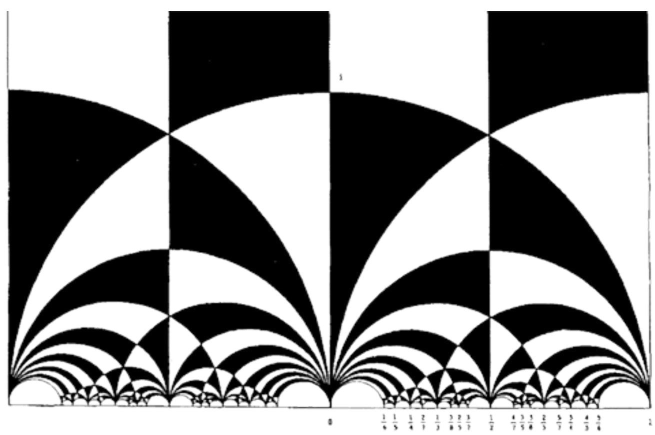
640:558 The theory of algebras: KAC-MOODY SYMMETRY IN MATHEMATICS AND PHYSICS



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KAC-MOODY SYMMETRY IN MATHEMATICS AND PHYSICS

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Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of *finite, affine, or hyperbolic* type, over K, a field, and let G be a Kac-Moody group associated to $\mathfrak{g}$. If $\mathfrak{g}$ is of finite type, then $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, and G is a semisimple Lie group. Almost all of these occur in 'space-time symmetries' and the development of the Standard Model of particle physics, which could not have progressed without an understanding of symmetries and group transformations.

Affine Kac-Moody algebras and their generalizations by Borcherds have concrete physical realizations and have wide applications in physical theories such as elementary particle theory, quantum field theory, gauge theory, conformal field theory, gravity and string theory. Affine Kac-Moody groups and algebras also give rise to a rich mathematical theory, are relevant to number theory and modular forms, and they occur in the relation between the sporadic simple Monster group and symmetries of codes, lattices and conformal fields theories.

The theory of hyperbolic Kac-Moody groups and algebras naturally generalizes the theory of affine Kac-Moody groups and algebras which is itself the most natural generalization to infinite dimensions of finite dimensional Lie theory. Recently hyperbolic and Lorentzian Kac-Moody groups and algebras have been discovered as symmetries in high-energy physics, and they have been shown to serve as duality symmetries of a proposed theory, known as *M-theory*, which unifies all superstring theories.

In particular hyperbolic Kac-Moody groups and algebras are known to be symmetries of 'dimensionally reduced' supergravity, a theory incorporating general relativity and supersymmetry and to parametrize the scalar fields of supergravity theories via their coset spaces. They are conjectured to be symmetries of full supergravity theories, to encode geometrical objects of M-theory as well as the dynamics of certain supergravity theories near a space-like singularity.

However fundamental questions about the structure of hyperbolic groups and algebras remain open.

Extensions of Kac-Moody algebras constructed by Borcherds, called *generalized Kac-Moody algebras*, have a physical realization in terms of the space of physical states of the vertex algebra of a lattice and they provide a natural framework for understanding the structure of their embedded maximal Kac-Moody algebras. However, recent conjectures about unifying superstring theories all point to symmetries of *hyperbolic and Lorentzian Kac-Moody algebras*.

Prerequisites and course objectives

In this course we study the mathematics suggested by the development of certain high energy physical theories and their symmetries, focussing on the occurence of hyperbolic Kac-Moody groups and algebras and their properties.

Some familiarity with finite dimensional Lie groups or Lie algebras is preferable though not required. Background in theoretical physics will not be assumed. A brief overview of classical and quantum field theories, general relativity and supersymmetry will be covered in order to explore the open problems connecting Kac-Moody groups and algebras with high energy physical theories such as string theory, supergravity and M-theory.

We adopt a utilitarian approach to this highly theoretical subject, focussing on concepts and examples. Detailed references to proofs and a more technical treatment will be given.

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