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A sup + inf Inequality for Some Nonlinear Elliptic Equations Involving Exponential Nonlinearities

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Let u be a solution of the equation $-\Delta u = V(x)\, e^u$ in a domain $\Omega \subset \mathbb{R}^2$, where $0 \leqslant a < V \leqslant b$ and V is Lipschitz continuous. We prove that $\sup u$ can be controlled in terms of $\inf u$. More precisely, $\sup_K u + \inf_\Omega u \leqslant C(a, b, K, \Omega, \|\nabla V\|_{L^\infty})$ for any compact subset $K \subset \Omega$. This extends an earlier result of Shafrir who obtained a similar conclusion when $V \equiv 1$. © 1993 Academic Press, Inc.

1. INTRODUCTION

In this paper we are concerned with the equation

$$-\Delta u = V(x)\, e^u \qquad \text{in } \Omega, \tag{1}$$

where Ω is a domain in $\mathbb{R}^2$ and V satisfies

$$0 < a \leqslant V(x) \leqslant b < \infty \tag{2}$$

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for some constants a and b. This type of nonlinear equation appears in connection with scalar curvature problems and has been extensively studied (see, e.g., [3, 4, 8, 9]). However, the estimate we present here seems to be of a new kind.

The starting point of our analysis is a result from [1] which asserts that for every compact subset $K \subset \Omega$, $\sup_K u$ can be controlled in terms of $\inf_\Omega u$; that is,

$$\sup_K u \leqslant C(a, b, K, \Omega, \inf_\Omega u).$$

However, no explicit dependence of C in terms of $\inf_\Omega u$ was established. It was conjectured in [1] that the dependence is linear. In the case in which $V(x) \equiv a$ is a constant, it was proved in [13] that indeed any solution of (1) satisfies

$$\sup_K u + \inf_\Omega u \leqslant C(a, K, \Omega). \tag{3}$$

The proof of (3) given in [13] relies heavily on the Liouville representation formula; i.e., if V is constant any solution of (1) may be written (locally) as

$$u = \log \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} + C$$

for some constant C and some holomorphic function f with $f' \neq 0$. For the general case (2) the linear dependence was also proved in [13]; i.e., there exist two constants $C_1 > 1$ and C_2 depending on a, b, K, and Ω such that

$$\sup_K u + C_1 \inf_\Omega u \leqslant C_2. \tag{4}$$

Moreover, inequality (4) fails if we take $C_1 = 1$ and if we consider the full class of *all* functions V satisfying (2).

The purpose of this paper is to prove that (3) holds under an additional Lipschitz condition on V. More precisely, our main result is the following.

THEOREM 1. *Assume V is a Lipschitz function satisfying* (2) *and let K be a compact subset of Ω. Then any solution u of* (1) *satisfies*

$$\sup_K u + \inf_\Omega u \leqslant C(a, b, \|\nabla V\|_{L^\infty}, K, \Omega). \tag{5}$$

One of the ingredients in the proof is the blow-up analysis from [1, Theorem 3]. Another ingredient is the Alexandrov moving plane technique (as developed by Serrin [12] and Gidas, Ni, and Nirenberg [6]).

Let us note that Schoen [11] originally used this method in proving that any positive C^2 solution u of

$$-\Delta u = V(x)\, u^{(n+2)/(n-2)} \qquad \text{in} \quad \Omega \subset \mathbb{R}^n, \text{ with } n = 3, 4, \tag{6}$$

satisfies

$$\sup_K u \cdot \inf_\Omega u \leqslant C(a, b, \|\nabla V\|_{L^\infty}, K, \Omega) \quad \text{when} \quad n = 3.$$

and

$$\sup_K u \cdot \inf_\Omega u \leqslant C(a, b, \|\nabla V\|_{L^\infty}, \|\nabla^2 V\|_{L^\infty}, K, \Omega) \quad \text{when} \quad n = 4.$$

However, there is a major difference between problem (1) and problem (6); in both problems we are led, after a blow-up, to classical entire solutions of the equations

$$-\Delta u = e^u \qquad \text{in } \mathbb{R}^2 \tag{7}$$

and

$$-\Delta u = u^{(n+2)/(n-2)} \qquad \text{in } \mathbb{R}^n, \quad u > 0. \tag{8}$$

All solutions of (8) are classified; i.e.,

$$u = C\left(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2}\right)^{(n-2)/2}$$

(see Obata [10], Gidas *et al.* [6, 7], Caffarelli *et al.* [2], and Chen and Li [5]). By contrast, the solutions of (7) have a more complicated structure; it is only the solutions of (7) such that $\int_{\mathbb{R}^n} e^u < \infty$ which have a simple form; i.e.,

$$u(x) = \log \frac{8\varepsilon^2}{(\varepsilon^2 + |x - x_0|^2)^2}$$

(see Chen and Li [5]).

