CHEVALLEY GROUPS AND INFINITE DIMENSIONAL GENERALIZATIONS

LISA CARBONE

1. Course description

The Chevalley groups give an alternate construction of simple Lie groups. Namely, Chevalley defined generators for Lie groups in terms of the root system and automorphisms of the underlying Lie algebra ([Ch]). With some additional external data including the universal enveloping algebra and highest weight representations, this gave the first unified construction of classical and exceptional groups over arbitrary fields, and also over $\mathbb{Z}$. The Steinberg presentation for Chevalley groups gives a way to describe these groups in terms of generators and relations. The advantage of this approach is that it is able to be generalized to infinite dimensions.

In the affine case, the Chevalley theory was developed in detail by Garland. We study Chevalley groups and the Steinberg presentation for finite dimensional Lie groups, and we build the necessary machinery to generalize the Chevalley construction and Steinberg presentation to infinite dimensions.

Course objectives To obtain a concrete and computational approach to Lie group theory in both finite and infinite dimensions.

Motivation It is useful and computationally effective to have a formulation of Kac-Moody groups as infinite dimensional Chevalley groups. Many questions in both finite and infinite dimensions come directly from physical symmetries of supergravity and superstring theories.

Approach We discuss Chevalley's construction of finite dimensional simple Lie groups over arbitrary fields, including all classical and exceptional types of equal root length (also called 'simply laced' types). We develop the Steinberg presentation in these cases with as many explicit examples as possible. We generalize the Chevalley construction and Steinberg presentation to infinite dimensions in the affine and hyperbolic Kac-Moody cases. In the affine case, the Chevalley theory was developed in detail by Garland. In the hyperbolic case, the topic has been developed by Carbone and Garland [CG2].

We also follow computational methods in [Bri], [BH], [BH2], [CHM], [CMT], [R], who have developed effective techniques for computation in Lie theory including algorithms and implementation for Lie theory computation in the computer algebra system Magma. Most of the algorithms implemented work in finite dimensions.

2. Topics in approximate order

2.1. **Chevalley bases for simple Lie algebras.** Explicit examples SL_2, SL_3, SL_n, E_6, E_7, E_8

2.2. **Computation of structure constants.** Properties of structure constants, the Frenkel-Kac formula, fast algorithms for computing signs of structure constants, explicit examples SL_3, SL_n, E_6, E_7, E_8

2.3. **Chevalley groups.** Adjoint group, highest weight modules, universal enveloping algebra and $\mathbb{Z}$-form, simply connected group

2.4. **Fundamental representations and construction exceptional groups.** 27 dim representation of E_6, 56 dim representation of E_7, 248 dim representation of E_8

2.5. **Steinberg presentation for Chevalley groups.** Examples SL_2, GL_2, E_7, general reductive groups

2.6. ***BN*-pairs for Chevalley groups.** Parabolic subgroups and Bruhat decomposition

2.7. **Chevalley groups over $\mathbb{Z}$.** Basic construction, criteria for integrality, Steinberg presentation, explicit examples SL_2, SL_3, E_7

2.8. **Chevalley bases for Kac-Moody algebras.** Affine case, the Frenkel-Kac formula, structure constants for real roots, investigation for imaginary roots, explicit examples $A_1^{(1)}$, E_9

2.9. **Kac-Moody groups as infinite dimensional Chevalley groups.** The Tits functor over fields, generators and relations, explicit examples $\mathcal{H}(3)$, $\widehat{A}_1^{(1)}$, E_{10}

2.10. **Kac-Moody groups over $\mathbb{Z}$.** The Tits functor over $\mathbb{Z}$, generators and relations, E_{10}

References

[AM] Abramenko, P. and Mühlherr, B. *Présentations de certaines BN-paires jumeles comme sommes amalgamées* (French) [Presentation of certain twin BN-pairs as amalgamated sums] C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 7, 701–706

[A] Allcock, D., *Private communication*,

[AB] Alperin, J. L. and Bell, R. *Groups and representations*, Graduate Texts in Mathematics, 162. Springer-Verlag, New York, (1995). x+194 pp. ISBN: 0-387-94525-3

[BH] Brigitte Brink and Robert B. Howlett, *A finiteness property and an automatic structure for Coxeter groups*, Math. Ann. **296** (1993), no. 1, 179–190.

