

The Haagerup property, Property (T) and the Baum-Connes conjecture for locally compact Kac-Moody groups

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Abstract

We indicate which symmetrizable locally compact affine or hyperbolic Kac-Moody groups satisfy Kazhdan's Property (T), and those that satisfy its strong negation, the Haagerup property. This reveals a new class of hyperbolic Kac-Moody groups satisfying the Haagerup property, namely symmetrizable locally compact Kac-Moody groups of rank 2 or of rank 3 noncompact hyperbolic type. These groups thus satisfy the strongest form of the Baum-Connes conjecture, namely the conjecture with coefficients in any C^* -algebra.

For symmetrizable locally compact Kac-Moody groups G of rank 3 compact hyperbolic type or of affine or hyperbolic type and rank ≥ 4 we show that G has Property (T) and we deduce that the Baum-Connes assembly map on equivariant K -homology of G is both injective and surjective. Thus G satisfies the Baum-Connes conjecture without coefficients.

We show that Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups can be determined from the Dynkin diagram or equivalently from the generalized Cartan matrix. Our results give a dichotomy for hyperbolic Kac-Moody groups of noncompact type, with rank 3 Kac-Moody groups of noncompact hyperbolic type such as $\widehat{A}_1^{(1)}$ satisfying the Haagerup property and hence the Baum-Connes conjecture with coefficients, and groups of rank $4 \leq r \leq 10$, such as E_{10} , satisfying Property (T) and the Baum-Connes conjecture without coefficients.

For certain Kac-Moody lattices Γ satisfying the Haagerup property, we exhibit a proper action of Γ on a simplicial tree that is embedded in a space with a Lorentzian metric.

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1 Introduction

Let G be a second countable, locally compact and Hausdorff topological group. Let $C_{red}^*(G)$ denote the reduced C^* -algebra of G . Then there exists a universal space $\underline{E}G$ for proper G -actions, unique up to G -equivariant homotopy. Using the KK -theory of Kasparov we may form the equivariant K -homology $K_*^G(\underline{E}G)$ ([Ka], [HK1]). There is then an assembly map from K -homology to K -theory

$$\mu : K_*^G(\underline{E}G) \longrightarrow K_*(C_{red}^*(G)),$$

which is conjectured by Baum and Connes to be an isomorphism (see for example [BCH] and [HK1]). This has become known as the celebrated *Baum-Connes conjecture* in non-commutative geometry, relating the analytic and topological properties of a group and proven for large classes of groups.

A deep result of Higson and Kasparov gives a sufficient condition for discrete groups to satisfy the Baum-Connes conjecture. They showed that if a countable discrete group Γ admits an affine, isometric and metrically proper action on a Euclidean space, that is, Γ has the *Haagerup property*, then Γ satisfies the Baum-Connes conjecture with coefficients in any Γ - C^* -algebra ([HK1], [HK2]).

It is known that if a lattice subgroup Γ of a locally compact group G has the Haagerup property, then G has the Haagerup property ([CCJJV]). It is also easy to verify that the Haagerup property for a locally compact group G implies the Haagerup property for closed subgroups. In particular this applies to lattice subgroups $\Gamma \leq G$.

However, the Haagerup property is not equivalent to the Baum-Connes conjecture, as is implied by results of Lafforgue who exhibited certain discrete groups satisfying both the Baum-Connes conjecture and Property (T) which is a strong negation of the Haagerup property ([L2], see also Section 5).

In this work, we study locally compact forms of affine and hyperbolic Kac-Moody groups. These are constructed as completions of the *Tits functor* over finite fields (see Section 2) associated to (infinite dimensional) Kac-Moody algebras. Complete Kac-Moody groups over finite fields are locally compact and totally disconnected. Even though these groups have exponential growth and behave in many ways like infinite dimensional groups, they contain lattice subgroups which may be studied in analogy with lattices in Lie groups (see [CG], [CC], [Re1] and [RR] for example). Locally compact forms of Kac-Moody groups were anticipated by Tits ([Ti1]) but have only appeared in the literature recently ([CG] and [RR]). The main objective of this work is to explore the representation theoretic and K -theoretic properties of these groups.

We determine which symmetrizable locally compact affine or hyperbolic Kac-Moody groups satisfy Kazhdan's Property (T), and those that satisfy the Haagerup property. By the work of Higson and Kasparov, our symmetrizable locally compact Kac-Moody groups with the Haagerup property satisfy the Baum-Connes conjecture with coefficients in any C^* -algebra. For symmetrizable locally compact Kac-Moody groups G with Property (T) we deduce that the Baum-Connes assembly map on equivariant K -homology of G is both injective and surjective. Thus G satisfies the Baum-Connes conjecture without coefficients.

We show that Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups can be determined from the Dynkin diagram, equivalently from the generalized Cartan matrix (Section 7).

Our results give a dichotomy for hyperbolic Kac-Moody groups of noncompact type, with rank 3 Kac-Moody groups of noncompact hyperbolic type such as $\widehat{A}_1^{(1)}$ satisfying the Haagerup property and hence the Baum-Connes conjecture with coefficients, and groups of rank $4 \leq r \leq 10$, such as E_{10} , satisfying Property (T) and the Baum-Connes conjecture without coefficients.

The results here reveal a new class of locally compact groups satisfying the Baum-Connes conjecture without ever computing K -theory of classifying spaces for proper actions, or K -homology of group C^* -algebras associated to our Kac-Moody groups. Since Kac-Moody groups are amalgams of their parabolic subgroups we believe that it should be straightforward to compute K -homology in terms of the K -homology of the parabolic subgroups. We have not yet determined the reduced C^* -algebra $C_{red}^*(G)$ of G . We hope to take these questions up elsewhere.