2. Proof of Theorem 1

Set $A = \|\nabla V\|_{L^\infty}$. The proof is divided into 5 steps.

Step 1. *Reduction to* $\Omega = B_2$ (*the disc of radius* 2 *centered at* 0) *and*

$$u(0) + \inf_{B_2} u \leqslant C(a, b, A). \tag{9}$$

The general case follows from (9). Indeed, suppose (9) holds and let v be a solution of

$$-\Delta v = Ve^v \qquad \text{on } B_R.$$

Then

$$u(x) = v\left(\frac{R}{2}x\right) + 2\log(R/2)$$

satisfies

$$-\Delta u = V\left(\frac{R}{2}x\right)e^{u} \qquad \text{on } B_2$$

and therefore, by (9),

$$v(0) + \inf_{B_R} v \leqslant C(a, b, RA/2) - 4\log(R/2)$$
$$= C(a, b, A, R). \tag{10}$$

Theorem 1 is clearly implied by (10).

In what follows we argue by contradiction and we assume that (9) fails. More precisely we assume that there is a sequence (u_n) of solutions of

$$-\Delta u_n = V_n e^{u_n} \qquad \text{on } B_2 \tag{11}$$

with

$$a \leqslant V_n \leqslant b, \qquad \|\nabla V_n\|_{L^\infty} \leqslant A \tag{12}$$

such that

$$u_n(0) + \inf_{B_2} u_n \to +\infty. \tag{13}$$

By passing to a subsequence we may assume that $V_n \to V$ uniformly on $\bar{B}_2$ with $V(0) = K \geqslant a > 0$. Set

$$\delta_n = e^{-u_n(0)/2}. \tag{14}$$

STEP 2. *We have*

$$\delta_n \to 0 \tag{15}$$

and

$$\overline{\lim} \int_{B_{R\delta_n}} V_n e^{u_n} \leqslant 8\pi \qquad \textit{for all} \quad R > 0. \tag{16}$$

Proof. We have

$$u_n(0) + \inf_{B_2} u_n \leqslant 2u_n(0)$$

and thus, by (13), $u_n(0) \to +\infty$. As in [13] we introduce the function

$$G(r) = u_n(0) + \frac{1}{2\pi r}\int_{\partial B_r} u_n\, ds + 4\log r, \qquad 0 < r \leqslant 2.$$

Since

$$G'(r) = \frac{1}{2\pi r}\int_{\partial B_r} \frac{\partial u_n}{\partial r}\, ds + \frac{4}{r}$$

and

$$\int_{\partial B_r} \frac{\partial u_n}{\partial r}\, ds = \int_{B_r} \Delta u_n\, dx = -\int_{B_r} V_n e^{u_n},$$

we conclude that

$$G'(r) \geqslant 0 \Leftrightarrow \int_{B_r} V_n e^{u_n} \leqslant 8\pi \tag{17}$$

and

$$G'(r) = 0 \Leftrightarrow \int_{B_r} V_n e^{u_n} = 8\pi. \tag{18}$$

The function $G(r)$ achieves its maximum on $[0, 2]$ at some point $0 < \mu_n \leqslant 2$. If $\mu_n < 2$ we have

$$\int_{B_{\mu_n}} V_n e^{u_n} = 8\pi.$$

Otherwise, $\mu_n = 2$ and we have

$$\int_{B_{\mu_n}} V_n e^{u_n} \leqslant 8\pi.$$

Thus, in all cases,

$$\int_{B_{\mu_n}} V_n e^{u_n} \leqslant 8\pi.$$

Since u_n is superharmonic we have

$$\begin{aligned} 2(u_n(0) + 2\log \mu_n) &\geqslant u_n(0) + \frac{1}{2\pi\mu_n}\int_{\partial B_{\mu_n}} u_n\, ds + 4\log \mu_n \\ &= G(\mu_n) \geqslant G(2) = u_n(0) + \frac{1}{4\pi}\int_{\partial B_2} u_n\, ds + 4\log 2 \\ &\geqslant u_n(0) + \inf_{\partial B_2} u_n + 4\log 2 \geqslant u_n(0) + \inf_{B_2} u_n + 4\log 2. \end{aligned}$$

