H}_{(\widehat{k, \psi)}}$ is given by

$$
\begin{equation*}
\Delta_{\widehat{(i, \psi)}}=\frac{1}{\lambda} \Delta_{i}+\frac{c}{24}\left(\lambda-\frac{1}{\lambda}\right)+\frac{\psi}{\lambda} \tag{4.21}
\end{equation*}
$$

with $\psi=0 \ldots \lambda-1$. Up to an integer contribution, this form is equivalent to the set of conformal weights $\hat{\Delta}_{r}$ in (3.3) of the principal primary fields. As a consequence, the elements of the modular matrix $T$ for the primary fields in the twisted sector read

$$
\begin{equation*}
T_{(\widehat{i, \psi)}}=\mathrm{e}^{2 \pi \mathrm{i}\left(\Delta_{\overparen{(i, \psi)}}-\hat{c} / 24\right)} \tag{4.22}
\end{equation*}
$$

where $\Delta_{\overparen{(i, \psi)}}$ are the conformal weights in (4.21). Equation (4.20) shows us that

$$
\begin{equation*}
\chi_{k}\left(\frac{\tau+n}{\lambda}, z\right)=T_{k}^{n / \lambda} \sum_{\psi=0}^{\lambda-1} \mathcal{X}_{\widehat{(k, \psi)}}(\tau, z) \mathrm{e}^{2 \pi \mathrm{in} \psi / \lambda} \tag{4.23}
\end{equation*}
$$

where the quantity $T_{k}^{n / \lambda}$ in (4.23) is a particular root of the modular matrix $T_{k}$ of the mother theory

$$
\begin{equation*}
T_{k}^{n / \lambda}=\exp \left(2 \pi \mathrm{i} \frac{n}{\lambda}\left(\Delta_{k}-\frac{c}{24}\right)\right) . \tag{4.24}
\end{equation*}
$$

Inverting relation (4.23), we find the characters of the primary fields of the twisted sector

$$
\begin{equation*}
\mathcal{X}_{(\widehat{k}, \psi)}(\tau, z)=\operatorname{Tr}_{\mathcal{H}}{ }_{(k, \psi)} \mathrm{e}^{2 \pi \mathrm{i} \tau\left(\hat{L}^{(0)}(0)-\hat{c} / 24\right)} \mathrm{e}^{2 \pi \mathrm{i} z \cdot \hat{J}(0)}=\frac{1}{\lambda} \sum_{n=0}^{\lambda-1} \mathrm{e}^{-2 \pi \mathrm{i} n \psi / \lambda} T_{k}^{-n / \lambda} \chi_{k}\left(\frac{\tau+n}{\lambda}, z\right) \tag{4.25}
\end{equation*}
$$

where $k$ labels a primary field of the mother theory $\mathcal{C}$ and $\psi=0 \ldots \lambda-1$. The relation (4.25) is the analogue of the monodromy sums which were introduced earlier to decompose operators on the sphere, so the characters of the twisted sector are obtained by essentially the same orbifold induction procedure presented in Sections 2 and 3.

As a check on this result, we may compute a partition function for the twisted sectors of the orbifold:

$$
\begin{align*}
Z_{\text {twisted }}^{\prime}(\tau, z) & =\sum_{k} \sum_{\psi=0}^{\lambda-1}\left|\mathcal{X}_{\widehat{(k, \psi)}}(\tau, z)\right|^{2} \\
& =\sum_{k} \frac{1}{\lambda^{2}} \sum_{\psi} \sum_{n, n^{\prime}=0}^{\lambda-1} \chi_{k}\left(\frac{\tau+n}{\lambda}, z\right) \chi_{k}\left(\frac{\tau+n^{\prime}}{\lambda}, z\right)^{*} T_{k}^{-n / \lambda} T_{k}^{n^{\prime} / \lambda} \mathrm{e}^{-2 \pi \mathrm{in} \psi / \lambda} \mathrm{e}^{2 \pi \mathrm{in} n^{\prime} \psi / \lambda} \\
& =\sum_{k} \frac{1}{\lambda} \sum_{n}\left|\chi_{k}\left(\frac{\tau+n}{\lambda}, z\right)\right|^{2} \tag{4.26}
\end{align*}
$$