[BH2] Brigitte Brink and Robert B. Howlett, *Normalizers of parabolic subgroups in Coxeter groups*, Invent. Math. **136** (1999), no. 2, 323–351.

[Bri] Brigitte Brink, *The set of dominance-minimal roots*, J. Algebra **206** (1998), no. 2, 371–412.

[Ca] Caprace, P.-E., *On 2-spherical Kac-Moody groups and their central extensions*, Forum Math. 19 (2007), no. 5, 763–781

[CM] Caprace, P.-E. and Mühlherr, B *Isomorphisms of Kac-Moody groups*, Invent. math. 161 (2005), 361-388

[C] Carbone, L. *Chevalley groups and infinite dimensional generalizations*, In preparation (2012)

[CCCMNNP] Carbone, L., Chung, S., Cobbs, L., McRae, R., Nandi, D., Naqvi Y. and Penta, D., *Classification of hyperbolic Dynkin diagrams, root lengths and Weyl group orbits*, Journal of Physics. A: Math. Theor. 43 155209, (2010)

[CG1] Carbone, L. and Garland, H., *Existence of lattices in Kac-Moody groups over finite fields*, Communications in Contemporary, Math, Vol 5, No.5, 813–867, (2003)

[CG2] Carbone, L. and Garland, H., *Infinite dimensional Chevalley groups and Kac-Moody groups over* $\mathbb{Z}$, Preprint (2012)

[CER] Carbone, L., Ershov, M. and Ritter, G., *Abstract simplicity of complete Kac-Moody groups over finite fields*, Journal of Pure and Applied Algebra, No. 212, (2008), 2147–2162

[CCG] Carbone, L., Cartier, P. and Garland, H. *Integral points of Chevalley groups and their flag varieties*, Preprint (2012)

[Car] Carter, R. W. *Simple groups of Lie type.* Pure and Applied Mathematics, Vol. 28. John Wiley and Sons, London-New York-Sydney, (1972). viii+331 pp.

[Cas] Casselman, B. *The computation of structure constants according to Jacques Tits*, Preprint, (2012)

[Ch] Chevalley, C. *Sur certains groupes simples*, (French) Tôhoku Math. J. (2) 7 (1955), 14–66.

[CGL] Cohen, A. M., Griess, R. L., Jr., Lisser, B. *The group L(2,61) embeds in the Lie group of type*E_8. Comm. Algebra 21 (1993), no. 6, 1889–1907

[CS] Conway, J. H. and Sloane, N. J. A., *Sphere packings, lattices and groups*, Third edition. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 290. Springer-Verlag, New York, (1999)

[CHM] Arjeh M. Cohen, Sergei Haller, and Scott H. Murray, *Computing in unipotent and reductive algebraic groups*, LMS J. Comput. Math. 11 (2008), pp. 343–366.

[CMT] Cohen, A. M., Murray, S. H., and Taylor, D. E. *Computing in groups of Lie type*, Mathematics of computation, Volume 73, Number 247, Pages 1477–1498, (2003)

[DGH] De Medts, T., Gramlich, R., and Horn, M., *Iwasawa decompositions of split Kac-Moody groups*, Journal of Lie Theory 19 (2009), No. 2, 311–337

[D] Dirac, P. *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. (London) A 133, 60 (1931)

[FF] Feingold, A. J. and Frenkel, I. B., *A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus* 2, Math. Ann. 263 (1983), no. 1, 87–144

[FKN] Feingold, A. J., Kleinschmidt, A. and Nicolai, H., *Hyperbolic Weyl groups and the four normed division algebras*, J. Algebra 322 (2009), no. 4, 1295–1339

[FLM] Frenkel, I., Lepowsky J. and Meurman, A. *Vertex operator algebras and the Monster*, Pure and Appl. Math., 134, Academic Press, New York (1988).