When G is a symmetrizable locally compact Kac-Moody group of rank 2 or of rank 3 noncompact hyperbolic type, we are able to deduce that G satisfies the Haagerup property using the following ingredients. We use the work of Dymara and Januszkiewicz giving criteria for groups with a BN -pair to have the Haagerup property and Property (T) in terms of cohomology vanishing theorems and the action on the corresponding Tits building ([DJ]), the existence of nonuniform lattice subgroups ([CG] and [Re1]), and certain aspects of the classification of hyperbolic Kac-Moody algebras ([Sa], [Li] and Section 6).

We have the following.

Theorem 1.1. *Let G be a symmetrizable locally compact affine or hyperbolic Kac-Moody group over a finite field \mathbb{F}_q . Assume that q is sufficiently large. If $\text{rank}(G) = 2$, or if $\text{rank}(G) = 3$ and G has noncompact hyperbolic type, then G has the Haagerup property. Hence G satisfies the Baum-Connes conjecture with coefficients in any C^* -algebra.*

In certain cases, this assumption of large thickness of the Tits building can be removed.

Theorem 1.2. *Let G be a locally compact affine or hyperbolic Kac-Moody group over a finite field \mathbb{F}_q . Suppose that q is sufficiently large. If $r = \text{rank}(G) = 3$ and G has compact hyperbolic type, or if $\text{rank}(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type then*

- (i) G has Property (T),
- (ii) All entries in the Coxeter matrix for G are finite.

Corollary 1.3. *Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups over sufficiently large finite fields can be determined from the Dynkin diagram, or equivalently from the generalized Cartan matrix.*

For certain hyperbolic Kac-Moody lattices with the Haagerup property, we construct a proper action on a space \mathcal{Y} with measured walls in the sense of [CMV] (Section 9). In this case, \mathcal{Y} is a tree embedded in the Tits building, a hyperbolic building.

The Kac-Moody Tits buildings described here are all $CAT(0)$ spaces. Thus they are ‘bolic spaces’ in the sense of Kasparov and Skandalis ([KS]). By the work of Lafforgue, groups acting on $CAT(0)$ spaces are ‘strongly bolic’ and for these groups, the Baum-Connes assembly map is both injective and surjective ([L1]-[L3]). Thus we have the following (see Section 8).

Theorem 1.4. *Let G be a symmetrizable locally compact Kac-Moody group of rank 3 compact hyperbolic type or of affine or hyperbolic type and rank ≥ 4 . Then the Baum-Connes assembly map on equivariant K -homology of G is both injective and surjective. Thus G satisfies the Baum-Connes conjecture without coefficients.*

The above results can be summarized in the following table, where G is a symmetrizable locally compact affine or hyperbolic Kac-Moody group over a finite field \mathbb{F}_q .

Rank r of G	Assumptions on G	Properties of G
$r = 2$	affine or hyperbolic type	Haagerup property Baum-Connes conjecture with coefficients
$r = 3$	noncompact hyperbolic type q sufficiently large	Haagerup property Baum-Connes conjecture with coefficients
$r = 3$	compact hyperbolic type q sufficiently large	Property (T) Baum-Connes conjecture without coefficients
$r \geq 3$	affine type q sufficiently large	Property (T) Baum-Connes conjecture without coefficients
$4 \leq r \leq 10$	hyperbolic type q sufficiently large	Property (T) Baum-Connes conjecture without coefficients

The Baum-Connes conjecture has not yet been formulated for topological Kac-Moody groups defined over \mathbb{C} since these groups are not locally compact, however for these groups Kitchloo has recently defined the equivariant K -homology in terms of the K -homology of parabolic subgroups ([K2]). A discussion of the Baum-Connes conjecture for such groups G would require a universal space $\underline{E}G$ for proper G -actions. Kitchloo has shown that the space $\underline{E}G$ is the topological Tits building of G , a smooth stack when G is a Kac-Moody group of affine or compact hyperbolic type ([K1]).

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2 Locally compact Kac-Moody groups and Tits buildings

2.1 Introduction to Kac-Moody algebras

The initial construction by Cartan and Killing of finite dimensional simple Lie algebras from the Cartan integers was type dependent. In 1966 Serre showed how to remove this dependence by giving defining presentations using data from the Cartan matrix.

Victor Kac and Robert Moody, in 1967, independently enlarged the class of classical Lie algebras, by dropping the assumption that the Cartan matrix is positive definite, resulting in new Lie algebras which are infinite dimensional.

The data for constructing a Kac-Moody algebra includes a *generalized Cartan matrix*. This is a square matrix $A = (a_{ij})$, $i, j \in \{1, 2, \dots, \ell\}$ whose entries are integers such that:

- (1) $a_{ii} = 2$,
- (2) $a_{ij} \leq 0$, $i \neq j$,
- (3) $a_{ij} = 0$ implies $a_{ji} = 0$.

A generalized Cartan matrix A is called *indecomposable* if there is no rearrangement of the indices so that A can be written in block diagonal form. A generalized Cartan matrix A is called *symmetrizable* if there exist nonzero rational numbers d_1, \dots, d_ℓ , such that the matrix DA is symmetric, where $D = \text{diag}(d_1, \dots, d_\ell)$. We call DA a *symmetrization* of A . Such a symmetrization always exists and is unique up to a scalar multiple. Symmetrizability is an important property of a generalized Cartan matrix, necessary for the existence of a well-defined symmetric invariant bilinear form $(\cdot | \cdot)$ on the Kac-Moody algebra which plays the role of ‘squared length’ of a root.