The sum over $k$ in $Z_{\text {twisted }}^{\prime}$ is a sum over all primary fields in the mother theory $\mathcal{C}$. Comparing with the known [11] partition function $Z_{\text {twisted }}$ of the twisted sector of the orbifold, we find that $Z_{\text {twisted }}=(\lambda-1) Z_{\text {twisted }}^{\prime}$. This shows that for $\lambda=2$ the result $Z_{\text {twisted }}^{\prime}$ in (4.26) is the correct partition function, but for higher prime $\lambda$ one must in fact include $\lambda-1$ copies of the same module of the orbifold algebra. This parallels the situation found in the untwisted sector. Before we explain this in more detail, we want to continue with the case $\lambda=2$, where this additional complication does not arise.

### 4.4 Modular transformations for $\lambda=2$

Having computed the characters for the twisted and untwisted sectors at $\lambda=2$, we may now study the modular matrices $S^{\text {orb }}$ and $T^{\text {orb }}$ of the orbifold. We have already determined the
modular matrix $T^{\text {orb }}$ which describes the behavior of the characters under $\tau \rightarrow \tau+1$. We now compute the modular matrix $S^{\text {orb }}$ which describes the transformation $\tau \rightarrow-\frac{1}{\tau}$. Via the Verlinde formula

$$
\begin{equation*}
\mathcal{N}_{i j k}^{\text {orb }}=\sum_{n} \frac{S_{i n}^{\text {orb }} S_{j n}^{\text {orb }} S_{k n}^{\text {orb }}}{S_{0 n}^{\text {orb }}} \tag{4.27}
\end{equation*}
$$

the modular matrices $S^{\text {orb }}$ will also give the fusion rules of the orbifold theory. We denote by 0 the vacuum $(k, \psi)=(0,0)$ of the orbifold theory; this is not the ground state $\widehat{(k, \psi)}=\widehat{(0,0)}$ of the twisted sector of the orbifold, which was called $|0\rangle$ in Subsection 2.7.

To compute the $S$-matrix elements, we begin with the characters (4.5) of the off-diagonal fields in the untwisted sector. One finds that

$$
\begin{equation*}
\mathcal{X}_{(i j)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right)=\sum_{p<q}\left(S_{i p} S_{j q}+S_{i q} S_{j p}\right) \mathcal{X}_{(p q)}(\tau, z)+\sum_{p} S_{i p} S_{j p} \sum_{\psi=0}^{1} \mathcal{X}_{(p, \psi)}(\tau, z), \tag{4.28}
\end{equation*}
$$

which implies the following orbifold $S$-matrix elements,

$$
\begin{align*}
S_{(i j),(p q)} & =S_{i p} S_{j q}+S_{i q} S_{j p} \\
S_{(i j),(p, \psi)} & =S_{i p} S_{j p}  \tag{4.29}\\
S_{(i j),(\widehat{p, \chi)}} & =0 .
\end{align*}
$$

Similarly, for the characters (4.8) of the diagonal fields, we find that

$$
\begin{align*}
\mathcal{X}_{(i, \psi)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) & =\frac{1}{2} \sum_{p<q} S_{i p} S_{j q} \chi_{p}(\tau, z) \chi_{q}(\tau, z)+\frac{1}{2} \mathrm{e}^{2 \pi \mathrm{i} \psi / 2} \sum_{p} S_{i p} \chi_{p}\left(\frac{\tau}{2}, z\right) \\
& =\sum_{p<q} S_{i p} S_{j q} \mathcal{X}_{(p q)}(\tau, z)+\sum_{p} \frac{1}{2} S_{i p}^{2} \sum_{\psi=0}^{1} \mathcal{X}_{(p, \psi)}(\tau, z)  \tag{4.30}\\
& +\sum_{p} \frac{1}{2} \mathrm{e}^{\pi \mathrm{i} \psi} S_{i p} \sum_{\psi=0}^{1} \mathcal{X}_{(p, \psi)}(\tau, z) .
\end{align*}
$$

