Tree lattice subgroups

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1 Introduction

Let X be a locally finite tree and let $G = Aut(X)$. Then G is naturally a locally compact group. A discrete subgroup $\Gamma \leq G$ is called an *X-lattice*, or a *tree lattice* if

$$Vol(\Gamma\backslash\backslash X) \quad := \quad \sum_{x\in V(\Gamma\backslash X)} \frac{1}{|\Gamma_x|}$$

is finite, and a *uniform X-lattice* if $\Gamma\backslash X$ is a finite graph, *non-uniform* otherwise. In [C2] and [BCR] we gave the necessary and sufficient conditions for the existence of X-lattices.

In this work we determine how to construct *pairs* of tree lattice subgroups. In the setting of topological covering theory, there is a correspondence between coverings, $p : X \to A$, and subgroups of $\pi_1(A)$, and this gives essentially 'one' subgroup up to conjugacy. Here with lattices given by their quotient graphs of groups, we are in an 'orbifold' setting where coverings have an extra ingredient, namely isotropy groups. We make use of this additional data to construct lattice subgroups by using subgroups of isotropy (vertex) groups. This will give rise to zero, one, finitely many, or infinitely many possible subgroups up to isomorphism. In fact in [CR1] the authors exhibited infinite ascending chain of lattice subgroups, all with the same quotient graph.

We describe several methods (sections 3-5) for constructing a pair of X-lattices (Γ', Γ) with $\Gamma \leq \Gamma'$, starting from 'edge-indexed graphs' (A', i') and (A, i) which will correspond to the edge-indexed quotient graphs of their (common) universal covering tree by Γ' and Γ respectively.

Our techniques are a combination of topological graph theory, covering theory for graphs of groups ([B]), and covering theory for edge-indexed graphs developed in [C1] and [BCR]. As an application, we show (section 5) that a non-uniform X-lattice Γ contains an infinite chain of subgroups $\Lambda_1 < \Lambda_2 < \Lambda_3 < \dots$ where each Λ_k is a uniform X_j-lattice, X_k a subtree of X.

Let $\phi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs (defined in section 2). We wish to determine if it is possible to extend ϕ to a covering morphism of graphs of groups as in [B]. In sections 6-8 we give a local necessary condition for extending ϕ to a covering $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$ of graphs of groups, where $\mathbb{A}'$ and $\mathbb{A}$ are abelian groupings of (A', i') and (A, i) respectively. For coverings $\phi : (A, i) \longrightarrow (A', i')$ satisfying suitable conditions (sections 6-8 and [BL], [BCR], [C1]) these will give pairs of lattice subgroups $\Gamma \leq \Gamma'$ with abelian stabilizers.

Finally in section 9 we provide some examples to illustrate some of the constructions proposed here.

2 Tree lattices, edge-indexed graphs, volumes and coverings

An *edge-indexed graph* (A, i) consists of an underlying graph A, and an assignment of a positive integer $i(e) > 0$ to each oriented edge $e \in EA$. Here we assume that the underlying graph A is locally finite. We use $\partial_0 e$ and $\partial_1 e$ to denote the initial and terminal vertices of an edge $e \in EA$.

Let $\mathbb{A} = (A, \mathcal{A})$ be a graph of groups, with underlying graph A, vertex groups $(\mathcal{A}_a)_{a \in VA}$, edge groups $(\mathcal{A}_e = \mathcal{A}_{\overline{e}})_{e \in EA}$ and monomorphisms $\alpha_e \colon \mathcal{A}_e \hookrightarrow \mathcal{A}_{\partial_0 e}$. A graph of groups $\mathbb{A}$ naturally gives rise to an edge-indexed graph $I(\mathbb{A}) = (A, i)$ where $i : EA \to \mathbb{Z}_{>0}$ is defeined as $i(e) = [\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e]$, which we assume to be finite, for all $e \in EA$.

Given an edge-indexed graph (A, i), a graph of groups $\mathbb{A}$ such that $I(\mathbb{A}) = (A, i)$, is called a *grouping* of (A, i). We call $\mathbb{A}$ a *finite grouping* if the vertex groups $\mathcal{A}_a$ are finite and a *faithful grouping* if $\mathbb{A}$ is a faithful graph of groups, that is if $\pi_1(\mathbb{A}, a)$ acts faithfully on $X = \widetilde{(\mathbb{A}, a)}$.