The generalized Cartan matrix A is *affine* if A is positive semi-definite but not positive definite. If A is neither positive definite nor positive semi-definite, but every proper indecomposable submatrix is either positive definite or positive semi-definite, we say that A has *hyperbolic type*. If every proper indecomposable submatrix of A is positive definite, we say that A has *compact hyperbolic type*. Thus if A has a proper indecomposable affine submatrix, we say that G has *noncompact hyperbolic type*.

Given a generalized Cartan matrix and a finite dimensional vector space \mathfrak{h} satisfying some natural conditions, Gabber and Kac defined a Kac-Moody algebra by generators and relations in analogy with the Serre presentation for finite dimensional simple Lie algebras ([K]).

Roughly speaking, a Lie algebra has *polynomial growth* if the dimension of the subspace spanned by all monomials of length $\leq n$ is a polynomial in n , and *exponential growth* if the dimension of such a subspace is exponential in n . Affine Kac-Moody algebras have polynomial growth, while hyperbolic Kac-Moody algebras have exponential growth.

2.2 Examples

Rank 2 hyperbolic type The rank 2 hyperbolic generalized Cartan matrices, infinite in number are:

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}_{ab > 4}$$

Rank 2 affine type The only 2×2 affine generalized Cartan matrices are:

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

Rank 3 affine type Every proper indecomposable submatrix is of finite type:

$$\tilde{A}_2 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Rank 3 compact hyperbolic type If $\text{rank}(A) = 3$ then for any rank 2 proper indecomposable submatrix, $a_{ij}a_{ji} < 4$ for $i \neq j$:

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix}$$

Rank 3 noncompact hyperbolic type If $\text{rank}(A) = 3$ then there is a rank 2 proper indecomposable submatrix with $a_{ij}a_{ji} = 4$ for $i \neq j$:

$$\widehat{A}_1^{(1)} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Rank 10 noncompact hyperbolic type $E_{10} = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & & & & \\ 0 & 0 & & E_8 & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{pmatrix}$

2.3 Introduction to Kac-Moody groups and their properties

Subsequent to the discovery of infinite dimensional generalizations of Lie algebras by dropping the assumption that the matrix of Cartan integers is positive definite, the problem of associating groups to Kac-Moody algebras arose, the difficulty being that there is no obvious definition of a general ‘Kac-Moody group’.

Several appropriate definitions of a Kac-Moody group have been discovered, many of them using a variety of techniques as well as additional external data, such as a \mathbb{Z} -form for the universal enveloping algebra. Most constructions use some version of the *Tits functor*.

Though there is no obvious infinite dimensional generalization of finite dimensional Lie groups, Tits associated a group functor G_A on the category of commutative rings, such that for any symmetrizable generalized Cartan matrix A and any ring R there exists a group $G_A(R)$ ([Ti1], [Ti2]). Tits showed that if R is a field, then $G_A(R)$ is characterized uniquely up to isomorphism, apart from some degeneracy in the case of small fields. Tits defined not one group, but rather *minimal* and *maximal* groups. The value of the Tits functor G_A over a field k is called a *minimal Kac-Moody group*. The *maximal* or *complete Kac-Moody group* is defined relative to a completion of the Kac-Moody algebra and contains $G_A(k)$ as a dense subgroup.

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. The existence of a completion $G = G_A(\mathbb{F}_q)$ of the Tits functor associated to A and the finite field \mathbb{F}_q was noted by Tits ([Ti1]).

Explicit completions have been constructed using distinct methods by Carbone and Garland ([CG]) and by Rémy and Ronan ([RR]). A complete Kac-Moody group G over a finite field is locally compact and totally disconnected.

Locally compact Kac-Moody groups and their lattices have appeared in the literature only recently ([CG], 2003 and [RR], 2006). The main aim of this work is to give a detailed account of the representation theoretic and K -theoretic properties of these groups. We mention here some of the known results concerning Property (T) for locally compact Kac-Moody groups.

In [CG], the authors showed that for locally compact Kac-Moody groups G of rank greater than or equal to 3 over sufficiently large finite fields, if G contains a cocompact lattice, then G has Property (T).

This result was generalized considerably by Dymara and Januszkiewicz ([DJ]) who obtained vanishing theorems for various cohomologies on the Bruhat-Tits building X associated with a BN -pair of our Kac-Moody group G (see subsection 2.4), and on discrete subgroups $\Gamma \leq G$. They also gave criteria for lattices in G to have Property (T) and the Haagerup property. Much of our work in Section 7 involves showing that these (nontrivial) criteria of [DJ] are satisfied.

In [EJ-Z] the authors establish a new spectral criterion for Kazhdan's Property (T) which gives new examples of Property (T) groups. They also apply it to discrete subgroups of Kac-Moody groups over finite fields giving precise Kazhdan constants and generalizing the results of [DJ].

While Kac-Moody groups may be described axiomatically, by generators and relations, by analytic, topological and geometric methods, for our purposes it will be convenient and sufficient to characterize these groups by describing their (B, N) -pairs.

2.4 The (B, N) -pair and Tits building of a Kac-Moody group

A Kac-Moody group G may be described by certain group theoretic data, called a *Tits system* or (B, N) -pair. This data carries a great deal of information about the group and its subgroups, and in particular determines a simplicial complex, a *Tits building* X on which the group acts faithfully and cocompactly.

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field \mathbb{F}_q ([CG] and [RR]). The Tits building X of a complete Kac-Moody group G over a finite field is locally finite. In this section we give a brief description of the Tits system for G and its corresponding Tits building.

