

FUNDAMENTAL DOMAINS FOR CONGRUENCE SUBGROUPS OF SL_2 IN POSITIVE CHARACTERISTIC

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ABSTRACT. Morgenstern ([Mor95]) claimed to have constructed fundamental domains for congruence subgroups of the lattice group $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$, and subgraphs providing the first known examples of linear families of bounded concentrators. His method was to construct the fundamental domain for a congruence subgroup as a ‘ramified covering’ of the fundamental domain for Γ on the Bruhat-Tits tree $X = X_{q+1}$ of $G = \mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))$. We prove that Morgenstern’s constructions do not yield the desired ramified coverings, and in particular yield graphs that are not connected in characteristic 2. It follows that Morgenstern’s graphs cannot be quotient graphs by the action of congruence subgroups on the Bruhat-Tits tree. Moreover, subgraphs of Morgenstern’s graphs which he claims to be expanders are also not connected.

We clarify the construction of Morgenstern and we prove that his full graphs are connected only in odd characteristic. We also repair his constructions of ramified coverings. We construct fundamental domains for congruence subgroups of $SL_2(\mathbb{F}_q[t])$ and $\mathrm{PGL}_2(\mathbb{F}_q[t])$ as ramified coverings, and we give explicit graphs of groups for a number of congruence subgroups. We thus provide new families of graphs whose level $0 - 1$ subgraphs potentially have the expansion properties claimed by Morgenstern.

1. INTRODUCTION

Morgenstern ([Mor95]) claimed to have constructed fundamental domains of lattices for congruence subgroups of the group $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$ which is a nonuniform lattice subgroup of $G = \mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))$. These congruence subgroups have the form

$$\Gamma(g) = \{A \in \mathrm{PGL}_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \pmod{g}\}$$

for some $g \in \mathbb{F}_q[t]$. His method was to construct the fundamental domain for $\Gamma(g)$ as a ‘ramified covering’ of the fundamental domain for Γ on the Bruhat-Tits tree $X = X_{q+1}$ of $G = \mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))$. This method for producing the fundamental domain as a ramified covering is consistent with the theory of branched topological coverings and, in Morgenstern’s setting, coincides with a method suggested by Drinfeld in his theory of modular curves over function fields ([Dri77]). Gekeler and Nonnengardt ([GN95]) and Rust ([Rus98]) give similar constructions of fundamental domains of lattices for congruence subgroups using essentially the same method.

Morgenstern’s motivation was to provide the first known examples of linear families of bounded concentrators. These are claimed in [Mor95] to be subgraphs $D_g(0 - 1)$ induced by the levels $0 - 1$ in the fundamental domains of lattices for congruence subgroups $\Gamma(g)$ for $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q[t])$.

We prove however that Morgenstern’s constructions do not yield the desired ramified coverings, and in particular yield graphs that are not connected in characteristic 2. Since the

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fundamental domain is the quotient graph by the action of a group on a tree, the quotient must be connected. It follows that Morgenstern's graphs cannot be quotient graphs by the action of congruence subgroups on the Bruhat-Tits tree. Moreover the subgraphs at levels 0 – 1, which he claims are expanders, are not even connected in characteristic 2. We prove that Morgenstern's full graphs are connected in odd characteristic, but we have not verified that the subgraphs at levels 0 – 1 are connected in odd characteristic, nor that they have the claimed expansion properties.

The main source of Morgenstern's error was his incorrect assumption that $\Gamma/\Gamma(g) \cong \text{PGL}_2(R_g)$ where $R_g = \mathbb{F}_q[t]/(g)$. The correct formula for $\Gamma/\Gamma(g)$ is somewhat more complicated and is given in section 4. In sections 4 and 5, we repair Morgenstern's method of constructing ramified coverings to give fundamental domains of congruence subgroups of $\text{PGL}_2(\mathbb{F}_q[t])$ and $\text{SL}_2(\mathbb{F}_q[t])$. When Morgenstern's fundamental domains for congruence subgroups as ramified coverings are not connected, we show that all connected components of the graphs are isomorphic and that there is a group acting freely by permuting the components. We give a number of explicit examples and we provide full graphs of groups descriptions of congruence subgroups of SL_2 . We thus provide new families of subgraphs which potentially have the expansion properties claimed by Morgenstern, though we have not verified this. Some of our results on the construction of graphs of groups for congruence subgroups of SL_2 are included in the work of Gekeler-Nonnengardt ([GN95]) and Rust ([Rus98]). However Gekeler-Nonnengardt and Rust did not verify that their constructions yield connected graphs, though this will follow from the methods we give here.

We remark that the method of constructing a fundamental domain for a congruence subgroup as a ramified covering is unclear in the settings of Morgenstern, Gekeler-Nonnengardt and Rust. It is straightforward to verify that correctly constructing a ramified covering for the action of $\Gamma(g)$ on X over $\Gamma \backslash X$ gives rise to a graph of groups $\Gamma(g) \backslash\backslash X$ with fundamental group isomorphic to $\Gamma(g)$, quotient graph $\Gamma(g) \backslash X$, and universal covering tree of groups isomorphic to X . Applying the Bass-Serre theory for reconstructing group actions on trees thus gives an equivalence between the graph of groups $\Gamma(g) \backslash\backslash X$ and the action of $\Gamma(g)$ on X . Thus the ramified covering for $\Gamma(g)$ on X over $\Gamma \backslash X$ should coincide with the quotient graph, $\Gamma(g) \backslash X$, of X by the action of $\Gamma(g)$, though we have not included a detailed proof here.

The structural properties of the quotient graphs obtained as ramified coverings are difficult to determine and detailed drawings of these graphs are non-trivial to obtain. We use the Magma computer algebra system ([BC97]) to construct explicit examples and to carry out computations. This involves a number of features of Magma including finite matrix groups, graph isomorphism ([McK81]), and finite geometries ([JL04]). We drew some of the resulting graphs with the program `dot` which is part of the Graphviz graph visualization system ([GN00]).

Work of Lubotzky ([Lub91] and [Lub90]) and Ragunathan ([Rag89]) indicates that the quotient $\Gamma \backslash X$ of the Bruhat-Tits tree $X = X_{q+1}$ by a nonuniform lattice subgroup Γ of $G = \text{SL}_2(\mathbb{F}_q((t^{-1})))$ consists of a finite core graph together with finitely many cusps, which are semi-infinite rays. The work of Morgenstern, Gekeler-Nonnengardt and Rust shows that, for congruence subgroups of $\text{PGL}_2(\mathbb{F}_q[t]) \leq \text{PGL}_2(\mathbb{F}_q((t^{-1})))$, the core graph is a $(q + 1)$ -regular bipartite graph.