[GaN] Gabber, O. and Kac, V., *On defining relations of certain infinite-dimensional Lie algebras*, Bull. Amer. Math. Soc. (N.S.) 5 (1981), no. 2, 185–189

[Ga1] Garland, H., *The arithmetic theory of loop algebras*, J. Algebra 53 (1978), no. 2, 480–551.
[Ga2] Garland, H., *The arithmetic theory of loop groups*, Inst. Hautes tudes Sci. Publ. Math. No. 52 (1980), 5–136.
[GP] Gilkey, P. B. and Seitz, G. M., *Some representations of exceptional Lie algebras*, Geometries and groups (Noordwijkerhout, 1986), Geom. Dedicata 25 (1988), no. 1-3, 407–416
[GR] Golubitsky, M. and Rothschild, B. *Primitive subalgebras of exceptional Lie algebras*, Pacific J. Math. 39 (1971), 371–393.
[H-C] Harish-Chandra *On some applications of the universal enveloping algebra of a semisimple Lie algebra*, Trans. Amer. Math. Soc. 70, (1951), 28–96
[HRT] Howlett, R. B., Rylands, L. J., and Taylor, D. E. *Matrix generators for exceptional groups of Lie type*, J. Symbolic Comput. 31 (2001), no. 4, 429–445
[HT] C. M. Hull and P. K. Townsend, *Unity of superstring dualities*, Nuclear Phys. **B 438** (1995), no. 1-2, 109–137.
[Hu] Humphreys, J. E. *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, (1978). xii+171 pp
[K] Kac, V., *Infinite dimensional Lie algebras*, Cambridge University Press, (1990)
[KP] Kac, V. and Peterson, D., *Defining relations of certain infinite-dimensional groups*, Astérisque Numéro Hors Série (1985), 165–208.
[Kan] Kantor, I. L. *Graded Lie algebras* (Russian) Trudy Sem. Vektor. Tenzor. Anal. 15 (1970), 227–266
[Ki] Kirillov, A., Jr. *An introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics, 113. Cambridge University Press, Cambridge, (2008). xii+222 pp. ISBN: 978-0-521-88969-8
[Ko] Kostant, B., *Groups over* $\mathbb{Z}$, 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., (1965)) pp. 90–98 Amer. Math. Soc., Providence, R.I.
[Ku] Kumar, S., *Kac-Moody Groups, Their Flag Varieties and Representation Theory*, Birkhauser, Boston Progress in Mathematics Series, 204, (2002)
[Ma] Marcuson, R. *Tits' systems in generalized nonadjoint Chevalley groups* J. Algebra 34 (1975), 84–96
[M] Mathieu, O. *Sur la construction de groupes associés aux algébres de Kac-Moody.* (French) [On the construction of groups associated with Kac-Moody algebras] C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 6, 161–164
[Mi] Milnor, J., *Introduction to algebraic* K*-theory*, Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971. xiii+184 pp.
[Mi] Mitzman, D., *Integral bases for affine Lie algebras and their universal enveloping algebras*, Contemporary Mathematics, 40. American Mathematical Society, Providence, RI, (1985). vii+159 pp. ISBN: 0-8218-5043-1
[Miy] Miyake, T., *Modular forms*, Translated from the Japanese by Yoshitaka Maeda. Springer-Verlag, Berlin, (1989) x+335 pp.
[M1] Moody, R. V. , *Lie algebras associated with generalized Cartan matrices*, Bull. Amer. Math. Soc., 73 (1967) 217–221
[M2] Moody, R. V. *A new class of Lie algebras*, J. Algebra 10 (1968) 211–230
[MP] Moody, R. V. and Pianzola, A. *Lie algebras with triangular decompositions*, Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1995. xxii+685 pp. ISBN: 0-471-63304-6
[MT] Moody, R. V. and Teo, K. L. *Tits' systems with crystallographic Weyl groups*, J. Algebra 21 (1972), 178–190
[Mo] Moore, C.C. *Group extensions of p-adic and adelic linear groups*, I.H.E.S., No. 35 (1968)
[Pr] Prevost, S. A., *Vertex algebras and integral bases for the enveloping algebras of affine Lie algebras*, Mem. Amer. Math. Soc. 96 (1992), no. 466, viii+97 pp
[R] Rylands, L. J. *Fast calculation of structure constants*, Preprint, (2000)
[S] Saçlioğlu, C., *Dynkin diagrams for hyperbolic Kac-Moody algebras*, Journal of Physics A: Mathematics and General **22** 3753–3769, (1989)
[Sa] Samelson, H. *Notes on Lie algebras* Second edition. Universitext. Springer-Verlag, New York, (1990) xii+162 pp. ISBN: 0-387-97264-1