A completion G of Tits' functor over the finite field \mathbb{F}_q has subgroups $B^\pm \subseteq G$, $N \subseteq G$, and Weyl group $W = N/H$, where $H = N \cap B^\pm$ is a normal subgroup of N . We have $B^\pm = HU^\pm$ where U^+ is generated by all positive real root groups, U^- is generated by all negative real root groups, B^+ is compact, in fact a profinite neighborhood of the identity in G , and B^- is discrete. Then (G, B^+, N) and (G, B^-, N) are BN -pairs, and

$$G = B^+NB^- = B^-NB^+.$$

It follows that G has Bruhat decomposition

$$G = \sqcup_{w \in W} B^\pm w B^\pm.$$

Let S be the standard generating set for the Weyl group W consisting of simple root reflections. Let $U \subsetneq S$. The *standard parabolic subgroups* are

$$P_U = \sqcup_{w \in \langle U \rangle} B^\pm w B^\pm.$$

A *parabolic* subgroup is any subgroup containing a conjugate of B^\pm . The Tits building of G is a simplicial complex X of dimension $\dim(X) = |S| - 1$. In fact we associate a building X^\pm to each BN -pair (G, B^+, N) and (G, B^-, N) . The buildings X^+ and X^- are isomorphic as chamber complexes and have constant thickness $q + 1$ (see [DJ, Appendix KMT]).

The vertices of X are given by the cosets of G by the maximal parabolic subgroups of G . The incidence relation is described as follows. The $r + 1$ vertices Q_1, \dots, Q_{r+1} span an r -simplex if and only if the intersection $Q_1 \cap \dots \cap Q_{r+1}$ is parabolic, that is, contains a conjugate of B^\pm . In our case, the Weyl group W is infinite, so by the Solomon-Tits theorem, X is contractible. The group G acts by left multiplication on cosets.

Since the vertex stabilizers for the action of G on X are compact and open, the action of G on X is metrically proper (see Section 4 for a definition of a metrically proper action).

2.5 Lattices in Kac-Moody groups

Let G be a locally compact group and let μ be a (left) Haar measure on G . Let $\Gamma \leq G$ be a discrete subgroup with quotient $p : G \rightarrow \Gamma \backslash G$. We call Γ a *lattice* in G if $\mu(\Gamma \backslash G) < \infty$, and a *cocompact* lattice if $\Gamma \backslash G$ is compact.

Symmetrizable locally compact Kac-Moody groups over finite fields \mathbb{F}_q are known to contain lattice subgroups. If q is sufficiently large then G contains nonuniform lattice subgroups ([CG] and [Re1]) for $\ell \geq 3$, and both cocompact and nonuniform lattice subgroups if $\ell = 2$, with no restriction on q ([CG], [CC] and [RR]).

The discreteness of a lattice subgroup Γ in a locally compact Kac-Moody group G is equivalent to the property that Γ acts on the Tits building X with finite vertex stabilizers ([BL]). Thus locally compact Kac-Moody groups G come equipped with lattices that act properly and isometrically on the locally finite Tits building X .

3 The Baum-Connes conjecture

Let G be a second countable, locally compact and Hausdorff topological group. The reduced C^* -algebra $C_{red}^*(G)$ of G is the completion in the operator norm of the convolution algebra $L^1(G)$ viewed as an algebra of operators on $L^2(G)$.

If G is a discrete and torsion free group, it is relatively straightforward to formulate the Baum-Connes conjecture for G . For such G , there is a natural homomorphism, or ‘assembly map’ on K -homology

$$\mu : K_*(BG) \rightarrow K_*(C_{red}^*(G)),$$

where BG is the classifying space for G , and $C_{red}^*(G)$ is the reduced C^* -algebra of G . The Baum-Connes conjecture states that μ is an isomorphism of abelian groups ([BCH]).

Now suppose that G is non-discrete or discrete with torsion. There exists a universal space \underline{EG} for proper G -actions, unique up to G -equivariant homotopy. Using the KK -theory of Kasparov ([Ka]), we may form the equivariant K -homology $K_*^G(\underline{EG})$. There is then an assembly map from K -homology to K -theory

$$\mu : K_*^G(\underline{EG}) \longrightarrow K_*(C_{red}^*(G)),$$

which is conjectured by Baum and Connes to be an isomorphism.

3.1 The Baum-Connes conjecture with coefficients

A more general version of the Baum-Connes conjecture can be formulated with coefficients. Let A be a C^* -algebra on which G acts as C^* -algebra automorphisms. Let $C_{red}^*(G, A)$ denote the reduced crossed-product C^* -algebra. By [BCH], there is a homomorphism of abelian groups

$$\mu : K_*^G(\underline{EG}, A) \longrightarrow K_*(C_{red}^*(G, A)),$$

where $K_*(C_{red}^*(G, A))$ denotes the K -theory of $C_{red}^*(G, A)$ and $K_*(\underline{EG}, A)$ denotes the G -equivariant K -homology of \underline{EG} . As above the Baum-Connes conjecture states that μ is an isomorphism of abelian groups ([BCH]).

4 Metrically proper, isometric, affine actions of discrete groups

Let Γ be a discrete group. The reduced C^* -algebra $C_{red}^*(\Gamma)$ is a completion of the complex group algebra $\mathbb{C}[\Gamma]$. In [BCH], the authors give a model for the classifying space $\underline{E}\Gamma$ for proper actions which has a geometric realization as a simplicial complex whose p -simplices are all $(p+1)$ -element subsets of Γ . Using this model for $\underline{E}\Gamma$, the authors give an interpretation of the Γ -equivariant homology of $\underline{E}\Gamma$ in terms of the group homology of Γ .