Note that primary fields in the twisted sector have now appeared on the right. This can be traced back to the fact that we had to split each diagonal field in the untwisted sector into two fields with different characters. Equation (4.30) implies the following orbifold $S$-matrix elements:

$$
\begin{align*}
S_{(i, \psi),(p q)} & =S_{i p} S_{i q} \\
S_{(i, \psi),(j, \chi)} & =\frac{1}{2} S_{i j}^{2}  \tag{4.31}\\
S_{(i, \psi), \widehat{(p, x)}} & =\frac{1}{2} \mathrm{e}^{2 \pi \mathrm{i} \psi / 2} S_{i p} .
\end{align*}
$$

To compute the modular transformation properties of the twisted sector, we remark first
that

$$
\begin{align*}
\chi_{i}\left(\frac{-1+\tau}{2 \tau}, \frac{z}{\tau}\right) & =T_{i} \chi_{i}\left(\frac{-1-\tau}{2 \tau}, \frac{z}{\tau}\right) \\
& =\sum_{j} T_{i} S_{i j} \chi_{j}\left(\frac{2 \tau}{\tau+1}, \frac{2 z}{\tau+1}\right) \\
& =\sum_{j} T_{i} S_{i j} T_{j}^{2} \chi_{j}\left(\frac{-2}{\tau+1}, \frac{2 z}{\tau+1}\right)  \tag{4.32}\\
& =\sum_{j l} T_{i} S_{i j} T_{j}^{2} S_{j l} \chi_{l}\left(\frac{\tau+1}{2}, z\right) \\
& =\sum_{l}\left(T S T^{2} S\right)_{i l} \chi_{l}\left(\frac{\tau+1}{2}, z\right) .
\end{align*}
$$

Using this equation, we find the modular transformation of the characters (4.25) of the primary fields in the twisted sector

$$
\begin{align*}
\mathcal{X}_{\widehat{(k, \psi)}}\left(-\frac{1}{\tau}\right) & =\sum_{p} \frac{1}{2} S_{k p} \chi_{p}(2 \tau, 2 z)+\mathrm{e}^{\pi \mathrm{i} \psi} \frac{1}{2} T_{k}^{-1 / 2}\left(T S T^{2} S\right)_{k l} \chi_{l}\left(\frac{\tau+1}{2}, z\right) \\
& =\sum_{p} \frac{1}{2} S_{k p} \sum_{\chi=0}^{1} \mathrm{e}^{\mathrm{i} \pi \chi} \mathcal{X}_{(p, \chi)}(\tau, z)+\mathrm{e}^{\pi \mathrm{i} \psi} \frac{1}{2} \sum_{p}\left(T^{1 / 2} S T^{2} S T^{1 / 2}\right)_{k p} \sum_{\chi=0}^{1} \mathrm{e}^{\mathrm{i} \pi \chi} \mathcal{X}_{\widehat{(p, \chi)}}(\tau, z) . \tag{4.33}
\end{align*}
$$

It is useful to introduce the symmetric, unitary matrix $P$ as

$$
\begin{equation*}
P \equiv T^{1 / 2} S T^{2} S T^{1 / 2}, P^{2}=S^{2} \tag{4.34}
\end{equation*}
$$

and then we can write our result as

$$
\begin{align*}
& S_{\widehat{(k, \psi),(p q)}}=0 \\
& S_{\widehat{(k, \psi),(j, \chi)}}=\mathrm{e}^{\pi \mathrm{i} \chi \frac{1}{2} S_{k j}}  \tag{4.35}\\
& S_{\widehat{(k, \psi),(\overline{q, \chi)}}}=\mathrm{e}^{\pi \mathrm{i}(\psi+\chi) \frac{1}{2} P_{p q}}
\end{align*}
$$

This completes the computation of the modular matrices of the orbifold for induction order $\lambda=2$.

The matrix $P$ introduced in (4.34) has also arisen in the theory of open and unoriented strings [46], where it describes the transition from horizontal to vertical time on a Möbius strip. This is more than a coincidence: Given a closed string theory, the construction of the open string theory can be thought of as a 'parameter space orbifold' in the sense that its worldsheet can be obtained from an oriented closed Riemann surface by dividing out an anticonformal involution. This amounts to modding out the $\mathbb{Z}_{2}$ symmetry which permutes the chiral and the anti-chiral algebra of a left-right symmetric theory. It is therefore not surprising that we encounter the same quantities in our orbifolds as in the construction of open strings. However, in the case of the $\mathbb{Z}_{2}$ permutation orbifold, the interpretation is quite different: In our case the orbifold is associated to a transformation (see Section 3) which is still locally conformal. Moreover, we still have left and right movers and the orbifolds we consider are still defined only on closed oriented Riemann surfaces.