Let $\mathbb{A}'$ and $\mathbb{A}$ be groupings of (A, i). Then $\mathbb{A}' = (A, \mathcal{A}')$ is called a *full graph of subgroups* of $\mathbb{A} = (A, \mathcal{A})$ (as in ([B], (1.14)) if $\mathcal{A}'_a \leq \mathcal{A}_a$ for $a \in A'$, and for $e \in EA'$, $\mathcal{A}'_e \leq \mathcal{A}_e$, and $\alpha'_e = \alpha_e|_{\mathcal{A}'_e}$. We further assume that for $e \in EA'$, with $\partial_0 e = a$, $\mathcal{A}'_a \cap \alpha_e \mathcal{A}_e = \alpha_e \mathcal{A}'_e$, that is $\mathcal{A}'_a / \alpha_e \mathcal{A}'_e \longrightarrow \mathcal{A}_a / \alpha_e \mathcal{A}_e$ is injective, and hence bijective. This assumption implies that $I(\mathbb{A}') = (A, i)$, and that $\pi_1(\mathbb{A}', a') \leq \pi_1(\mathbb{A}, a)$ ([B], (1.14)).

Let (A, i) be an edge-indexed graph. A *tower of groupings* on (A, i) is a semi-infinite sequence $(\mathbb{A}_i)_{i \in \mathbb{Z}_{>0}}$ of groupings of (A, i) such that each $\mathbb{A}_i$ is a full graph of proper subgroups of $\mathbb{A}_{i+1}$. A tower of faithful groupings induces an infinite ascending chain of fundamental groups:

$$\pi_1(\mathbb{A}_1, a_0) < \pi_1(\mathbb{A}_2, a_0) < \pi_1(\mathbb{A}_3, a_0) < \dots$$

For an edge $e \in EA$, define:

$$\Delta(e) = \frac{i(\overline{e})}{i(e)}.$$

If $\gamma = (e_1, \dots, e_n)$ is a path, set:

$$\Delta(\gamma) = \Delta(e_1) \dots \Delta(e_n).$$

Definition 1 *An indexed graph* (A, i) *is unimodular if* $\Delta(\gamma) = 1$ *for all closed paths* γ *in* A.

Now assume that (A, i) is unimodular. Pick a base point $a_0 \in VA$, and define, for $a \in VA$,

$$N_{a_0}(a) = \frac{\Delta a}{\Delta a_0}(= \Delta(\gamma) \text{ for any path } \gamma \text{ from } a_0 \text{ to } a) \in \mathbb{Q}_{>0}.$$

For $e \in EA$, put

$$N_{a_0}(e) \quad := \quad \frac{N_{a_0}(\partial_0(e))}{i(e)}.$$

Following ([BL], (2.6)), we say that (A, i) has *bounded denominators* if

$$\{N_{a_0}(e) \mid e \in EA\}$$

has bounded denominators, that is, if for some integer $D > 0$, $D \cdot N_{a_0}$ takes only integer values on edges. This condition is automatic if A is finite, and since

$$N_{a_1} \quad = \quad \frac{\Delta a_0}{\Delta a_1} N_{a_0},$$

this condition is independent of $a_0 \in VA$. As in [BK] the functions $N : A \longrightarrow \mathbb{Q}^{\times}_{>0}$ as above are called *vertex orderings* of (A, i). We call N *integral* if for all $e \in EA$, we have $N(\partial_0(e))/i(e) \in \mathbb{Z}$ and hence $N(a) \in \mathbb{Z}$ for $a \in VA$.

Theorem 1 *([BK], (2.4)) The following conditions on an edge-indexed graph* (A, i) *are equivalent.*

(a) (A, i) *admits a finite grouping.*

(b) (A, i) *is unimodular and has bounded denominators.*

(c) (A, i) *admits an integral vertex ordering.*

The grouping in (a) can further be taken to be faithful. Under these conditions there is a unique integral vertex ordering N_{min} *such that every other one is of the form* $N = m \cdot N_{min}$ *for some integer* $m > 0$.