We are indebted to Gunther Cornelissen for extremely helpful discussions which led us to the completion of this work. We are also grateful to Gunther for notifying us of the work of Gekeler-Nonnengardt and Rust, and for referring us to Max Gebhardt whose independent computations done a number of years ago also show that Morgenstern's graphs are not connected. We thank

Max Gebhardt for providing us with the details of his unpublished computations ([Geb08]). We thank Dimitri Leemans for helping us with theory and computation for coset graphs.

2. PRELIMINARIES ON FUNDAMENTAL DOMAINS

Our objective in this work is to describe the fundamental domains of certain congruence subgroups of $\mathrm{SL}_2(\mathbb{F}_q[t])$ and $\mathrm{PGL}_2(\mathbb{F}_q[t])$ with respect to the action of these groups on a tree. In this section we give a brief summary of the definition of a fundamental domain of a group acting on a tree and we establish some notation.

2.1. Fundamental Domains and Ramified Coverings. For our purposes, a *graph* A consists of vertices $V(A)$, oriented edges $E(A)$, initial and terminal functions that pick out the endpoints of an edge, and an involution on the edge set that is fixed point free and reverses the orientation. Our graphs are connected and locally finite.

Suppose a group Γ acts on a tree X . Taking a quotient by the action of Γ yields a graph $\Gamma \backslash X$ whose vertices and edges correspond to the Γ -orbits of vertices and edges of X . Initial and terminal vertices of edges in the orbit $\Gamma \cdot e$ are the Γ -orbits for the initial and terminal vertices of e in X . We call $\Gamma \backslash X$ the *quotient graph* of Γ with respect to its action on X . There is a natural quotient morphism $X \rightarrow \Gamma \backslash X$. We will often identify $\Gamma \backslash X$ with a preimage in X ; this subgraph is called the *fundamental domain* of Γ on X .

Given a normal subgroup N in Γ , we can define the quotient graph $N \backslash X$ by

$$\begin{aligned} V(N \backslash X) &= N \backslash V(X) = \{N \cdot v \mid v \in V(X)\}, \\ E(N \backslash X) &= N \backslash E(X) = \{N \cdot e \mid e \in E(X)\}. \end{aligned}$$

Then Γ/N acts on $N \backslash X$ by

$$\gamma N(N \cdot x) = N \cdot \gamma x,$$

where x denotes either a vertex or an edge of X .

Equivalently, we can take the graph $N \backslash X$ to have vertices (respectively edges) given by cosets of $\mathrm{Stab}_\Gamma(x)\Gamma/N$ in Γ/N , $x \in V(\Gamma \backslash X)$ (respectively cosets of $\mathrm{Stab}_\Gamma(e)\Gamma/N$ in Γ/N , $e \in E(\Gamma \backslash X)$). Cosets are adjacent as vertices in the graph $N \backslash X$ if and only if their intersection is non-empty. We call this construction of $N \backslash X$ a *ramified covering* over $\Gamma \backslash X$. In all of our applications, $\Gamma \backslash X$ will be a semi-infinite ray.

2.2. Fundamental Domain for $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$. Let Γ denote the subgroup $\mathrm{SL}_2(\mathbb{F}_q[t])$ of $G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$. The Bruhat-Tits building of G is the $(q+1)$ -homogeneous tree $X = X_{q+1}$ ([Ser03]). The vertices of X are the conjugates of the standard parabolic subgroups P_1 and P_2 in G , which are defined as

$$P_1 = \mathrm{SL}_2(\mathcal{O}), \quad P_2 = \left\{ \begin{pmatrix} a & tb \\ c/t & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \right\},$$

where

$$\mathcal{O} = \mathbb{F}_q[[t^{-1}]] = \left\{ \sum_{n \geq 0} a_n t^{-n} \mid a_n \in \mathbb{F}_q \right\}.$$

The minimal parabolic, or *Iwahori* subgroup is defined as

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \mid c \equiv 0 \pmod{(t^{-1})} \right\}.$$

If Q_1 and Q_2 are vertices, then there is an edge connecting Q_1 and Q_2 if and only if $Q_1 \cap Q_2$ contains a conjugate of B . We have an action of G on X by conjugation. The conjugates of P_1 , P_2 and B in G are in bijective correspondence with the cosets of P_1 , P_2 and B in G . We obtain the following description of the Bruhat-Tits tree X ([Ser03]):

$$\begin{aligned} V(X) &= G/P_1 \sqcup G/P_2, \\ E(X) &= G/B \sqcup \overline{G/B}, \end{aligned}$$

where $\overline{G/B}$ is a copy of the set G/B , giving an orientation to $E(X)$, so that positively oriented edges come from G/B , and negatively oriented edges come from $\overline{G/B}$.

The graph $\mathrm{SL}_2(\mathbb{F}_q[t]) \backslash X$ is a semi-infinite ray of vertices with ascending chains of vertex and edge stabilizers, characterized by the following.

Proposition 2.1 (Proposition 3, p. 87, [Ser03]). *Let $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t]) \leq G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$. Let $X = X_{q+1}$ be the Bruhat-Tits tree of G . Let $\Lambda_0, \Lambda_1, \Lambda_2, \dots$ be the vertices of the semi-infinite ray which is a fundamental domain for Γ on X . Let $\Gamma_0 = \mathrm{SL}_2(\mathbb{F}_q)$ and, for $n \geq 1$, let*

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q[t], \deg(b) \leq n \right\}.$$

- (a) *The vertices Λ_n are pairwise inequivalent mod Γ .*
- (b) *Γ_n is the stabilizer of Λ_n in Γ .*
- (c) *Γ_0 acts transitively on the set of edges with origin Λ_0 .*
- (d) *For $n \geq 1$, Γ_n leaves the edge $(\Lambda_n \rightarrow \Lambda_{n+1})$ fixed, and acts transitively on the set of edges in X with origin Λ_n which are distinct from $\Lambda_n \rightarrow \Lambda_{n+1}$.*

Proposition 2.1 gives the fundamental domain for Γ on X as a semi-infinite ray with vertex sequence

$$\Lambda_0, \Lambda_1, \Lambda_2, \dots$$

We will construct fundamental domains for congruence subgroups of Γ as ramified coverings over $\Gamma \backslash X$. Since these subgroups are normal, we have an action by the quotient groups, as indicated in the following lemma.