### 4.5 Checks

In the following checks on the modular matrices of the orbifold, we have assumed the familiar properties of the modular matrices of the mother theory. It is obvious that the resulting modular matrix $S$ for the orbifold theory is symmetric, $S=S^{t}$. Moreover, a straightforward calculation shows that the $S$-matrix of the orbifold theory is unitary, given the unitarity of the original $S$-matrix. One can also show that the square of the new $S$-matrix is a permutation (of order two) of the primary fields, as it should be. Explicit computation shows that $(S T)^{3}=S^{2}$, i.e. $S$ together with the modular matrix $T$ gives a (projective) representation of the modular group. We have also checked that the $S$-matrix of the orbifold theory obeys the required relations $S_{0, k} \geq S_{00}>0$, so that the quantum dimensions have the usual properties.

As another check of our result we have considered the $\mathbb{Z}_{2}$-orbifold of the tensor product of two Ising models: The critical Ising model corresponds to a free massless fermion, and the twisted sectors of the Ising orbifolds at $\hat{c}=\lambda / 2$ have been discussed explicitly in Subsection 3.5. For $\lambda=2$, the cyclic orbifold of the tensor product of two Ising models is a conformal field theory with Virasoro central charge $\hat{c}=1$, and this cyclic orbifold can be identified [11] as the rational $\mathbb{Z}_{2}$ orbifold of the free boson with 13 primary fields. Our formulas correctly reproduce the modular matrix $S$ [10] of the $\hat{c}=1$ orbifold.

### 4.6 Orbifold fusion rules

We can now use the Verlinde formula (4.27) to obtain the fusion rules of the orbifold theory. It is straightforward to check that one obtains the expected twist selection rules: The fusion product of two fields in the untwisted sector contains only fields in the untwisted sector, the fusion of two fields in the twisted sector contains only fields in the untwisted sector, and the fusion product of a field in the untwisted sector with one in the twisted sector of the orbifold lies in the twisted sector.

The non-vanishing fusion coefficients in the untwisted sector can be computed explicitly from the Verlinde formula (4.27) and expressed in terms of the fusion coefficients $\mathcal{N}_{i j k}$ of the mother theory (which are non-negative integers). The results are:

$$
\begin{array}{ll}
\mathcal{N}_{(i j)(k l)(q p)} & =\mathcal{N}_{i k p} \mathcal{N}_{j l q}+\mathcal{N}_{i k q} \mathcal{N}_{j l p}+\mathcal{N}_{i l p} \mathcal{N}_{j k q}+\mathcal{N}_{i l q} \mathcal{N}_{j k p} \\
\mathcal{N}_{(i j)(p q)(r, \psi)} & =\mathcal{N}_{i p r} \mathcal{N}_{j q r}+\mathcal{N}_{i q r} \mathcal{N}_{j p r}  \tag{4.36}\\
\mathcal{N}_{(i j)(p, \psi)(q, \phi)} & =\mathcal{N}_{i p q} \mathcal{N}_{j p q} \\
\mathcal{N}_{(i, \psi)(j, \phi)(k, \chi)} & =\frac{1}{2} \mathcal{N}_{i j k}\left(\mathcal{N}_{i j k}+\mathrm{e}^{\pi \mathrm{i}(\psi+\phi+\chi)}\right)
\end{array}
$$

The first three expressions are manifestly non-negative integers; moreover, the last expression is also a non-negative integer, since the factor $\exp (\mathrm{i} \pi(\psi+\phi+\chi))$ can only take the values -1 and 1 when $\lambda=2$.