We define the *volume* of an indexed graph (A, i) at a basepoint $a_0 \in VA$:

$$Vol_{a_0}(A, i) \quad := \quad \sum_{a \in VA} \frac{1}{(\frac{\Delta a}{\Delta a_0})} \quad = \quad \sum_{a \in VA} (\frac{\Delta a_0}{\Delta a}).$$

Then

$$Vol_{a_1}(A, i) \quad = \quad \frac{\Delta a_0}{\Delta a_1} Vol_{a_0}(A, i),$$

as in ([BL], Ch. 2) We write $Vol(A, i) < \infty$ if $Vol_a(A, i) < \infty$ for some, and hence every $a \in VA$.

If $\mathbb{A}$ is a finite grouping of (A, i), then we have ([BL], (2.6.15)):

$$Vol(\mathbb{A}) \quad = \quad \frac{1}{|\mathcal{A}_a|} Vol_a(A, i),$$

which is automatically finite if $Vol(A, i) < \infty$.

We now describe a method for constructing X-lattices which follows naturally from the fundamental theory of Bass-Serre ([B], [S]), and was first suggested in ([BK]). We begin with an edge-indexed graph (A, i). Then (A, i) determines $X = (\widetilde{A, i, a_0})$ up to isomorphism ([BL], Ch. 2).

We say that (A, i) *admits a lattice* if (A, i) admits a grouping $\mathbb{A}$ such that $\pi_1(\mathbb{A}, a_0)$ is an X-lattice. This is the case if and only if $\mathbb{A}$ is a (faithful) graph of finite groups of finite volume. Note that

$$Vol(\Gamma\backslash\backslash X) = Vol(\mathbb{A})(:= \sum_{a \in VA} \frac{1}{|\mathcal{A}_a|} = \frac{1}{|\mathcal{A}_a|} Vol_a(A, i)) < \infty.$$

Finally, we explain the notion of a *covering* of edge-indexed graphs ([BL], (2.5)),

$$p : (B, j) \longrightarrow (A, i).$$

Here $p : B \longrightarrow A$ is a graph morphism such that for all $e \in EA$, $\partial_0(e) = a$, and $b \in p^{-1}(a)$, we have

$$i(e) = \sum_{f \in p_{(b)}^{-1}(e)} j(f),$$

where $p_{(b)} : E_0^B(b) \longrightarrow E_0^A(a)$ is the local map on stars $E_0^B(b)$ and $E_0^A(a)$ of vertices $b \in VB$ and $a \in VA$ (*cf.* [BL], (2.5)). If $b \in VB$, $p(b) = a \in VA$, then we can identify

$$(\widetilde{A, i, a}) = X = (\widetilde{B, j, b})$$

so that the diagram of natural projections

$$\begin{tikzcd} & X \arrow[dl, "p_B"'] \arrow[dr, "p_A"] & \\ B \arrow[rr, "p"] & & A \end{tikzcd}$$

commutes. Hence $G_{(B,j)} \leq G_{(A,i)}$. If instead we have

$$i(e) \leq \sum_{f \in p_{(b)}^{-1}(e)} j(f),$$

then we call p an *immersion* of edge-indexed graphs.

Let X be a locally finite tree, and Λ a lattice subgroup of Aut(X). Suppose that $\Gamma \leq Aut(X)$ and that $\Lambda \leq \Gamma$. We may ask whether or not Γ is also a lattice subgroup of $Aut(X)$. It is easy to see that if Λ has finite index in Γ, then Γ is discrete if and only if Λ is discrete. Then from ([BL], (1.7)), Λ is a lattice ($Vol(\Lambda\backslash\backslash X) < \infty$) if and only if Γ is a lattice ($Vol(\Gamma\backslash\backslash X) < \infty$) and Λ has finite index in Γ. In this case Λ is a uniform X-lattice if and only if Γ is a uniform X-lattice. In this work we focus on the existence of lattice *subgroups*, not '*overlattices*' as Γ is here. A study of the asymptotics of overlattices may be found in ([BK]) and in ([L]).