Using (13) we conclude that

$$u_n(0) + 2 \log \mu_n \to +\infty,$$

i.e.,

$$\log(\mu_n/\delta_n) \to +\infty$$

so that $\mu_n/\delta_n \to +\infty$. Hence for any given R and for n sufficiently large, $R\delta_n \leqslant \mu_n$ and thus

$$\int_{B_{R\delta_n}} V_n e^{u_n} \leqslant \int_{B_{\mu_n}} V_n e^{u_n} \leqslant 8\pi.$$

STEP 3. *There exist a sequence $x_n \to 0$ and a sequence $R_n > 0$ such that (for a subsequence)*

$$|x_n| < R_n \leqslant 1, \tag{20}$$

$$x_n \text{ is a maximum point of } u_n \text{ on } B_{R_n}(x_n), \tag{21}$$

$$R_n e^{u_n(x_n)/2} \to \infty \tag{22}$$

and

$$\overline{\lim} \int_{B_{R_n}(x_n)} V_n e^{u_n} \leqslant 8\pi. \tag{23}$$

Proof. Set

$$v_n(x) = u_n(\delta_n x) + 2 \log \delta_n \qquad \text{for} \quad |x| \leqslant 1/\delta_n.$$

Consider (v_n) restricted to B_1; it satisfies

$$-\Delta v_n = V_n(\delta_n x)\, e^{v_n} \qquad \text{on } B_1. \tag{24}$$

From (16) (applied with $R = 1$) and (12) we deduce that

$$\overline{\lim} \int_{B_1} e^{v_n} \leqslant \frac{8\pi}{a}. \tag{25}$$

We are now in a situation where we may apply the blow-up analysis of [1, Theorem 3], i.e., there are only three possibilities (after choosing a subsequence):

Case 1. (v_n) is bounded in $L^\infty_{\text{loc}}(B_1)$.

Case 2. $v_n \to -\infty$ uniformly on compact subsets of B_1.

Case 3. There is a non-empty, finite, blow-up set S in B_1 such that $v_n \to -\infty$ uniformly on compact subsets of $B_1 \backslash S$ and for each point $a \in S$ there is a sequence (a_n) such that $a_n \to a$ and $v_n(a_n) \to +\infty$.

Since $v_n(0) = 0$, Case 2 is excluded. We examine Cases 1 and 3 separately.

Case 1. Consider (v_n) restricted to B_R for some fixed $R > 1$. For n sufficiently large, (v_n) satisfies (24) and (25) (with B_1 replaced by B_R). Applying [1] in B_R we see that (v_n) is bounded in $L^\infty_{\text{loc}}(B_R)$. By elliptic regularity theory we deduce that (v_n) is bounded in $W^{2,p}_{\text{loc}}(B_R)$ for every $p < \infty$. Hence, by passing to a subsequence, we may assume that (v_n) converges in $C^1_{\text{loc}}(\mathbb{R}^2)$ to some v satisfying

$$v \in L^\infty_{\text{loc}}(\mathbb{R}^2),$$

$$-\Delta v = Ke^v \text{ on } \mathbb{R}^2 \qquad (K = \lim V_n(0)),$$

$$\int_{\mathbb{R}^2} e^v \leqslant \frac{8\pi}{a},$$

and

$$v(0) = 0.$$

It follows that (see [5]) v is of the form

$$v(x) = \log\left\{\frac{8\lambda^2/K}{(1+\lambda^2|x-y_0|^2)^2}\right\}$$

for some point $y_0 \in \mathbb{R}^2$ and some constant $\lambda > 0$. For any fixed $\rho > |y_0|$ the maximum of v_n on $\bar{B}_\rho$ is achieved at some y_n. Clearly $y_n \to y_0$ since $v_n \to v$ uniformly on B_ρ. In particular, for each integer k (large) we have some (v_{n_k}) and (y_{n_k}) such that