For the twisted sector, we obtain the following results for the non-vanishing fusion coefficients:

$$
\begin{equation*}
\mathcal{N}_{(i j) \widehat{(p, \psi)(q, \chi)}}=\sum_{s} \frac{S_{i s} S_{j s} S_{p s} S_{q s}}{S_{0 s}^{2}}=\sum_{s} \mathcal{N}_{i j s} \mathcal{N}_{p q s^{+}}, \tag{4.37}
\end{equation*}
$$

where the sums are over primary fields $s$ in the mother theory and $s^{+}$denotes the primary field that is conjugate to $s$. In the last step we used the Verlinde formula and the unitarity of the $S$-matrix of the mother theory. From the last expression in (4.37) we see that the number $\mathcal{N}_{(i j)(p, \psi)(q, \chi)}$ is just the dimension of the space of conformal blocks of the mother theory for the four-point function with the primary fields $i, j, p$ and $q$ as the insertions. Finally, we obtain

$$
\begin{equation*}
\mathcal{N}_{(i \phi) \widehat{(j, \psi)(k, \chi)}}=\frac{1}{2} \sum_{s} \frac{S_{i s}^{2} S_{j s} S_{k s}}{S_{0 s}^{2}}+\frac{1}{2} \mathrm{e}^{\pi \mathrm{i}(\phi+\psi+\chi)} \sum_{s} \frac{S_{i s} P_{j s} P_{k s}}{S_{0 s}} . \tag{4.38}
\end{equation*}
$$

The sum in the first term is the dimension of a space of conformal blocks of four point functions of the mother theory, this time with insertions $i, i, j$ and $k$. The second term in (4.38) has also arisen in open string theory, and it has been argued [47] that this term is an integer as well, since it describes oriented fusion rules in front of a crosscap. These results imply that the fusion coefficients (4.38) of the orbifold theory are integer or half-integer. Since $\mathcal{N}_{(i \phi)(\overline{j, \psi)}(\widehat{k, \chi)}}$ is a fusion coefficient of the orbifold theory, and therefore must be a non-negative integer, we are led to the conjecture that the expression in (4.38) is a non-negative integer in any rational conformal field theory.

## 5 Discussion

Many of the aspects of cyclic orbifolds presented in this paper can be generalized to arbitrary values of $\lambda$ and even further to orbifolds of tensor products of a conformal field theory by the full permutation group $S_{n}$ or any of its subgroups. A complete treatment of these issues is, however, beyond the scope of this paper, and we briefly sketch here only some aspects of these extensions.

## The new fixed point problem

The first generalization of our results is to cyclic orbifolds where $\lambda>2$ is still prime. In this case, however, the comparison of (4.26) with the modular invariant partition function of the twisted sector derived in [11] showed that one must include $\lambda-1$ copies of the characters (4.25) of the twisted sector. Sets of $\lambda-1$ identical characters were also found in the untwisted sector. In this situation one has to face the so-called fixed point problem [48], where one cannot read off the modular matrix $S$ from the modular transformation properties of the characters as functions of $\tau$ and $z$, as we have done for $\lambda=2$. (Similar problems occur in the description of coset conformal field theories [49] and of integer spin simple current modular invariants [50].)

## Copies of the twisted sector on the sphere

Another implication of this observation is that we must include $\lambda-1$ copies of the twisted
sector of the orbifold on the sphere, and in what follows, we discuss the interpretation of these copies.

For brevity, we discuss only $\lambda=3$ and we choose the simplest case, where the chiral algebra of the mother theory is an affine Lie algebra at level $k$. In this case, the chiral algebra $\mathcal{A}$ of the tensor product theory decomposes into three subspaces which transform under cyclic permutations with the same eigenvalue. The invariant subspace $\mathcal{A}^{(0)}$ is the subalgebra spanned by the operators that are invariant under cyclic permutations

$$
\begin{equation*}
J_{a}(z) \otimes \mathbf{1} \otimes \mathbf{1}+\mathbf{1} \otimes J_{a}(z) \otimes \mathbf{1}+\mathbf{1} \otimes \mathbf{1} \otimes J_{a}(z) \tag{5.39}
\end{equation*}
$$

which span an affine Lie algebra at level $3 k$. Again, the analogue of equation (3.105) implies that the abstract algebra generated by these operators should be represented in the twisted sector of the orbifold by integer moded operators,

$$
\begin{equation*}
J_{a}(z) \otimes \mathbf{1} \otimes \mathbf{1}+\mathbf{1} \otimes J_{a}(z) \otimes \mathbf{1}+\mathbf{1} \otimes J_{a}(z) \quad \rightarrow \quad \hat{J}_{a}^{(0)}(z) \tag{5.40}
\end{equation*}
$$