Lemma 2.2 ([DD89]). *Let Γ be a group and X a tree. Suppose Γ acts on X . If N is a normal subgroup of Γ , then Γ/N acts on the connected graph $N \backslash X$. Each $Nx \in N \backslash X$ has stabilizer*

$$\mathrm{Stab}_{\Gamma/N}(Nx) = N \mathrm{Stab}_\Gamma(x)/N.$$

Therefore, given a normal subgroup N of Γ , we may describe the vertices (respectively edges) of $N \backslash X$ not only as N -orbits with respect to the action of N on X , but as Γ/N -orbits of $\{Nv : v \in V(\Gamma \backslash X)\}$ (respectively of $\{Ne : e \in E(\Gamma \backslash X)\}$). That is, we may construct $N \backslash X$ as a ramified covering of $\Gamma \backslash X$.

Although our graphs are directed, for all vertices u and v there is an edge $v \rightarrow u$ if and only if there is an edge $u \rightarrow v$. So we can usually treat the graphs as if they were undirected. Note however that the orientation is required for graphs of groups.

2.3. Graphs of Groups. A *graph of groups* $\mathbb{A} = (A, \mathcal{A}_v, \mathcal{A}_e, \alpha_e)$ over a connected graph A consists of an assignment of vertex groups \mathcal{A}_v for each $v \in V(A)$ and edge groups $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$ for each $e \in E(A)$, together with monomorphisms $\alpha_e : \mathcal{A}_e \rightarrow \mathcal{A}_u$ for each edge $e : u \rightarrow v$.

Conversely, when a group G acts on tree X there is a graph of groups $G \backslash \backslash X$ built on the quotient $A = G \backslash X$. For each vertex $v \in V(A)$ (respectively edge $e \in E(A)$) there is an

associated group, namely the stabilizer in G of a lifting of v (respectively e) to X . The edge-monomorphisms α_e are inclusion maps. For example, the graph of groups for $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$ is the infinite ray whose vertex and edge groups are given by Γ_i as described in Proposition 2.1.

3. FUNDAMENTAL DOMAINS FOR CONGRUENCE SUBGROUPS OF $\mathrm{SL}_2(\mathbb{F}_q[t])$ AS RAMIFIED COVERINGS

In this section we construct quotient graphs for congruence subgroups of $\Gamma = \mathrm{SL}_2(\mathbb{F}_q[t])$ acting on the Bruhat-Tits tree $X = X_{q+1}$ of $G = \mathrm{SL}_2(\mathbb{F}_q((t^{-1})))$ as ramified coverings over $\Gamma \backslash X$.

Fix $g \in \mathbb{F}_q[t]$ of degree n . Since

$$\Gamma(g) = \{A \in \mathrm{PGL}_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \pmod{g}\}$$

is normal in Γ , the quotient graph $X_g = \Gamma(g) \backslash X$ may be viewed as a ramified covering of the quotient graph $\Gamma \backslash X$, which is a semi-infinite ray.

3.1. The levels of X_g . Since $\Gamma \backslash X$ is a semi-infinite ray of vertices Λ_i (see Proposition 2.1), we can partition $V(X)$ into Γ -orbits of the Λ_i . Since $\Gamma_i = \mathrm{Stab}_\Gamma(\Lambda_i)$ by Proposition 2.1, the orbit $\Gamma \cdot \Lambda_i$ is in one-to-one correspondence with Γ/Γ_i . Similarly the edges between $\Gamma \cdot \Lambda_i$ and $\Gamma \cdot \Lambda_{i+1}$ correspond to $\Gamma/(\Gamma_i \cap \Gamma_{i+1})$. So we make the identifications

$$\begin{aligned} V(X) &= \bigsqcup_{i \geq 0} \Gamma/\Gamma_i, \\ E(X) &= \bigsqcup_{i \geq 0} \Gamma/(\Gamma_i \cap \Gamma_{i+1}). \end{aligned}$$

We can now describe the vertices and edges of $X_g = \Gamma(g) \backslash X$ as follows:

$$\begin{aligned} V(X_g) &= \bigsqcup_{i \geq 0} \Gamma(g) \backslash (\Gamma/\Gamma_i), \\ E(X_g) &= \bigsqcup_{i \geq 0} \Gamma(g) \backslash (\Gamma/(\Gamma_i \cap \Gamma_{i+1})). \end{aligned}$$

Define groups $H = \Gamma/\Gamma(g)$ and $H_i = \Gamma_i\Gamma(g)/\Gamma(g)$, and coset spaces $L_i = H/H_i$. By Proposition 2.1 we have $\mathrm{Stab}_\Gamma(\Lambda_i) = \Gamma_i$ and so

$$\mathrm{Stab}_H(\Gamma(g) \cdot \Lambda_i) = (\Gamma_i\Gamma(g))/\Gamma(g) = H_i.$$

So we can identify the set of vertices $\Gamma(g) \backslash (\Gamma/\Gamma_i)$ with L_i . We call these the vertices at *level* i . Similarly the edges $\Gamma(g) \backslash (\Gamma/(\Gamma_i \cap \Gamma_{i+1}))$ are identified with $H/(H_i \cap H_{i+1})$.

3.2. The structure of H . In this subsection, we show that

$$H = \Gamma/\Gamma(g) \cong \mathrm{SL}_2(R_g)$$

where $R_g = \mathbb{F}_q[t]/(g)$. The argument is the same as in ([Shi94]) for the classical setting $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.1. *The map $\mathrm{SL}_2(\mathbb{F}_q[t]) \rightarrow \mathrm{SL}_2(R_g)$ given by $A \mapsto A \pmod{(g)}$ is surjective.*

Proof. Let $\bar{A} \in \mathrm{SL}_2(R_g)$ and let $A \in M_2(\mathbb{F}_q[t])$ be a matrix such that $A \bmod (g) = \bar{A}$. We seek a matrix in $\mathrm{SL}_2(\mathbb{F}_q[t])$ which is congruent to $A \bmod g$. By the Smith normal form ([DF04]) there exist matrices $U, V \in \mathrm{SL}_2(\mathbb{F}_q[t])$ such that UAV is diagonal. Then $UAV = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, with $ad \equiv 1 \pmod{(g)}$. Let $B = \begin{pmatrix} a & ad-1 \\ 1-ad & 2d-ad^2 \end{pmatrix}$. Then $\det(B) = 1$, so $B \in \mathrm{SL}_2(\mathbb{F}_q[t])$. Moreover $B \equiv UAV \pmod{(g)}$. Therefore $U^{-1}BV^{-1} \in \mathrm{SL}_2(\mathbb{F}_q[t])$ and $U^{-1}BV^{-1} \equiv A \pmod{(g)}$, as desired. \square