Let G be a locally compact group. We say that there is a continuous, isometric action of G on some affine Hilbert space H if there is a continuous map $G \longrightarrow Isom(H)$. We say that the action of G on H is metrically proper if for any bounded subset B in H the set

$$K(G, B) := \{g \in G \text{ s.t. } gB \cap B \neq \emptyset\}$$

has compact closure in G . The locally compact group G satisfies the *Haagerup property*, (or is *a-T-menable*) if it admits a continuous, isometric, proper action on an affine Hilbert space.

Higson and Kasparov have shown that if a countable discrete group Γ has the Haagerup property, then Γ satisfies the Baum-Connes conjecture with coefficients in any Γ - C^* -algebra ([HK1], [HK2]).

5 Property (T), the Haagerup property and rapid decay

A locally compact group G has Property (T) if and only if every continuous action of G by isometries on a Hilbert space has a fixed point. Other equivalent definitions in representation

theory and ergodic theory all indicate that the Haagerup property is a strong negation of Kazhdan's Property (T). In addition, if Γ is a discrete group that has both Kazhdan's Property (T) and the Haagerup property, then Γ is finite. Furthermore, a theorem of Wang ([W]) shows that Property (T) for a lattice subgroup Γ of a locally compact group G is equivalent to Property (T) for G itself.

It is known that if a lattice subgroup Γ of a locally compact group G has the Haagerup property, then G has the Haagerup property ([CCJJV]). It is also easy to verify that the Haagerup property for a locally compact group G implies the Haagerup property for a lattice subgroup $\Gamma \leq G$.

Lafforgue has shown that there exist groups satisfying both Property (T) and the Baum-Connes conjecture. In [L1]-[L3] he showed that if a discrete group Γ acts properly, isometrically and with the analytical property of 'rapid decay' on a space with non-positive curvature then Γ satisfies the Baum-Connes conjecture without coefficients. Thus Lafforgue's work reveals groups satisfying both Property (T) and the Baum-Connes conjecture, such as cocompact lattices in $SL_3(\mathbb{R})$ (see [L1] for a proof of rapid decay for lattices in $SL_3(\mathbb{R})$, adapted from earlier work of Ramagge, Robertson and Steger on rapid decay for lattices in $SL_3(\mathbb{Q}_p)$ ([RRS]).)

In [CR], the authors give new examples of groups with the rapid decay property, including finitely generated nonuniform lattices in rank one Lie groups over \mathbb{R} or \mathbb{C} . Rapid decay is known to fail for all nonuniform lattices in simple Lie groups of rank > 1 , and to hold for only certain cocompact lattices in rank > 1 .

If G is a locally compact rank 2 Kac-Moody group, then G contains cocompact lattices ([CG], [CC] and [RR]). These lattices are Gromov hyperbolic groups and hence have rapid decay. It is not currently known if lattices in Kac-Moody groups of rank $r > 2$ have the rapid decay property.

We ask the following:

Question Let G be a locally compact hyperbolic Kac-Moody group of rank ≥ 3 . Let Γ be a lattice subgroup of G . Does Γ have the rapid decay property? Assume now that G has rank 2 and Γ is nonuniform. Does Γ have rapid decay?

6 Classification of hyperbolic Dynkin diagrams

It is known that the maximal rank of a hyperbolic Kac-Moody algebra is 10. This is determined by the following restrictive conditions:

(1) The fundamental chamber \mathcal{C} of the Weyl group, viewed as a hyperbolic reflection group, must be a Coxeter polyhedron. The dihedral angles between adjacent walls must be of the form π/k , where $k \geq 2$.

(2) The fundamental chamber \mathcal{C} of the Weyl group must be a simplex, which gives a bound on the number of faces.

Such a 'Coxeter simplex' \mathcal{C} exists in hyperbolic n -space \mathbb{H}^n for $n \leq 9$. The bound on the rank of a hyperbolic Dynkin diagram can also be deduced by purely combinatorial means ([K], [Li], [Sa]).

There are 18 hyperbolic algebras of rank 7-10, classified by Kac [Ka]. There are 142 possible symmetric or symmetrizable hyperbolic Dynkin diagrams between the ranks 3 and 10 (136 of these were classified and exhibited by Saçlioğlu, but he omitted 6, as was observed by [dBS], p 4491). Li independently classified the hyperbolic Dynkin diagrams without the assumption of symmetrizability ([Li]). The rank 2 hyperbolic generalized Cartan matrices are:

$$A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}_{ab>4}$$

The only 2×2 affine generalized Cartan matrices are:

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}, \quad A_2^{(2)} = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

Proposition 6.1. *A symmetrizable hyperbolic generalized Cartan matrix contains an $A_1^{(1)}$ or $A_2^{(2)}$ proper indecomposable submatrix if and only if rank $A = 3$ and A has noncompact type.*

Proof: A symmetrizable hyperbolic generalized Cartan matrix cannot contain an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable proper submatrix if rank $A = l > 3$, since $S(A)$ has l vertices, and the 3 vertex connected subdiagram consisting of $A_1^{(1)}$ or $A_2^{(2)}$ plus an additional vertex would then be neither affine nor finite. Thus if A is a symmetrizable hyperbolic generalized Cartan matrix with an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrix, then the rank of A is 3, and A has noncompact type.

Conversely, every hyperbolic diagram of rank 3 of noncompact type must contain contain an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable subdiagram, since it must contain a subdiagram of affine type with 2 vertices. \square

The following corollary summarizes the properties of rank 3 hyperbolic generalized Cartan matrix of noncompact type.