3 Finite sheeted topological coverings

In this section we will construct a lattice subgroup pair $\Gamma \leq \Gamma'$ using topological coverings of the underlying edge-indexed graphs. We begin with an edge-indexed graph (A', i') that admits an X-lattice, where $X = \widetilde{(A', i')}$. Let $\mathbb{A}'$ be a finite faithful grouping of finite volume of (A', i'). Let $\Gamma' = \pi_1(\mathbb{A}', a')$, $a' \in VA'$, then Γ' is an X-lattice, uniform if A' is finite, non-uniform if A' is infinite. We assume that $\pi_1(A) \neq \{1\}$. Let $p : (A, i) \to (A', i')$ be a finite sheeted (n-fold) topological covering; that is, locally (on the star of each vertex) p is an isomorphism, and p is index-preserving. Then $X = \widetilde{(A, i)}$ (section 2 and [BL]). We seek a lattice grouping of (A, i). As in ([R], p108) we observe that the grouping $\mathbb{A}'$ on (A', i') induces a finite faithful grouping $\mathbb{A}$ on (A, i) with $Vol(\mathbb{A}) = n \cdot \mathbb{A}'$. It follows ([BK], 2.4) that (A, i) admits a lattice, namely $\Gamma = \pi_1(\mathbb{A}, a)$ for $a \in VA$.

Since $\Gamma' = \pi_1(\mathbb{A}', a')$ and $\Gamma = \pi_1(\mathbb{A}, a)$ are X-lattices, the inclusion $\Gamma \leq \Gamma'$ is automatically of finite k where

$$k = \frac{Vol(\Gamma\backslash\backslash X)}{Vol(\Gamma'\backslash\backslash X)} = \frac{Vol(\mathbb{A})}{Vol(\mathbb{A}')}.$$

The following lemma is easily verified directly, and also follows from the observations above.

Lemma 1 *Let (A', i') be an edge-indexed graph. If (A', i') admits a lattice and $p : (A, i) \to (A', i')$ is a finite sheeted topological covering, then (A, i) admits a lattice. That is,*
(a) if (A', i') is unimodular, then (A, i) is unimodular

(b) if (A', i') has finite volume, then (A, i) has finite volume

(c) if (A', i') has bounded denominators, then (A, i) has bounded denominators.

4 Full graphs of subgroups

In this section, we produce a pair of lattice subgroups ($\Gamma \leq \Gamma'$) by constructing a 'full graph of subgroups' encoding Γ from a graph of groups encoding a given lattice Γ'. This method was used in ([B], 1.14) and ([CR], section 1) and has no analog in classical topological covering theory.

Let (A, i) be an edge-indexed graph. Let $\mathbb{A}'$ and $\mathbb{A}$ be groupings of (A, i). Then $\mathbb{A} = (A, \mathcal{A})$ is called a *full graph of subgroups* of $\mathbb{A}' = (A, \mathcal{A}')$ (as in ([B], (1.14)) if $\mathcal{A}_a \leq \mathcal{A}'_a$ for $a \in A$, and for $e \in EA$, $\mathcal{A}_e \leq \mathcal{A}'_e$, and $\alpha_e = \alpha'_e|_{\mathcal{A}_e}$. We further assume that for $e \in EA$, with $\partial_0 e = a$, $\mathcal{A}_a \cap \alpha_e \mathcal{A}'_e = \alpha_e \mathcal{A}_e$, that is $\mathcal{A}_a/\alpha_e\mathcal{A}_e \longrightarrow \mathcal{A}'_a/\alpha_e\mathcal{A}'_e$ is injective, and hence bijective. This assumption implies that $\pi_1(\mathbb{A}, a) \leq \pi_1(\mathbb{A}', a')$ ([B], (1.14)).

If $\mathbb{A}'$ is a graph of finite groups of finite volume, then this yields a lattice subgroup pair $\Gamma \leq \Gamma'$ with $\Gamma = \pi_1(\mathbb{A}, a)$ and $\Gamma' = \pi_1(\mathbb{A}', a')$, $a, a' \in VA$.

As an example of an application this, in [CR], the authors constructed an *infinite tower of groupings* on (A, i), that is, a semi-infinite sequence $(\mathbb{A}_i)_{i \in \mathbb{Z}_{>0}}$ of groupings of (A, i) such that each $\mathbb{A}_i$ is a full graph of proper subgroups of $\mathbb{A}_{i+1}$. A tower of faithful groupings then induces an infinite ascending chain of fundamental groups:

$$\pi_1(\mathbb{A}_1, a_0) < \pi_1(\mathbb{A}_2, a_0) < \pi_1(\mathbb{A}_3, a_0) < \ldots$$

with $Vol(\mathbb{A}_i) \to 0$ as $i \to \infty$.