Write $g = \prod_{i=1}^s g_i^{n_i}$ where the g_i are distinct irreducible polynomials with $\deg(g_i) = d_i$ and $\sum_i n_i d_i = n$. Then

$$R_g \cong \bigoplus_{i=1}^s R_i \quad \text{where} \quad R_i := R_{g_i^{n_i}} \cong \mathbb{F}_{q^{d_i}}[t_i]/(t_i^{n_i}).$$

By Corollary 2.4 of [Han],

$$R_g^\times \cong \prod_i R_i^\times \quad \text{and} \quad \mathrm{GL}_2(R_g) \cong \prod_i \mathrm{GL}_2(R_i^\times).$$

Using Theorem 2.7(3) of [Han], we now get

$$|\mathrm{GL}_2(R_g)| = q^{4n} \prod_i \left(1 - \frac{1}{q^{2d_i}}\right) \left(1 - \frac{1}{q^{d_i}}\right) \quad \text{and}$$

$$|R_g^\times| = q^n \prod_i \left(1 - \frac{1}{q^{d_i}}\right).$$

Hence

$$|H| = |\mathrm{SL}_2(R_g)| = \frac{|\mathrm{GL}_2(R_g)|}{|R_g^\times|} = q^{3n} \Pi(q),$$

where $\Pi(q) := \prod_i \left(1 - \frac{1}{q^{2d_i}}\right)$.

3.3. The structure of X_g . We can describe X_g as a *levelled coset graph* given by

$$H_0, H_1, \dots \leq H.$$

That is the vertices at level i correspond to cosets hH_i , for $h \in H$, and hH_i and kH_{i+1} are connected by an edge if and only if $hH_i \cap kH_{i+1} \neq \emptyset$. From level n on, X_g becomes a collection of disjoint infinite rays beginning at each vertex of level L_{n-1} , so it suffices to describe the graph induced by the first n levels. It is also useful to note that $\Gamma_i \cap \Gamma(g) = \{1\}$ for $i \leq n-1$. Thus for $i \leq n-1$ we may treat the H_i as matrix groups $H_i = \Gamma_i$ and consider the levelled coset graph given by $H_0, H_1, \dots, H_{n-1} \leq H = \mathrm{SL}_2(R_g)$.

Proposition 3.2. *The levelled coset graph given by $H_0, H_1, \dots, H_{n-1} \leq H$ with*

$$H_1 \leq H_2 \leq \dots \leq H_{n-1}$$

has $|H : \langle H_0, H_{n-1} \rangle|$ connected components.

Proof. Suppose that $H = \langle H_0, H_{n-1} \rangle$. Clearly there is a path connecting $H_i a$ and $H_j a$ for every $i, j = 0, \dots, n-1$. Let $H_i a$ and $H_j b$ be two vertices, with a and b in the same coset of $\langle H_0, H_{n-1} \rangle$. Write

$$ba^{-1} = h_1 k_1 h_2 k_2 \cdots h_m k_m$$

for $h_l \in H_0$ and $k_l \in H_{n-1}$. Then we have a path from $H_i a$ to $H_{n-1} a = H_{n-1} k_m a$, to $H_0 k_m a = H_0 h_m k_m a$, to $H_{n-1} h_m a = H_{n-1} k_{m-1} h_m k_m a$, and so on, to $H_0 h_1 k_1 \cdots h_m k_m a = H_0 b$, and so to $H_j b$.

Conversely, given a path from $H_i a$ and $H_j b$, we can construct a path from $H_0 a$ to $H_0 b$ since $H_1 \leq H_2 \leq \cdots$. This second path gives us $a^{-1} b \in H$ as a word in elements of the groups H_0 and H_{n-1} . \square

Using this description of the levelled coset graph we can now prove:

Theorem 3.3. *The graph X_g is connected.*

Proof. Let $K = \langle H_0, H_{n-1} \rangle$. For all $a \in R_g^\times$ we have

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in K.$$

We now know that K contains all elementary matrices (ie, H_{n-1} , the elements just described, together with $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). Since an arbitrary element of $H = \text{SL}_2(R_g)$ can be reduced to the identity by row and column operations, this shows that $H = K$. \square

Now $H_0 = \text{SL}_2(q)$, and H_i is a semidirect product of $(\mathbb{F}_q^+)^{\min(n, i+1)}$ by \mathbb{F}_q^\times . So we have formulas for the number of vertices in each level:

$$|L_i| = \begin{cases} q^{3n-3} \Pi(q) \left(1 - \frac{1}{q^2}\right)^{-1} & \text{for } i = 0, \\ q^{3n-2-i} \Pi(q) \left(1 - \frac{1}{q}\right)^{-1} & \text{for } 0 < i < n, \\ q^{2n-2} \Pi(q) \left(1 - \frac{1}{q}\right)^{-1} & \text{for } i \geq n. \end{cases}$$

We can also compute the sizes of subgroups $H_i \cap H_{i+1}$ to find the number of edges between vertices in levels i and $i+1$, but we omit the details.

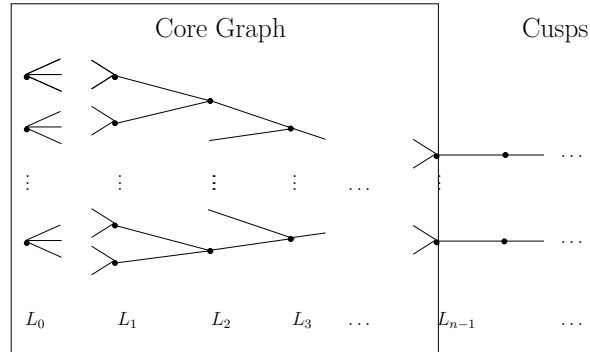


FIGURE 1. Schematic drawing of X_g

Figure 1 gives a schematic drawing of the graph X_g , whose properties are summarized below:

Remark 3.4.

- (1) X_g is bipartite since it is a quotient of the bipartite graph X . The sets $\bigsqcup_{i \text{ even}} L_i$ and $\bigsqcup_{i \text{ odd}} L_i$ form a bipartition of the vertex set.