Corollary 6.2. *Let A be a rank 3 symmetrizable hyperbolic generalized Cartan matrix of noncompact type. Let $W = W(A)$ be the corresponding Weyl group, let $G = G_A(\mathbb{F}_q)$ be a complete Kac-Moody group corresponding to A over the finite field \mathbb{F}_q , and let X denote the Tits building of G . Then we have the following equivalent conditions.*

- (a) *The Dynkin diagram for A has an $A_1^{(1)}$ or $A_2^{(2)}$ proper connected subdiagram.*
- (b) *A has a proper indecomposable affine submatrix $B = (b_{ij})$ such that for some i and j , $i, j \in \{1, \dots, l\}$, $i \neq j$, $b_{ij}b_{ji} = 4$.*
- (c) *Every infinite parabolic subgroup of W is dihedral.*
- (d) *X has a noncompact link which contains a linear subtree $\cong \mathbb{Z}$.*

7 The Haagerup property for locally compact Kac-Moody groups

Let A be an $\ell \times \ell$ symmetrizable generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field \mathbb{F}_q ([CG], [Ti1], [Ti2]). Let X be the Tits

building of G . If q is sufficiently large we recall that G contains nonuniform lattice subgroups ([CG] and [Re1]) for $\ell \geq 3$, and both cocompact and nonuniform lattice subgroups if $\ell = 2$, with no restriction on q ([CG] and [CC]).

We recall the following conditions from [DJ], in order to define the class of group actions on buildings denoted $\mathcal{B}+$ of [DJ].

B1. 0-dimensional links in X (i.e., links of faces of codimension 1) are finite.

B2. Links of simplices in X are connected in the following sense: for any two chambers in a link, there exists a path of chambers connecting them, such that each pair of consecutive elements of the path share a face of codimension 1.

B3. The link of every simplex in X is either compact or contractible; in particular, this holds for $X = Lk(\emptyset)$.

B4. The group G acts transitively on chambers in X and the quotient map $X \rightarrow X/G$ restricts to an isomorphism on each chamber.

B δ . 1-dimensional links are compact and the nonzero eigenvalues of the Laplacian on 1dimensional links are $\geq 1 - \delta$,

The class $\mathcal{B}+$ of [DJ] is then defined to be all pairs (X, G) for groups G acting on buildings X satisfying **B1** – **B4** and **B δ** for $\delta = \frac{13}{28^n}$, where n is the dimension of the Tits building X .

Lemma 7.1. *Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field \mathbb{F}_q . Let X be the Tits building of G . If A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices, and q is sufficiently large, then (X, G) is in the class $\mathcal{B}+$ of [DJ].*

Proof: The hypothesis that A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices ensures that all entries of the Coxeter matrix associated to A are finite. We then apply Proposition A of [DJ] to conclude that (X, G) is in the class $\mathcal{B}+$. \square

Proposition 7.2. *Let A be a symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field \mathbb{F}_q . Let $\Gamma \leq G$ be a lattice subgroup of G .*

(1) *If A has an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrix (not necessarily proper), then Γ does not have Property (T).*

(2) *If A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices, and q is sufficiently large, then Γ has Property (T).*

Proof: For (1), the hypothesis that A has an $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrix (not necessarily proper) implies that X has a noncompact 1-dimensional link by Corollary 6.2. We then apply part (1) of Corollary G of [DJ] to conclude that Γ does not have Property (T). Part (2) follows from Lemma 7.1 and (2) of Corollary G of [DJ]. \square

One of our main tools will be the following theorem of [DJ].

Theorem 7.3. ([DJ], Thm 10.1) *Let X be the building of a BN-pair (G, B, N) such that X has sufficiently large finite thickness. The following conditions are equivalent.*

- (a) G acts with a proper orbit on a product of trees.
- (b) G has the Haagerup property.
- (c) Every closed Property (T) subgroup of G is compact.
- (d) For every simplex σ of X with noncompact link, there is a codimension 2 simplex containing σ with noncompact link.
- (e) Every infinite parabolic subgroup of the Weyl group of G contains an infinite parabolic dihedral subgroup.

We now deduce our main theorems :

Theorem 7.4. *Let A be an $\ell \times \ell$ symmetrizable hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field \mathbb{F}_q , with q sufficiently large. Let Γ be a lattice subgroup of G . If the rank of A is 2, or if the rank of A is 3 and A has noncompact hyperbolic type, then Γ has the Haagerup property. Hence G satisfies the Baum-Connes conjecture with coefficients in any C^* -algebra.*

Proof: If the rank of A is 2, then (e) of [DJ] Theorem 10.1 holds, so Γ has the Haagerup property. If the rank of A is 3 and A is of noncompact hyperbolic type, by Proposition 6.1 and Corollary 6.2, if A is a 3×3 generalized Cartan matrix of noncompact hyperbolic type then every infinite parabolic subgroup of the Weyl group of G contains an infinite parabolic dihedral subgroup. By [DJ], Theorem 10.1, Γ has the Haagerup property. \square

Theorem 7.5. *Let A be an $\ell \times \ell$ symmetrizable affine or hyperbolic generalized Cartan matrix. Let $G = G_A(\mathbb{F}_q)$ be a completion of Tits' functor associated to A and the finite field \mathbb{F}_q , with q sufficiently large. Let Γ be a lattice subgroup of G . If $r = \text{rank}(G) = 3$ and G has compact hyperbolic type, or if $\text{rank}(G) \geq 3$ and G has affine type, or if $4 \leq r \leq 10$ and G has hyperbolic type then*

- (i) Γ has Property (T).
- (ii) All entries in the Coxeter matrix for G are finite.

Proof: For (i), if the rank of A is 3 and A is of compact hyperbolic type, then A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices. By (2) of Proposition 7.2, Γ has Property (T). If the rank of A is ≥ 4 , we use Proposition 6.1 to deduce that A has no $A_1^{(1)}$ or $A_2^{(2)}$ indecomposable submatrices. By (2) of Proposition 7.2, Γ has Property (T).