5 Subgraphs of subgroups

In this section, we produce a pair of lattice subgroups $(\Gamma \leq \Gamma')$ by constructing a 'subgraph of subgroups' $\mathbb{A}$ of $\mathbb{A}'$ on (A', i') as in ([B], 1.14).

We begin with an edge-indexed graph (A', i') that admits an X-lattice, and a lattice grouping $\mathbb{A}' = (A', \mathcal{A}')$ of (A', i'). Let $\Gamma' = \pi_1(\mathbb{A}', a')$. Let (A, i) be a subgraph of (A', i'), where $i = i'|_A$. We aim to construct a lattice subgroup $\Gamma \leq \Gamma'$ by constructing a graph of groups on (A, i) using subgroups of $\mathbb{A}'$.

Lemma 2 *If $Vol(A', i') < \infty$, then $Vol(A, i) < \infty$.*

Proof
Let $a_0 \in VA \cap VA'$. Let

$$\begin{aligned} V' &= Vol_{a_0}(A', i') \\ &= \sum_{a' \in VA'} \frac{1}{\left(\frac{\Delta a'}{\Delta a_0}\right)} \\ &< \infty. \end{aligned}$$

Then

$$\begin{aligned} V &= Vol_{a_0}(A, i) \\ &= \sum_{a \in VA} \frac{1}{\left(\frac{\Delta a}{\Delta a_0}\right)} \\ &< \infty \end{aligned}$$

□

Lemma 3 *Let (A', i') be a unimodular, edge-indexed graph with bounded denominators. Let (A, i) with $i = i'|_A$ be a subgraph of (A', i'). Then (A, i) also has bounded denominators.*

Proof Obvious.□

Corollary 1 *If (A', i') admits a lattice and (A, i) is a subgraph with $i = i'|_A$, then (A, i) also admits a lattice.*

To construct a lattice grouping $\mathbb{A}$ of (A, i), we choose subgroups

$$\mathcal{A}_a \leq \mathcal{A}'_a, \ a \in VA$$

$$\mathcal{A}_e \leq \mathcal{A}'_e, \ e \in EA$$

and $\alpha_e = \alpha'_e|_{\mathcal{A}_e}$.

We further assume that for $e \in EA$, $\partial_0 e = a$, we have

$$\mathcal{A} \cap \alpha'_e \mathcal{A}'_e = \alpha'_e \mathcal{A}_e$$

that is, that

$$\mathcal{A}_a / \alpha_e \mathcal{A}_\alpha \to \mathcal{A}'_a / \alpha_e \mathcal{A}'_e$$

is injective.

Then $\mathbb{A}$ is a 'subgraph of subgroups' of $\mathbb{A}'$ in the sense of ([B],(1.14)). If $\mathcal{A}_a = \mathcal{A}'_a$ for some, hence every, vertex $a \in VA$ then we call $\mathbb{A}$ a 'subgraph of full groups' of $\mathbb{A}'$. Let

$$\Gamma = \pi_1(\mathbb{A}, a), \ a \in VA$$

then we obtain an inclusion

$$\Gamma \leq \Gamma'$$

which is of finite index k, where

$$k = \frac{Vol(\Gamma \backslash\backslash X)}{Vol(\Gamma' \backslash\backslash X)} = \frac{Vol(\mathbb{A})}{Vol(\mathbb{A}')}.$$

Moreover, if A is a finite subgraph of A', then $\mathbb{A}$ is an *uniform* lattice.

An application of this method we have the following (Theorem 2) which gives us a way of approximating non-uniform tree lattices by uniform ones. We will make use of the following lemma.

Lemma 4 *Let (A, i) be an infinite edge-indexed graph. Let $\mathbb{A}$ be a finite faithful grouping of (A, i). Then there is a finite subgraph (A_0, i_0) of (A, i) on which the restriction $\mathbb{A} \mid_{A_0}$ is faithful.*