- (2) The edges run between consecutive levels, with the edges between L_i and L_{i+1} projecting to the edge $\Lambda_i \rightarrow \Lambda_{i+1}$ in X .
- (3) The subgraph induced by L_0 and L_1 is a $(q+1, q)$ -regular bipartite graph.
- (4) For $i = 1, \dots, n-1$, each vertex in L_i has q edges to vertices in L_{i-1} and only 1 edge to a vertex in L_{i+1} . Thus the graph ‘collapses’ in a q -fold manner until it reaches level L_{n-1} .
- (5) For $i \geq n$, each vertex in L_i has one edge to L_{i-1} and one edge to L_{i+1} . So there is a semi-infinite ray, called a cusp, attached to each vertex in L_{n-1} .

Finally we describe the graph of groups $\Gamma(g) \backslash X$. We label each vertex and edge with its stabilizer under the action of $\Gamma(g)$. By Proposition 2.1,

$$\text{Stab}_{\Gamma(g)}(\Lambda_i) = \Gamma_i \cap \Gamma(g) = \begin{cases} \{1\} & \text{if } i < n \\ U_i = \left\{ \begin{pmatrix} 1 & gf \\ 0 & 1 \end{pmatrix} \mid f \in \mathbb{F}_q[t], \deg(f) \leq i - n \right\} & \text{if } i \geq n \end{cases}$$

The stabilizer of any vertex in L_i is then conjugate to $\Gamma_i \cap \Gamma(g)$. Thus the ‘core’ vertices are labeled with the trivial group, and the ‘cusp’ vertex groups along each ray are of the form

$$s_j U_i s_j^{-1},$$

where $\{s_j \mid j = 1, \dots, k = (q+1)q^{2(n-1)}\}$ is a set of conjugacy class representatives. See Figure 2.

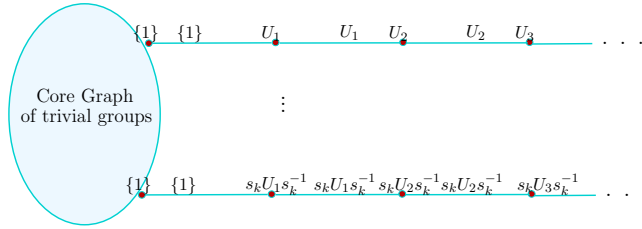


FIGURE 2. The core graph and the cusps

3.4. Detailed examples of fundamental domains for congruence subgroups. In this subsection we construct certain specific examples of the graph X_g for the congruence subgroups of SL_2 .

- (1) When g is linear, we have $|L_0| = 1$ and $|L_i| = q+1$ for $i \geq 1$. Thus X_g consists of a single core vertex plus $q+1$ cusps which are semi-infinite rays.
- (2) Let $g(t) = t^2$. Then $|L_0| = q^3$ and $|L_i| = (q+1)q^2$ for $i \geq 1$. The first two levels form a $(q+1, q)$ -regular bipartite graph, and semi-infinite rays are attached to each vertex in level L_1 . The graph X_g for $q = 2$ is given in Figure 3. The odd and even levels of vertices give the bipartition of Remark 3.4(1).
- (3) Let $g(t) = t^3$. Here, $|L_0| = q^6$, $|L_1| = (q+1)q^5$ and $|L_i| = (q+1)q^4$ for $i \geq 2$. The bipartite graph between the first two levels is $(q+1, q)$ -regular, and then the graph collapses once by a factor of q before extending onward as infinite rays. The core graph for $q = 2$ is given in Figure 4, with the rows of vertices top to bottom corresponding to L_0, L_1 and L_2 , respectively.

We used Magma to construct these graphs and many larger examples. The groups H and H_i are constructed as matrix groups of degree $2n$ over \mathbb{F}_q , and then the coset graphs are constructed using code due to Leemans ([JL04]). We used dot to draw the graphs ([GN00]). Due to the

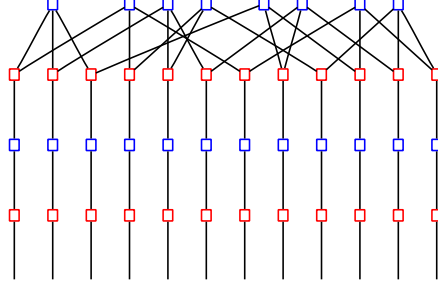


FIGURE 3. X_g for $g(t) = t^2$, $q = 2$

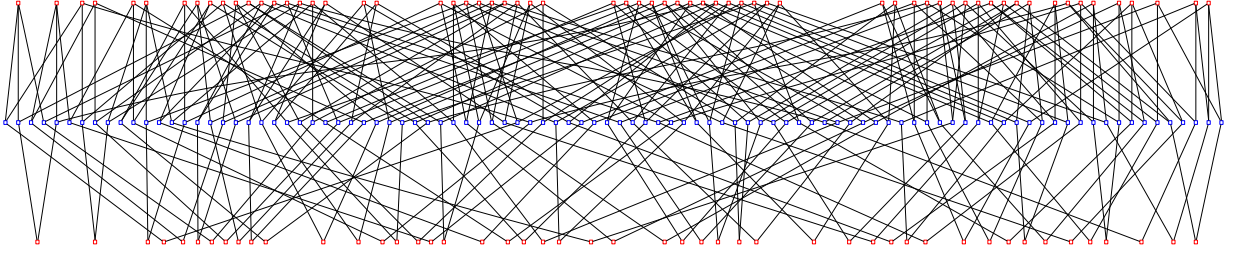


FIGURE 4. Core of X_g for $g(t) = t^3$, $q = 2$

large size of the core graph it is impractical to draw the larger examples generated by Magma, but a database of all our examples can be made available on request.

4. FUNDAMENTAL DOMAINS FOR CONGRUENCE SUBGROUPS OF $\mathrm{PGL}_2(\mathbb{F}_q[t])$

In this section we construct the fundamental domains for the congruence subgroups of $\overline{G} := \mathrm{PGL}_2(\mathbb{F}_q((t^{-1})))$. Let $\overline{\Gamma} = \mathrm{PGL}_2(\mathbb{F}_q[t])$ and let $\overline{\Gamma}(g) = \{A \in \overline{\Gamma} \mid A \equiv I_2 \pmod{(g)}\}$. Let \overline{X}_g be the graph defined for PGL in the analogous manner to the graph X_g from the previous section.

First we describe the structure of $\overline{H} := \overline{\Gamma}/\overline{\Gamma}(g)$.