[Sel] Seligman, George B. *Book Review: Infinite dimensional Lie algebras*, Bull. Amer. Math. Soc. (N.S.) 16 (1987), no. 1, 144–149

[Sl] Slodowy, P. *An adjoint quotient for certain groups attached to Kac-Moody algebras*, Infinite-dimensional groups with applications (Berkeley, Calif., 1984), 307–333, Math. Sci. Res. Inst. Publ., 4, Springer, New York, (1985)

[So] Soulé, C. *An introduction to arithmetic groups*, Frontiers in number theory, physics, and geometry. II, 247–276, Springer, Berlin, (2007)

[Sp] Springer, T. A. *Linear algebraic groups*, Reprint of the 1998 second edition. Modern Birkhuser Classics. Birkhuser Boston, Inc., Boston, MA, 2009. xvi+334 pp. ISBN: 978-0-8176-4839-8

[St] Steinberg, R. *Lectures on Chevalley groups*, Notes prepared by John Faulkner and Robert Wilson, Yale University, (1967)

[St2] Steinberg, R. *Générateurs, relations et revêtements de groupes algébriques*, (French) 1962 Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962) pp. 113–127 Librairie Universitaire, Louvain; Gauthier-Villars, Paris

[Ti] Tits, J., *Sur les analogues algébriques des groupes semi-simples complexes*, Colloque d'Algébre supérieure [1956, Bruxelles]

[Ti1] Tits, J., *Resume de Cours - Theorie des Groupes*, Annuaire du College de France, (1980-1981), 75–87

[Ti2] Tits, J., *Uniqueness and presentation of Kac-Moody groups over fields*, Journal of Algebra, 105, (1987), 542–573

[Ti3] Tits, J. *Sur les constantes de structure et le thòréme d'existence des algébres de Lie semi-simples* (French) Inst. Hautes Études Sci. Publ. Math. No. 31 (1966) 21–58.

[V1] Vavilov, N. A. *Do it yourself structure constants for Lie algebras of types E_ℓ*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 281 (2001), Vopr. Teor. Predst. Algebr. i Grupp. 8, 60–104, 281; translation in J. Math. Sci. (N. Y.) 120 (2004), no. 4, 1513–1548

[V2] Vavilov, N. A. *How is one to view the signs of structure constants?* (Russian) Algebra i Analiz 19 (2007), no. 4, 34–68; translation in St. Petersburg Math. J. 19 (2008), no. 4, 519–543

[VP] Vavilov, N. and Plotkin, E. *Chevalley groups over commutative rings. I. Elementary calculations*, Acta Appl. Math. 45 (1996), no. 1, 73–113.

Lisa Carbone, Department of Mathematics, Hill Center-Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Rd Piscataway, NJ 08854-8019
E-mail: *carbonel@math.rutgers.edu*