For (ii), since G is affine or hyperbolic, the Coxeter matrix for G contains an ∞ if and only if the generalized Cartan matrix A contains an $A_1^{(1)}$ or $A_2^{(2)}$ proper indecomposable submatrix. By Proposition 6.1 this occurs if and only if $\text{rank } A = 3$ and A has noncompact type, which is impossible under the assumptions here. \square

Since Γ (and hence G) has Property (T), a theorem of de la Harpe and Valette tells us that if a Property (T) group acts on a tree, then the group must fix a vertex ([HV]). Thus Property (T) for a group G implies property (FA) of Serre for G ([S]), namely that any action of G on a simplicial tree must fix a vertex.

Corollary 7.6. *Property (T) and the Haagerup property for symmetrizable locally compact affine or hyperbolic Kac-Moody groups over sufficiently large finite fields can be determined from the Dynkin diagram, or equivalently from the generalized Cartan matrix.*

8 The Baum-Connes conjecture for higher rank Kac-Moody groups

Let G be a symmetrizable locally compact Kac-Moody group of rank 3 compact hyperbolic type or of affine or hyperbolic type and rank ≥ 4 . As we have shown, G has Property (T). In this section, we deduce from the work of Lafforgue ([L1]-[L3]) that G satisfies the Baum-Connes conjecture without coefficients.

Lafforgue proved the first known cases of the Baum-Connes conjecture for infinite groups with Kazhdan's Property (T) ([L1]-[L3]). He constructed equivariant KK -theory for Banach algebras rather than C^* -algebras and established the conjecture for various completions of the L^1 algebras of these groups.

The Kac-Moody building X of G is a $CAT(0)$ space. Thus X is a 'bolic space' in the sense of Kasparov and Skandalis ([KS]). By [L1]-[L3], groups acting on $CAT(0)$ spaces are 'strongly bolic' which provides a strengthening of the Kasparov-Skandalis notion of bolicity. For these groups, the Baum-Connes assembly map is injective ([L1]-[L3]).

In order to discuss surjectivity of the Baum-Connes assembly map, we must make use of Kasparov's fundamental idempotent γ in Kasparov's representation ring for which $1-\gamma$ measures the deviation of the Baum-Connes assembly map from surjectivity. In order to prove that the Baum-Connes assembly map is surjective, one must prove that $\gamma = 1$ as an endomorphism of the K -theory of the reduced C^* -algebra $C_r^*(G)$ of G .

In [L3], Lafforgue establishes the equality $\gamma = 1$ in his equivariant Banach KK -theory for every locally compact group acting properly and isometrically on strongly bolic spaces, in particular for locally compact groups acting properly and isometrically on buildings.

Thus for all symmetrizable locally compact affine or hyperbolic Kac-Moody groups G , the proper isometric action of G on its Tits building leads to surjectivity of the Baum-Connes assembly map on equivariant K -homology of G . We have the following.

Theorem 8.1. *Let G be a symmetrizable locally compact Kac-Moody group of rank 3 compact hyperbolic type or of affine or hyperbolic type and rank ≥ 4 . Then the Baum-Connes assembly map on equivariant K -homology of G is both injective and surjective. Thus G satisfies the Baum-Connes conjecture without coefficients.*

We recall that for symmetrizable locally compact Kac-Moody group G of rank 2 or of rank 3 noncompact hyperbolic type, G satisfies the Haagerup property and hence the Baum-Connes conjecture with coefficients.

9 The Haagerup property, the Baum-Connes conjecture and actions on trees

Suppose that a group Γ acts on a tree X without inversions. In [Tu] the author introduces property (BC') for discrete groups, which implies the Baum-Connes conjecture with coefficients and the K -amenability of the group. He then shows that if Γ is a discrete group which acts on a tree X such that $\Gamma \backslash X$ is compact, and the stabilizers of the vertices and the stabilizers of the edges satisfy (BC'), then Γ itself satisfies (BC').

In [O-O] the author proves that the Baum-Connes conjecture with coefficients, for groups acting on oriented trees, is true if and only if the stabilizers of vertices satisfy the Baum-Connes conjecture.

Concerning the Haagerup property, a theorem of Haagerup ([H]) states that a free group Γ has the Haagerup property. It follows that if Γ is a discrete group acting freely on the vertices of a Bruhat-Tits building X of rank 2, that is, a homogeneous or bi-homogeneous tree, then Γ is a free group and hence has the Haagerup property. If Γ is a lattice in G , then G has the Haagerup property ([CCJJV]).

We may also ask: If a locally compact group G acts on a tree, when does G have the Haagerup property? This question has been answered in [CMV] where the authors introduce spaces with 'measured walls', generalizing Haglund and Paulin's spaces with 'walls' ([HP]). They show that if a locally compact group G acts on a space with measured walls, then G has the Haagerup property. They also conjecture that the converse is true, proving the converse in a number of cases, including the class of discrete groups with the Haagerup property.

This conjecture has been proven in [CDH] where the authors prove that a group G has the Haagerup property if and only if it admits a proper continuous action by isometries on a 'median space'. A median space of [CDH] is a metric space for which, given any triple of points, there exists a unique median point, that is a point which is simultaneously between any two points in that triple. Simplicial trees are examples of median spaces. Spaces with measured walls are naturally endowed with a (pseudo-)metric. In [CDH] the authors showed that a (pseudo-)metric on a space is induced by a structure of measured walls if and only if it is induced by an embedding of the space into a median space.

Having determined which symmetrizable locally compact affine or hyperbolic Kac-Moody groups satisfy Property (T) and those that satisfy the Haagerup property, we ask if we can *construct* proper actions on trees for groups with the Haagerup property. For a rank 2 Kac-Moody group G over a finite field \mathbb{F}_q , in [CG] the authors constructed a proper action of a nonuniform lattice $B^- \leq G$ on the Tits building of X , the homogeneous tree X_{q+1} .