Proposition 4.1. $\overline{H} \cong (\mathrm{SL}_2(R_g) \rtimes F)/Z$ where $F = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_q^\times \right\}$ and $Z = \mathbb{F}_q^\times I_2$.

Proof. First decompose the group of invertible matrices

$$\mathrm{GL}_2(\mathbb{F}_q[t]) = \mathrm{SL}_2(\mathbb{F}_q[t]) \rtimes F.$$

By taking the preimages of the projective matrices in $\mathrm{GL}_2(\mathbb{F}_q[t])$ we get

$$\begin{aligned} \overline{H} &\cong \mathrm{GL}_2(\mathbb{F}_q[t]) / \{A \in \mathrm{GL}_2(\mathbb{F}_q[t]) \mid A \equiv \lambda I_2 \pmod{(g)} \text{ for some } \lambda \in \mathbb{F}_q^\times\} \\ &= (\mathrm{SL}_2(\mathbb{F}_q[t]) \rtimes F) / \mathbb{F}_q^\times \{A \in \mathrm{GL}_2(\mathbb{F}_q[t]) \mid A \equiv I_2 \pmod{(g)}\} \\ &\cong (\mathrm{SL}_2(R_g) \rtimes F) / \mathbb{F}_q^\times I_2. \end{aligned} \quad \square$$

Note that $|\overline{H}| = |H|$ and the subgroup $\overline{H}_i := H_i F / Z$ has the same size as the corresponding subgroups H_i for SL_2 . Hence the vertex levels also have the same sizes.

Theorem 4.2. *The PGL graph \overline{X}_g is isomorphic to the SL graph X_g .*

Proof. We define a map ϕ from the vertices of X_g to the vertices of \overline{X}_g by

$$H_i x \mapsto H_i F x / Z.$$

Note that $H_i F = F H_i$ for all i . Recall that the edge between $H_i x$ and $H_{i+1} x$ corresponds to the coset $(H_i \cap H_{i+1})x$. Similarly the edge between $H_i F x / Z$ and $H_{i+1} F x / Z$ corresponds to the coset $(H_i F \cap H_{i+1} F)x / Z$. So to prove that ϕ takes every edge to an edge it suffices to show that

$$H_i F \cap H_{i+1} F = (H_i \cap H_{i+1}).$$

Clearly $(H_i \cap H_{i+1}) \subseteq H_i F \cap H_{i+1} F$. Conversely suppose $hf = kg$ for $h \in H_i, k \in H_{i+1}, f, g \in F$. Then $f = \text{diag}(a, 1) = g$ where $a = \det(hf) = \det(kg)$, and so $h = k \in H_i \cap H_{i+1}$. Finally we can conclude that ϕ is an isomorphism since the number of edges at level i is the same for the two graphs. \square

In particular, \overline{X}_g is always connected, unlike the graph constructed in ([Mor95]).

5. MORGENSTERN'S GRAPHS

5.1. Morgenstern's PGL graph. Let $\tilde{H} = \text{PGL}_2(R_g) = \text{GL}_2(R_g) / \tilde{Z}$, where $\tilde{Z} = R_g^\times I_2$. Let \tilde{H}_i be the subgroup $H_i F \tilde{Z} / \tilde{Z}$, and define levels $\tilde{L}_i = \text{PGL}_2(R_g) / \tilde{H}_i$. Morgenstern's graph \tilde{X}_g is now defined as the levelled coset graph for $\tilde{H}_0, \tilde{H}_1, \dots$ in \tilde{H} . This is analogous to the constructions of X_g in Section 3.1 and \overline{X}_g in Section 4. Furthermore

$$|H| = |\overline{H}| = |\tilde{H}|, \quad |H_i| = |\overline{H}_i| = |\tilde{H}_i|, \quad |H_i \cap H_{i+1}| = |\overline{H}_i \cap \overline{H}_{i+1}| = |\tilde{H}_i \cap \tilde{H}_{i+1}|$$

for all $i \geq 0$. Hence the properties of Remark 3.4 hold for all three graphs. We have already seen that $X_g \cong \overline{X}_g$. Morgenstern claims that the graphs \overline{X}_g and \tilde{X}_g are isomorphic, but we will see that this is not always the case. This is ultimately a consequence of the fact that Morgenstern fails to prove that he has the desired ramified covering.

We now consider connectedness properties of \tilde{X}_g .

Proposition 5.1. *Morgenstern's graph \tilde{X}_g has $|R_g^\times : \mathbb{F}_q^\times R_g^{\times 2}|$ connected components, where $R_g^{\times 2} = \{x^2 \mid x \in R_g^\times\}$.*

Proof. By Theorem 3.3, we know $\langle H_0, H_{n-1} \rangle = H$. Hence

$$\begin{aligned} \langle \tilde{H}_0, \tilde{H}_{n-1} \rangle &= \langle H_0 F \tilde{Z}, \tilde{H}_{n-1} F \tilde{Z} \rangle / \tilde{Z} \\ &= \langle H_0, H_{n-1} \rangle F \tilde{Z} / \tilde{Z} = H F \tilde{Z} / \tilde{Z}. \end{aligned}$$

Since \det maps $\text{GL}_2(R_g)$ onto R_n^\times with kernel H , we have

$$\text{GL}_2(R_g) / H F \tilde{Z} \cong \mathbb{F}_q^\times R_n^\times / \det(F \tilde{Z}) = R_n^\times / \mathbb{F}_q^\times R_n^{\times 2}. \quad \square$$

Lemma 5.2. *Let $R = \mathbb{E}[u]/(u^n)$ where $\mathbb{E} := \mathbb{F}_{q^d}$.*

- (1) *If q is odd, then $R^{\times 2} = \mathbb{E}^{\times 2} + \mathbb{E}u + \mathbb{E}u^2 + \dots$ and so $\mathbb{E}^\times R^{\times 2} = R^\times$.*
- (2) *If q is even, then $R^{\times 2} = \mathbb{E}^\times R^{\times 2} = \mathbb{E}^\times + \mathbb{E}u^2 + \mathbb{E}u^4 + \dots$.*

Proof. For q even, $(a_0 + a_1 u + a_2 u^2 + \dots)^2 = a_0^2 + a_1^2 u^2 + a_2^2 u^4 + \dots$, for all $a_i \in \mathbb{E}$. Using the fact that $\mathbb{E}^{\times 2} = \mathbb{E}^\times$, we get $R^{\times 2} = \mathbb{E}^\times + \mathbb{E}u^2 + \mathbb{E}u^4 + \dots$.