The other two subspaces $\mathcal{A}^{(+)}$and $\mathcal{A}^{(-)}$are spanned by the linear combinations

$$
\begin{array}{ll}
\mathcal{A}^{(+)}: & J_{a}(z) \otimes \mathbf{1} \otimes \mathbf{1}+\mathrm{e}^{2 \pi \mathrm{i} / 3} \mathbf{1} \otimes J_{a}(z) \otimes \mathbf{1}+\mathrm{e}^{-2 \pi \mathrm{i} / 3} \mathbf{1} \otimes \mathbf{1} \otimes J_{a}(z) \\
\mathcal{A}^{(-)}: & J_{a}(z) \otimes \mathbf{1} \otimes \mathbf{1}+\mathrm{e}^{-2 \pi \mathrm{i} / 3} \mathbf{1} \otimes J_{a}(z) \otimes \mathbf{1}+\mathrm{e}^{2 \pi \mathrm{i} / 3} \mathbf{1} \otimes \mathbf{1} \otimes J_{a}(z) . \tag{5.41}
\end{array}
$$

These two subspaces both transform in the adjoint representation with respect to $\mathcal{A}^{(0)}$.
It is clear that the representatives of $\mathcal{A}^{(+)}$and $\mathcal{A}^{(-)}$in the twisted sector appear as the orbifold currents $\hat{J}_{a}^{(1)}(z)$ and $\hat{J}_{a}^{(2)}(z)$. Since there is no principle by which to decide the precise identification of $\mathcal{A}^{( \pm)}$with the twisted currents, both possibilities must occur,

$$
\begin{array}{lll}
\text { sector 1: } & \mathcal{A}^{(+)} \rightarrow \hat{J}_{a}^{(1)}(z) & \mathcal{A}^{(-)} \rightarrow \hat{J}_{a}^{(2)}(z) \\
\text { sector 2: } & \mathcal{A}^{(-)} \rightarrow \hat{J}_{a}^{(1)}(z) & \mathcal{A}^{(+)} \rightarrow \hat{J}_{a}^{(2)}(z) \tag{5.42}
\end{array}
$$

and this is the interpretation of the two identical twisted sectors at $\lambda=3$. This observation is easily generalized to interpret the multiplicity $\lambda-1$ for arbitrary prime $\lambda$.

To study cyclic permutations for arbitrary $\lambda$, one should combine the fixed point resolution for $\lambda$ prime with an analysis that parallels the one given for the untwisted sector in Subsection 4.2 (which takes into account the different stabilizers in the theory).

## Non-abelian permutation orbifolds

We also expect that our analysis will be useful in the construction of arbitrary permutation orbifolds, since it is known [51] that the twisted sectors of these more general orbifolds are composed entirely of symmetrized combinations of the sectors of cyclic orbifolds.

In this connection, we also remark that the orbifold induction procedure includes cyclic orbifolds in which the mother theory is itself a cyclic orbifold or even a general orbifold. It
would be interesting to clarify the relation of non-abelian permutation orbifolds to these more complicated orbifolds.

## Generalized coset constructions

Using a construction related to our orbifold induction procedure, Kac and Wakimoto [13] and Bouwknegt [14] have proposed a new coset construction, which reads

$$
\begin{align*}
& \hat{T}_{(\mathrm{g} ; \mathrm{h})}=\frac{L_{\mathrm{g}}^{a b}(k)}{\lambda} \sum_{r=0}^{\lambda-1}: \hat{J}_{a}^{(r)} \hat{J}_{b}^{(-r)}:-L_{\mathrm{h}}^{a b}\left(\lambda k_{\mathrm{h}}\right): \hat{J}_{a}^{(0)} \hat{J}_{b}^{(0)}:+\frac{\hat{c}_{\mathrm{g}}(k)}{24 \lambda z^{2}}\left(1-\frac{1}{\lambda^{2}}\right)  \tag{5.43}\\
& \hat{c}_{(\mathrm{g} ; \mathrm{h})}=\hat{c}_{\mathrm{g}}(k)-c_{\mathrm{h}}\left(\lambda k_{\mathrm{h}}\right)
\end{align*}
$$

when written in terms of the orbifold currents $\hat{J}_{a}^{(r)}$ at level $\hat{k}=\lambda k$. Here $\hat{c}_{\mathrm{g}}(k)$ is the orbifold affine-Sugawara central charge $\hat{c}_{\mathrm{g}}$ in (3.51) and $c_{\mathrm{h}}\left(k_{\mathrm{h}}\right)$ is the affine-Sugawara central charge for level $k_{\mathrm{h}}$ of $\mathrm{h} \subset \mathrm{g}$. The construction (5.43) is a straightforward generalization of the one in Refs. [13, 14], which considered only the case $\mathrm{h}=\mathrm{g}$.