We mention also Proposition 2.1 of [DJ] where the authors showed that lattices in groups without Property (T) admit actions on trees without fixed points.

Proposition 2.1 of [DJ] *Let X be a building corresponding to a BN-pair for a group G . Suppose that one of the entries of the Coxeter matrix is ∞ . Let H be a subgroup of finite covolume in $G = \text{Aut}_+(X)$. Then H acts without a fixed point on an infinite tree.*

In this section we exhibit actions with finite vertex stabilizers of certain nonuniform lattices in rank 3 locally compact Kac-Moody groups of noncompact hyperbolic type on simplicial trees. We will consider 2 possible generalized Cartan matrices:

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

In each of these cases A has noncompact hyperbolic type and the Weyl group W is

$$W(A) = \langle w_1, w_2, w_3 \mid w_i^2, (w_1 w_2)^2 = 1, (w_2 w_3)^3 = 1, (w_1 w_3)^\infty = 1 \rangle,$$

which is isomorphic to $PGL_2(\mathbb{Z})$. In the corresponding Tits building, each apartment is a copy of the hyperbolic plane, tessellated by the action of $PGL_2(\mathbb{Z})$. The action of W on the upper-half plane is given by:

If $ad - bc = 1$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d},$$

and if $ad - bc = -1$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{a\bar{z} + b}{c\bar{z} + d}.$$

The building X_G has 3 types of vertices, denoted Q_1, Q_2 and Q_3 , corresponding to the maximal parabolic subgroups of G . We have $Q_1 = B\langle w_1, w_2 \rangle B$, $Q_2 = B\langle w_2, w_3 \rangle B$, $Q_3 = B\langle w_1, w_3 \rangle B$. For a vertex Q_i of type i , the intersection of the link of Q_i with the standard apartment \mathcal{A}_0 is a closed circuit of length 4 if $i = 1$ and of length 6 for $i = 2$. For $i = 3$, the intersection of \mathcal{A}_0 with the link ‘at ∞ ’ of Q_3 is a bi-infinite line isomorphic to \mathbb{Z} . We let P_1, P_2 and P_3 denote the parabolic subgroups $P_1 = B\langle w_1 \rangle B$, $P_2 = B\langle w_2 \rangle B$ and $P_3 = B\langle w_3 \rangle B$.

Let Y be the tree of $PSL_2(\mathbb{Z})$ in \mathbb{H}^2 , a barycentric subdivision of the trivalent tree. The group $PSL_2(\mathbb{Z})$ has torsion in Y at the vertices which are $PSL_2(\mathbb{Z})$ -translates of the arc from $e^{-i\pi/3}$ to i along the geodesic $x^2 + y^2 = 1$ bounding the fundamental chamber in \mathbb{H}^2 for the action of the extended modular group $PGL_2(\mathbb{Z})$. Every apartment \mathcal{A} of X_G contains an isomorphic copy of Y denoted $Y_{\mathcal{A}}$. Then

$$\mathcal{Y} = \bigcup_{\text{apartments } \mathcal{A} \text{ of } X_G} Y_{\mathcal{A}}$$

is connected, and is a tree on which G acts without inversions. We now consider the action of the group B^- on \mathcal{Y} . The tree \mathcal{Y} is locally finite over \mathbb{F}_q , except at its boundary, and has two types of vertices - those inherited from vertices of degree 2 and of degree 3 in Y . We denote these two types of vertices by $V\mathcal{Y}_I$ and $V\mathcal{Y}_{II}$ respectively.

We may identify the vertices $V\mathcal{Y}_I$ with the union of cosets $\sqcup_{i=1}^3 G/P_i$ which is the set of edges EX_G and $V\mathcal{Y}_{II}$ with $\sqcup_{i=1}^3 G/Q_i$ which is the set of vertices VX_G . We wish to show that the isotropy groups of the (non-boundary) vertices for the action of B^- on \mathcal{Y} are finite. It suffices to do this for the vertices on the standard apartment \mathcal{A}_0 since all other vertices (cosets gP_i and gQ_i) are conjugate to these. Each vertex on the interior of the standard apartment is a coset of the form wP_i or wQ_i , $w \in W$, $i = 1, 2$. We let Γ denote the group B^- , and let Γ_{wB} , $w \in W$, denote the isotropy group of the chamber wB :

$$\Gamma_{wB} = \{ \gamma \in \Gamma \mid \gamma wB = wB \}.$$

From [CG], we have that if G is constructed over the finite field \mathbb{F}_q , then

$$|\Gamma_{wB}| = q^{l(w)}(q-1)^3,$$

where $l(\cdot)$ is the length function of the Weyl group W . It remains to show that for a vertex wP_i or wQ_i , $i = 1, 2$, $w \in W$, the groups Γ_{wP_i} and Γ_{wQ_i} are finite. But each coset wP_i or wQ_i is its own stabilizer, so $\Gamma_{wP_i} = \Gamma \cap wP_i$ and $\Gamma_{wQ_i} = \Gamma \cap wQ_i$. Then $\Gamma_{wB} \leq \Gamma_{wP_i} \leq \Gamma_{wQ_i}$ and Γ_{wB} is a subgroup of finite index in both Γ_{wP_i} and Γ_{wQ_i} . Since $|\Gamma_{wB}| = q^{l(w)}(q-1)^3 < \infty$, Γ_{wP_i} and Γ_{wQ_i} are finite groups. Thus we have constructed a proper action of B^- on a locally finite tree which is a space with measured walls in the sense of [CMV].

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