Now let q be odd. It suffices to show that every element of the form $1 + a_1 u + \dots$ is in $R^{\times 2}$. Suppose this is not true, and take $a = 1 + a_i u^i + \dots \notin R^{\times 2}$ with i maximal such that $a_i \neq 0$. But $R^{\times 2}$ is a subgroup of R^\times , and so $a(1 - \frac{a_i}{2} u)^2 \notin R^{\times 2}$. Since the coefficients of u, u^2, \dots, u^i are all zero in this element, we have a contradiction. \square

Theorem 5.3. *Morgenstern's graph \tilde{X}_g is connected if and only if q is odd or g is squarefree.*

Proof. This follows immediately from the previous two results and the decomposition $R_g \cong \bigoplus_{i=1}^s \mathbb{F}_{q^{d_i}}[t_i]/(t_i^{n_i})$. \square

In particular, \tilde{X}_g is not isomorphic to \overline{X}_g when q is even and g is not squarefree. By Magma computation using the algorithm of [McK81], we found that X_{t^n} and \tilde{X}_{t^n} are also nonisomorphic for $q = 3$ and $n = 2, 3, 4$.

5.2. The subgraphs of levels 0 – 1. Morgenstern constructed \tilde{X}_g as a means of providing examples of linear families of bounded concentrators. These examples were obtained as the subgraph $\tilde{D}_g(0 - 1)$ induced by the vertices of \tilde{X}_g in the first two levels \tilde{L}_0 and \tilde{L}_1 . However, a necessary property for a bounded concentrator is connectedness. We will show in characteristic 2 that the subgraphs $\tilde{D}_g(0 - 1)$ are not connected. This contradicts the following claim of Morgenstern:

[Mor95], *Proposition 4.2:* If $q \geq 4$, or $q = 3$ and $g(x)$ is irreducible of degree greater than 2, then $\tilde{D}_g(0 - 1)$ is connected.

This in turn is based on an incorrect lower bound for $N_0(S)$, the set of vertices in \tilde{L}_0 which are adjacent to a subset $S \subseteq \tilde{L}_1$ of vertices in \tilde{L}_1 :

[Mor95], *Lemma 4.1:* For every $S \subseteq \tilde{L}_1$, $\frac{|N_0(S)|}{|S|} \geq \frac{q|\tilde{L}_1|}{(q-3)|S|+4|\tilde{L}_1|}$.

This bound fails if we take S to be a connected component of one of the disconnected graphs described below. We note that, when $\tilde{D}_g(0 - 1)$ is not connected, all the connected components are isomorphic. Furthermore H acts transitively on the set of components. This follows from general properties of coset graphs.

In the remainder of this section we consider connectedness properties of $\tilde{D}_g(0 - 1)$ and the corresponding subgraph $D_g(0 - 1)$ induced on the first two levels of X_g (or equivalently \overline{X}_g). By Proposition 3.2, the number of components of $D_g(0 - 1)$ is

$$C := |H : \langle H_0, H_1 \rangle|,$$

and the number of components $\tilde{D}_g(0 - 1)$ is

$$\tilde{C} := |\tilde{H} : \langle \tilde{H}_0, \tilde{H}_1 \rangle| = |\mathrm{GL}_2(R_g) : \langle H_0, H_1 \rangle F\tilde{Z}|.$$

This allows us to count components using Magma's matrix group machinery. These results, for even q and $g(t) = t^n$, are summarised in Table 1. For odd q we found both graphs to be connected in every example we computed.

Based on these experimental results, we conjecture formulas:

Conjecture 5.4. *For $g(t) = t^n$ over \mathbb{F}_q ,*

$$C = \begin{cases} q^{\lfloor (3n-5)/2 \rfloor} & \text{for } q = 2, n > 2, \\ 1 & \text{for } q > 2, \end{cases}$$

$$\tilde{C} = \begin{cases} q^{\lfloor (3n-5)/2 \rfloor + \lfloor (n+1)/4 \rfloor} & \text{for } q = 2, n > 2, \\ q^{\lfloor n/2 \rfloor} & \text{for } q > 2 \text{ even}, n > 1, \\ 1 & \text{for } q \text{ odd.} \end{cases}$$

q	2												
n	2	3	4	5	6	7	8	9	10	11	12	13	14
C	1	2^2	2^3	2^5	2^6	2^8	2^9	2^{11}	2^{12}	2^{14}	2^{15}	2^{17}	2^{18}
\tilde{C}	2^1	2^3	2^4	2^6	2^7	2^{10}	2^{11}	2^{13}	2^{14}	2^{17}	2^{18}	2^{20}	2^{21}
q	2												
n	15	16	17	18	19	20	21	22	23	24	25	26	
C	2^{20}	2^{21}	2^{23}	2^{24}	2^{26}	2^{27}	2^{29}	2^{30}	2^{32}	2^{33}	2^{35}	2^{36}	
\tilde{C}	2^{24}	2^{25}	2^{27}	2^{28}	2^{31}	2^{32}	2^{34}	2^{35}	2^{38}	2^{39}	2^{41}	2^{42}	
q	4												
n	2	3	4	5	6	7	8	9	10	11	12	13	
C	1	1	1	1	1	1	1	1	1	1	1	1	1
\tilde{C}	2^2	2^2	2^4	2^4	2^6	2^6	2^8	2^8	2^{10}	2^{10}	2^{12}	2^{12}	
q	8					16				32		64	
n	2	3	4	5	6	7	2	3	4	2	3	2	
C	1	1	1	1	1	1	1	1	1	1	1	1	1
\tilde{C}	2^3	2^3	2^6	2^6	2^9	2^9	2^4	2^4	2^8	2^5	2^5	2^6	

TABLE 1. Number of components of the first two levels for q even

We now give some theoretical results on the number of components of these graphs for arbitrary g .

Proposition 5.5.

$$C \cdot |R_g^\times : \mathbb{F}_q^\times R_g^{\times 2}| = \tilde{C} \cdot |S : T|$$

where $S := \{a \in R_g^\times \mid a^2 \in \mathbb{F}_q^\times\}$ and $T := \{a \in S \mid \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \langle H_0, H_1 \rangle\}$.

Proof. From Figure 5, we can see that

$$C \cdot |\mathrm{GL}_2(R_g) : HF\tilde{Z}| = \tilde{C} \cdot |H \cap F\tilde{Z} : \langle H_0, H_1 \rangle \cap F\tilde{Z}|$$

Since \det maps G onto R_g^\times with kernel H , we have $\mathrm{GL}_2(R_g)/HF\tilde{Z} \cong R_g^\times / \det(F\tilde{Z}) = R_g^\times / \mathbb{F}_q^\times R_g^{\times 2}$.