$$\operatorname*{Max}_{B_k} v_{n_k} \text{ is achieved at } y_{n_k}$$

and

$$y_{n_k} \to y_0.$$

Since $(\mu_n/\delta_n) \to +\infty$ we may in addition require that

$$k\delta_{n_k} \leqslant \tfrac{1}{2}\mu_{n_k}.$$

Set

$$x_{n_k} = \delta_{n_k} y_{n_k} \qquad \text{and} \qquad R_{n_k} = (k - |y_{n_k}|)\,\delta_{n_k}.$$

It is easily seen that the corresponding subsequence satisfies (20)–(23).

Case 3. Clearly $0 \in S$ (for otherwise we would have $v_n(0) \to -\infty$, but $v_n(0) = 0$). We may choose some $r_0 \in (0, 1)$ such that (v_n) has no other blow-up point in $\bar{B}_{r_0}$ except the origin. For each n let y_n be a maximum point of v_n on B_{r_0}. Then, by the blow-up assumption, $v_n(y_n) \to +\infty$ and $y_n \to 0$. Set $x_n = \delta_n y_n$ and $R_n = \frac{1}{2} r_0 \delta_n$. It is easily seen that properties (20)–(23) are satisfied.

To summarize, if we set, on B_1,

$$\bar{u}_n(x) = u_n(x + x_n) \qquad \text{and} \qquad \bar{V}_n(x) = V_n(x + x_n),$$

then we have

$$-\Delta \bar{u}_n = \bar{V}_n e^{\bar{u}_n} \qquad \text{on } B_1, \tag{26}$$

$$0 \text{ is a maximum point of } \bar{u}_n \text{ on } B_{R_n}, \text{ with } 0 < R_n \leqslant 1, \tag{27}$$

$$R_n e^{\bar{u}_n(0)/2} \to +\infty, \tag{28}$$

$$\overline{\lim} \int_{B_{R_n}} \bar{V}_n e^{\bar{u}_n} \leqslant 8\pi, \tag{29}$$

and

$$\bar{u}_n(0) + \inf_{B_1} \bar{u}_n \to +\infty. \tag{30}$$

STEP 4. *Set*

$$\eta_n = e^{-\bar{u}_n(0)/2}, \qquad \textit{so that} \quad \eta_n \to 0,$$

$$\bar{v}_n(x) = \bar{u}_n(\eta_n x) + 2 \log \eta_n, \qquad \textit{for} \quad |x| < 1/\eta_n,$$

and

$$\bar{w}_n(x) = \bar{v}_n(x) + 2 \log |x|, \qquad \textit{for} \quad |x| < 1/\eta_n.$$

Clearly, $\bar{v}_n$ satisfies

$$-\Delta \bar{v}_n = V_n(\eta_n x)\, e^{\bar{v}_n} \qquad \text{for} \quad |x| < 1/\eta_n,$$

$$\bar{v}_n(0) = 0,$$

and for each R,

$$\max_{B_R} \bar{v}_n \text{ is achieved at 0, for } n \text{ sufficiently large,}$$

$$\overline{\lim} \int_{B_R} e^{\bar{v}_n} \leqslant 8\pi/a \qquad \text{(by (28) and (29)).}$$

We may use once more Theorem 3 of [1] to conclude that $\bar{v}_n$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}^2)$. By standard elliptic estimates, we find that $\bar{v}_n$ is also bounded in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$. Therefore, for a subsequence, $\bar{v}_n$ converges in $L^\infty_{\text{loc}}(\mathbb{R}^2)$ to a function $\bar{v}$ satisfying

$$-\Delta \bar{v} = Ke^{\bar{v}} \qquad \text{on } \mathbb{R}^2,$$

$$\bar{v}(0) = 0,$$

and

$$\int_{\mathbb{R}^n} e^{\bar{v}} < \infty.$$

Thus, by [5], $\bar{v}$ is given by

$$\bar{v}(x) = \log \left\{ \frac{1}{(1+\gamma^2 |x|^2)^2} \right\}$$

with $\gamma = (K/8)^{1/2}$. It follows that

$$\bar{w}_n - \bar{w} \to 0 \qquad \text{in } C^2_{\text{loc}}(\mathbb{R}^2), \tag{31}$$

where

$$\bar{w}(x) = \log \left\{ \frac{|x|^2}{(1+\gamma^2 |x|^2)^2} \right\}.$$

Next, it is convenient to work in polar coordinates (r, θ) and to set $t = \log r$. Set, for $t < 0$ and $\theta \in [0, 2\pi]$,

$$\tilde{w}_n(t, \theta) = \bar{u}_n(e^t \cos\theta, e^t \sin\theta) + 2t. \tag{32}$$

