# INFINITE DIMENSIONAL LIE GROUPS AND APPLICATIONS

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## 1. Course description

Kac-Moody groups are natural generalizations to infinite dimensions of finite dimensional simple Lie groups. The subclass of real forms of hyperbolic Kac-Moody groups have recently been shown to appear in the study of algebraic symmetries of general relativity and a theory known as supergravity, which incorporates both general relativity and supersymmetry.

The discrete symmetries arising from forms of these groups over the integers, play a particularly important role, related to quantization of charges in particle physics.

In this course, we study the mathematics suggested by symmetries of supergravity, focusing on the occurrence of hyperbolic Kac-Moody groups and algebras and their properties. We draw many of our observations from analogy with finite dimensional Lie groups and their relation to classical spacetime symmetries.

The action of the Kac-Moody group $G$ on a simplicial complex, known as the Tits building, and the related structure theory for $G$ and its subgroups is an important tool that we will make use of.

We also consider the infinite dimensional symmetric space $G/K$, where $K$ is the fixed point subgroup of the Cartan involution. This space is an infinite dimensional analog of the Poincaré upper half plane and has a prominent role in the coset models of supergravity theories.

We also explore the open problems connecting hyperbolic Kac-Moody groups and algebras with physical theories such as supergravity and M-theory, which is a speculative theory which proposes to unify all superstring theories.

The basic structure theory of infinite dimensional Lie groups and Lie algebras will be covered. Some familiarity with finite dimensional Lie groups and Lie algebras is preferable though not required.

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