The results of our paper point to a conjecture that the constructions (5.43) are in fact an orbifold induction procedure for the twisted sectors of another type of orbifold: Consider the coset conformal field theory defined by embedding h diagonally into the tensor product of $\lambda$ copies of the affine Lie algebra $g$. It is possible to show that the coset theory has a residual action of the cyclic permutation symmetry $\mathbb{Z}_{\lambda}$ of the ambient algebra. Then it is natural to conjecture that the new coset constructions (5.43) give the twisted sectors of the orbifold theories obtained by modding out this $\mathbb{Z}_{\lambda}$ symmetry.

There is some support for this conjecture, at least in the case when $\mathrm{h}=\mathrm{g}$ : Generally, the chiral algebra of the coset theory $(\mathrm{g} \oplus \mathrm{g}) / \mathrm{g}$ is a Casimir W algebra [52,53], which includes a Virasoro subalgebra. On the other hand, it was observed in Ref. [14] that for $\hat{c}<1$ the primary fields of the new coset construction are typically those that are not present in the theory with W symmetry. This is exactly the role of twisted sectors, according to our general discussion of orbifolds in Subsection 3.8. Indeed, the extinction by orbifoldization of a $W_{3}$ symmetry was noted explicitly for the orbifold $\mathrm{W}_{3}$ algebra discussed in Subsections 2.5 and 2.7.

## Generalized Virasoro master equation

It is clear that the affine-Virasoro construction [21-23] can now be generalized to include the orbifold currents $\hat{J}_{a}^{(r)}, r=0 \ldots \lambda-1$ which satisfy the orbifold affine algebra (2.2). The general quadratic ansatz for the chiral stress tensor has the form

$$
\begin{equation*}
\hat{T}(\mathcal{L})=\sum_{r=0}^{\lambda-1} \mathcal{L}_{r}^{a b}: \hat{J}_{a}^{(r)} \hat{J}_{b}^{(-r)}:+c \text {-number term } \tag{5.44}
\end{equation*}
$$

where the coefficients $\mathcal{L}_{r}^{a b}, r=0 \ldots \lambda-1$ are to be determined by a generalized Virasoro master equation. The solutions of this system will therefore include

- The affine-Virasoro construction itself, when $\mathcal{L}_{r}^{a b}=L^{a b}$ is independent of $r$ and $L^{a b}$ is a solution of the Virasoro master equation,
- The orbifold integral Virasoro subalgebras $\hat{T}^{(0)}$ of this paper (see eq. 3.71) when $\mathcal{L}_{r}^{a b}=\frac{1}{\lambda} L^{a b}$,
- The new coset constructions (5.43),
as well as a presumably large class of Virasoro operators of (sectors of) new conformal field theories, beyond these listed above.

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After submission of this paper, we were informed that the orbifold Virasoro algebra has also appeared [54-57] in the context of conformal field theory on $\mathbb{Z}_{\lambda}$-symmetric higher-genus Riemann surfaces. Moreover, the integral affine subalgebras and vertex operators closely related to our higher-level vertex operator construction have been used in the construction of higher-level standard modules of affine $\mathrm{SU}(2)$ [58].

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[^0]:    ${ }^{1}$ The orbifold coset constructions are easily extended to reductive $\mathrm{g}=\oplus \mathrm{g}_{I}$ and $\mathrm{h}=\oplus \mathrm{h}_{i}$ with levels $k_{I}\left(\mathrm{~g}_{I}\right)$ and $k_{i}\left(\mathrm{~h}_{i}\right)$, which results in orbifold currents at levels $\hat{k}_{I}=\lambda k_{I}$ and $\hat{k}_{i}=\lambda k_{i}$ respectively.

[^1]:    ${ }^{2}$ Using the techniques developed in [44] it should be possible to prove directly that these modules are irreducible.