Clearly $\tilde{w}_n$ satisfies

$$-\Delta \tilde{w}_n = \tilde{V}_n(t, \theta)\, e^{\tilde{w}_n} \qquad \text{in } Q,$$

where

$$Q = \{(t, \theta);\ t \leqslant 0 \text{ and } 0 \leqslant \theta \leqslant 2\pi\},$$

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \theta^2},$$

and

$$\tilde{V}_n(t, \theta) = \bar{V}_n(e^t \cos\theta, e^t \sin\theta).$$

We also introduce, for $s \in \mathbb{R}$,

$$\tilde{w}(s) = \log\left\{\frac{e^{2s}}{(1+\gamma^2 e^{2s})^2}\right\} = 2s - 2\log(1+\gamma^2 e^{2s}).$$

Note that $\tilde{w}$ achieves its maximum at $s = -\log\gamma$, $\tilde{w}'(s) > 0$ for $s < -\log\gamma$, and $\tilde{w}(s)$ is symmetric about $s = -\log\gamma$. We shall also use the fact that

$$\tilde{w}(s) \leqslant 2s \qquad \forall s \in \mathbb{R}. \tag{33}$$

Clearly we have

$$\tilde{w}_n(s + \log\eta_n, \theta) - \tilde{w}(s) = \bar{w}_n(e^s \cos\theta, e^s \sin\theta) - \bar{w}(e^s \cos\theta, e^s \sin\theta).$$

In the new variables property (31) implies that, for every fixed $\alpha \in \mathbb{R}$, as $n \to \infty$,

$$\|\tilde{w}_n(s + \log\eta_n, \theta) - \tilde{w}(s)\|_{L^\infty(\{s \leqslant \alpha, \theta \in [0, 2\pi]\})} \to 0.$$

In particular we may choose n_0 sufficiently large so that, for $n \geqslant n_0$, we have

$$|\tilde{w}_n(t, \theta) - \tilde{w}(t - \log\eta_n)| \leqslant 1 \qquad \text{if} \quad t \leqslant 4 - \log\gamma + \log\eta_n,\ 0 \leqslant \theta \leqslant 2\pi, \tag{34}$$

and

$$\tilde{w}_n(-\log\gamma + \log\eta_n, \theta) > \tilde{w}_n(-\log\gamma + \log\eta_n + 4, \theta) \quad \text{if } 0 \leqslant \theta \leqslant 2\pi. \tag{35}$$

Finally, we introduce

$$\hat{w}_n(t, \theta) = \tilde{w}_n(t, \theta) - \frac{A}{a} e^t \qquad \text{in } Q. \tag{36}$$

We claim that

$$\frac{\partial}{\partial t}\left\{\tilde{V}_n(t, \theta)\, e^{Ae^t/a} e^{\xi} + \frac{A}{a} e^t\right\} \geqslant 0 \qquad \forall (t, \theta) \in Q,\ \forall \xi \in \mathbb{R}. \tag{37}$$

This follows easily from the fact that $\tilde{V}_n \geqslant a$ and the estimate $|(\partial/\partial t)\, \tilde{V}_n(t, \theta)| \leqslant Ae^t$.

STEP 5. (*Conclusion via the reflection method*). We now follow the standard reflection method (see [12, 6]). For $\lambda < 0$ and $\lambda \leqslant t \leqslant 0$ we set

$$t^\lambda = 2\lambda - t$$

and

$$\hat{w}_n^\lambda(t, \theta) = \hat{w}_n(t^\lambda, \theta).$$

We have

$$-\Delta(\hat{w}_n^\lambda - \hat{w}_n) = \hat{V}_n^\lambda(t, \theta)\, e^{\hat{w}_n^\lambda} - \hat{V}_n(t, \theta)\, e^{\hat{w}_n} + \frac{A}{a}(e^{t^\lambda} - e^t), \tag{38}$$

where $\hat{V}_n(t, \theta) = \tilde{V}_n(t, \theta)\, e^{Ae^t/a}$ and $\hat{V}_n^\lambda(t, \theta) = \hat{V}_n(t^\lambda, \theta)$.