An element of $F\tilde{Z}$ has the form $x = \begin{pmatrix} \lambda & 0 \\ 0 & a \end{pmatrix}$, for $\lambda \in \mathbb{F}_q^\times$ and $a \in R_g^\times$. And $x \in H$ is equivalent to $a^2 = \lambda^{-1} \in \mathbb{F}_q^\times$, so projection onto the bottom right entry gives an isomorphism from $H \cap F\tilde{Z}$ to S . Clearly the subgroup $\langle H_0, H_1 \rangle \cap F\tilde{Z}$ corresponds to the T under this isomorphism. \square

Proposition 5.6. *If q is odd and $g(t) = t^n$, then $C = \tilde{C}$.*

Proof. We have $\mathbb{F}_q^\times R_g^{\times 2} = R_g^\times$ by Lemma 5.2. If $a = a_0 + a_i t^i + \dots \in S$ with a_i the smallest nonzero coefficient other than a_0 , then $a^2 = a_0 + 2a_i t^i + \dots = 1$ and so $i \geq n$. Hence $S = \mathbb{F}_q^\times$, and it is now easy to prove that $T = S$. \square

Proposition 5.7. *If q is even and g is not squarefree, then $\tilde{C} > C$.*

Proof. By Lemma 9 and the decomposition $R = \bigoplus_r R_i$,

$$|R_g^\times : \mathbb{F}_q^\times R_g^{\times 2}| = \prod_i q^{d_i \lfloor n_i/2 \rfloor}.$$

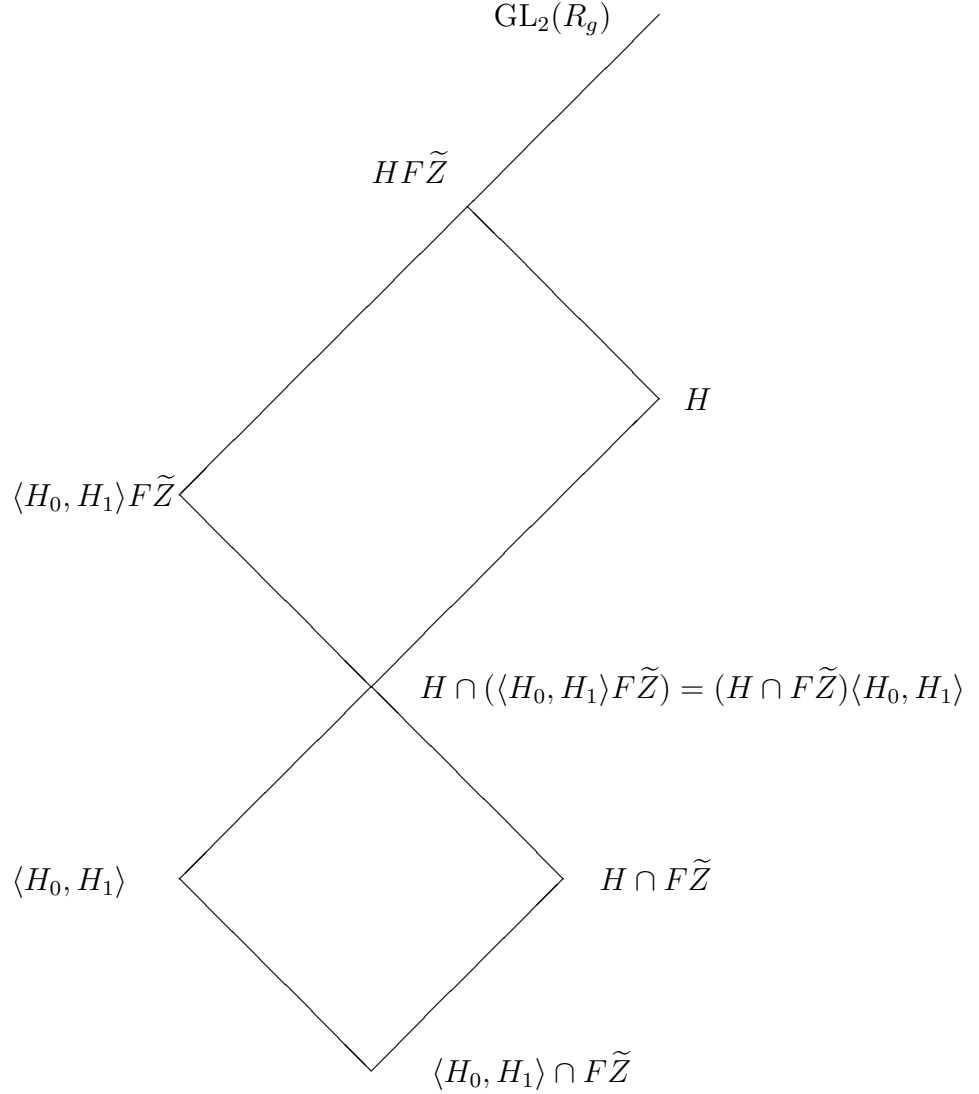


FIGURE 5. Subgroup lattice diagram

Now suppose $a = a_0 + a_1 t_i + a_2 t_i^2 + \dots \in R_i$ with $a^2 = 1$. This is equivalent to $a_0 = 1$, and $a_i = 0$ for all $j > 0$ with $2j < n_i$. Hence $|S| = \prod_i q^{d_i \lfloor n_i/2 \rfloor}$.

We now have $\tilde{C} = |T|C$. But if $2^e < n_i \leq 2^{e+1}$, then $a = 1 + t_i^{2^e}$ is a nontrivial element which squares to the identity. And $\langle H_0, H_1 \rangle$ contains

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \left[\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]^3,$$

so T is nontrivial. □

So, for q even and g not squarefree, we know that $\tilde{D}_g(0-1)$ is not connected, and also that it cannot be isomorphic to $D_g(0-1)$. By Magma computation using the algorithm of [McK81], we

found that $D_{t^n}(0-1)$ and $\tilde{D}_{t^n}(0-1)$ are also nonisomorphic for $q = 3$ and $n = 2, 3, 4$. However they are isomorphic for $q = 5, 7$ and $n = 2$.

What remains to be seen is whether or not the subgraphs $D_g(0-1)$ are connected in odd characteristic and if they have the claimed expansion properties. This is beyond the scope of the current work.

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