For λ very negative (depending on n), we have

$$\hat{w}_n^\lambda(t, \theta) - \hat{w}_n(t, \theta) < 0 \qquad \text{for} \quad \lambda < t \leqslant 0,\ 0 \leqslant \theta \leqslant 2\pi. \tag{39}$$

To prove (39) we just use the fact that, for fixed n, we have by (32) and (36),

$$\hat{w}_n(t, \theta) = 2t + a_n + O_n(e^t) \qquad \text{as} \quad t \to -\infty$$

and

$$\frac{\partial \hat{w}_n}{\partial t}(t, \theta) = 2 + O_n(e^t) \qquad \text{as} \quad t \to -\infty.$$

Define

$$\lambda_n = \sup\{\lambda < 0;\ \hat{w}_n^\lambda(t, \theta) - \hat{w}_n(t, \theta) < 0 \text{ for } \lambda < t \leqslant 0,\ 0 \leqslant \theta \leqslant 2\pi\}.$$

We claim that

$$\lambda_n \leqslant -\log \gamma + \log \eta_n + 2. \tag{40}$$

Indeed, if we choose $\lambda = -\log \gamma + \log \eta_n + 2$ and $t = -\log \gamma + \log \eta_n + 4$ then $t^\lambda = -\log \gamma + \log \eta_n$ and, by (35), $\hat{w}_n^\lambda(t, \theta) > \hat{w}_n(t, \theta)$, $\forall \theta \in [0, 2\pi]$.

On the other hand, we have, by (38), (37), and the definition of λ_n,

$$-\Delta(\hat{w}_n^\lambda(t, \theta) - \hat{w}_n(t, \theta)) \leqslant 0 \qquad \text{for} \quad \lambda \leqslant t \leqslant 0,\ \lambda \leqslant \lambda_n, \text{ and } 0 \leqslant \theta \leqslant 2\pi.$$

Now, we claim that,

$$\min_{0 \leqslant \theta \leqslant 2\pi} \hat{w}_n(0, \theta) \leqslant \max_{0 \leqslant \theta \leqslant 2\pi} \hat{w}_n(2\lambda_n, \theta). \tag{41}$$

Suppose not, that

$$\max_{0 \leqslant \theta \leqslant 2\pi} \hat{w}_n(2\lambda_n, \theta) < \min_{0 \leqslant \theta \leqslant 2\pi} \hat{w}_n(0, \theta);$$

then, by the maximum principle,

$$\hat{w}_n^{\lambda_n}(t, \theta) - \hat{w}_n(t, \theta) < 0 \qquad \text{for} \quad \lambda_n < t < 0,\ 0 \leqslant \theta \leqslant 2\pi$$

and by Hopf's Lemma

$$\frac{\partial}{\partial t}\{\hat{w}_n^{\lambda_n}(t,\theta)-\hat{w}_n(t,\theta)\}_{|t=\lambda_n}<0, \qquad 0\leqslant\theta\leqslant 2\pi.$$

This contradicts the definition of λ_n.

Using (34) we have

$$\max_{0\leqslant\theta\leqslant 2\pi}\hat{w}_n(2\lambda_n,\theta)\leqslant\tilde{w}(2\lambda_n-\log\eta_n)+1\leqslant 4\lambda_n-2\log\eta_n+1 \qquad \text{by (33).}$$

Hence, by (40), we obtain

$$\max_{0\leqslant\theta\leqslant 2\pi}\hat{w}_n(2\lambda_n,\theta)\leqslant 2\log\eta_n+C(\gamma). \tag{42}$$

Putting together (41) and (42) we see that

$$\min_{0\leqslant\theta\leqslant 2\pi}\hat{w}_n(0,\theta)\leqslant 2\log\eta_n+C(\gamma).$$

Going back to the definition of $\hat{w}_n$ (see (36) and (32)), we have

$$\min_{\partial B_1}\tilde{u}_n\leqslant 2\log\eta_n+C(a,A,\gamma). \tag{43}$$

Recall that $\eta_n=e^{-\tilde{u}_n(0)/2}$, and thus, by (43),

$$\tilde{u}_n(0)+\min_{\partial B_1}\tilde{u}_n\leqslant C(a,A,\gamma),$$

which contradicts (30).

3. Some Open Problems

In connection with Theorem 1 there are several natural questions:

Question 1. Can one replace in Theorem 1 the Lipschitz assumption on V by a uniform Hölder condition $|V(x)-V(y)|\leqslant A\,|x-y|^{\alpha}$, $0<\alpha<1$? We do not know the answer.

Question 2. Can one replace in Theorem 1 the Lipschitz assumption on V by a uniform modulus of continuity? The answer is negative as may be seen by the following example.

Example. There is a sequence $(\tilde{u}_n)$ of solutions of

$$-\Delta\tilde{u}_n=\tilde{V}_n e^{\tilde{u}_n} \qquad \text{on } B_1 \tag{44}$$

such that

$$\tilde{V}_n \to V \equiv 2 \qquad \text{in } C(\bar{B}_1)$$

but

$$\tilde{u}_n(0) + \inf_{B_1} \tilde{u}_n \to +\infty.$$

Choose a sequence $\beta_n \nearrow 1$ and define the sequence (u_n) by

$$u_n(r) = \begin{cases} 2\log\left\{\dfrac{2\beta_n (nr)^{\beta_n - 1}}{1 + (nr)^{2\beta_n}}\right\} + 2\log n & \text{if } 1/n < r \leqslant 1 \\ 2\log \beta_n + 2\log\left(\dfrac{2}{1 + (nr)^2}\right) + 2\log n & \text{if } 0 \leqslant r \leqslant 1/n. \end{cases}$$

It is easy to see that u_n satisfies

$$-\Delta u_n = V_n e^{u_n} \qquad \text{on } B_1$$

with

$$V_n = \begin{cases} 2 & \text{if } 1/n < r \leqslant 1, \\ \dfrac{2}{(\beta_n)^2} & \text{if } 0 \leqslant r \leqslant 1/n. \end{cases}$$

Clearly,

$$u_n(0) + u_n(1) \geqslant 4\log n + 2\log\left(\frac{n^{\beta_n - 1}}{1 + n^{2\beta_n}}\right) - C$$

$$= 2\log\left(\frac{1}{n^{-\beta_n - 1} + n^{\beta_n - 1}}\right) - C.$$

Thus $u_n(0) + u_n(1) \to +\infty$ if we choose (β_n) such that $(1 - \beta_n)\log n \to +\infty$.

We may now smooth the sequences (u_n) and (V_n). To do so we choose, for each *fixed* n, a sequence (V_n^j) of smooth functions such that

$$V_n^j \xrightarrow[j \to \infty]{} V_n \text{ a.e. on } B_1$$

and

$$2 \leqslant V_n^j \leqslant \frac{2}{\beta_n^2} \qquad \text{on } B_1.$$

Let u_n^j be the solution of the linear equation

$$-\Delta u_n^j = V_n^j e^{u_n} \quad \text{on} \quad B_1,$$
$$u_n^j = u_n \quad \text{on} \quad \partial B_1.$$

Clearly, $u_n^j \to u_n$, in $C(\bar{B}_1)$, as $j \to \infty$. Therefore, we may choose $j = j(n)$ such that

$$\|u_n^j - u_n\|_{L^\infty} \leqslant 1/n.$$

Since

$$-\Delta u_n^j = (V_n^j e^{u_n - u_n^j}) e^{u_n^j},$$

we see that $\tilde{u}_n = u_n^{j(n)}$ and $\tilde{V}_n = V_n^{j(n)} e^{u_n - u_n^{j(n)}}$ have all the required properties.

Question 3. Assume V is a *fixed* positive continuous (or even Hölder continuous) function on Ω. Let K be a compact subset of Ω and let u be a solution of (1). Does one have

$$\sup_K u + \inf_\Omega u \leqslant C(V, K, \Omega)?$$

We do not know the answer.

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