Proof Write $A = \cup_{k=1}^{\infty} A_k$ where $A_1 \subset A_2 \subset A_3 \subset \ldots$ are all finite graphs. Choose a basepoint $a \in A_1$. For each $k \geq 1$ let N_k be the kernel of the action of $\pi_1(\mathbb{A} \mid_{A_k}, a) < \pi_1(\mathbb{A}, a)$ on $\widetilde{(\mathbb{A} \mid_{A_k}, a)}$. Then N_k fixes the subtree $\widetilde{(\mathbb{A} \mid_{A_k}, a)}$ of $\widetilde{(\mathbb{A}, a)}$. So $N_1 \supset N_2 \supset N_3 \supset \ldots$. Since $\mathbb{A}$ is a finite grouping, all N_k are finite. As $\mathbb{A}$ is faithful we must have $\cap_{k=1}^{\infty} N_k = \{1\}$. Thus $N_k = \{1\}$ for some k. We now take A_0 to be A_k for such k. $\Box$

Theorem 2 *Let Γ be a non-uniform X-lattice.Then Γ contains an infinite chain of subgroups $\Lambda_1 < \Lambda_2 < \Lambda_3 < \ldots$ where each Λ_k is a uniform X_k-lattice, X_k a subtree of X.*

Proof Let $\mathbb{A}$ be the graph of groups for Γ and let $(A, i) = I(\mathbb{A})$. Then $\mathbb{A}$ is a finite faithful grouping of (A, i). Let (A_0, i_0) be as in Lemma 4. Let

$$A_0 \subset A_1 \subset A_2 \subset \ldots$$

be any infinite sequence of finite subgraphs of A. We define an indexing on each A_k for $k \geq 0$ by $i_k = i \mid_{A_k}$. Then each (A_k, i_k) is unimodular, since a closed path in (A_k, i_k) is closed in (A, i).

The universal covering X_k of (A_k, i_k) is a subtree of X. This defines a tower of subtrees of X:

$$X_0 \subset X_1 \subset X_2 \subset \ldots$$

For groupings of (A_k, i_k), choose subgraphs of full groups of $\mathbb{A}$, that is take $\mathbb{A}_k$ to be $\mathbb{A} \mid_{A_k}$.

We get inclusions of fundamental groups

$$\pi_1(\mathbb{A}_0, a) < \pi_1(\mathbb{A}_1, a) < \pi_1(\mathbb{A}_2, a) < \ldots < \Gamma.$$

Set $\Lambda_k = \pi_1(\mathbb{A}_k)$. Then we get a tower of inclusions:

$$\Lambda_0 < \Lambda_1 < \Lambda_2 < \ldots < \Gamma.$$

As in Lemma 4 the kernels of the actions of $\pi_1(\mathbb{A}_k, a)$ are descending and $\mathbb{A}_0$ is faithful hence we must have that each $\mathbb{A}_k$ is faithful. Thus each Λ_k is a uniform X_k - lattice, where X_k is the universal cover of (A_k, i_k), and X_k is a subtree of X. Of course Λ_k may not be uniform or even a lattice as a subgroup of Γ.$\Box$

6 Coverings of abelian groupings

Let $\phi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs, and assume that (A, i) and (A', i') admit finite groupings. In this section we consider the following:

Question 1 *Is it possible to find faithful finite groupings $\mathbb{A}$ and $\mathbb{A}'$ of (A, i) and (A', i') respectively in such a way that ϕ extends to a covering morphism $\Phi : \mathbb{A} \longrightarrow \mathbb{A}'$.*

A positive answer to Question 1 would then give rise to a pair $\Gamma \leq \Gamma'$ of discrete subgroups of $Aut(X)$, where $X = \widetilde{(A,i)} = \widetilde{(A',i')}$ and $\Gamma = \pi_1(\mathbb{A}, a)$, $\Gamma' = \pi_1(\mathbb{A}', a')$, $a \in VA$, and $a' = \phi(a)$.

To answer this question let us first recall what is meant by a covering morphism of graphs of groups originally defined by Bass in [B]. In [B] Bass noted that if there is a general covering morphism between two graphs of groups there exists a special one (which he referred to as $\delta\Phi$) which requires less information. Since we only care about the existence of a covering morphism we will make use of this fact and say that a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{A} \to \mathbb{A}'$ consists of:

(1) a graph morphism $\phi : A \to A'$;

(2) monomorphisms

$$\phi_a : \mathcal{A}_a \to \mathcal{A}'_{\phi(a)} \quad (a \in A), \qquad \phi_e = \phi_{\bar{e}} : \mathcal{A}_e \to \mathcal{A}'_{\phi(e)} \quad (e \in EA);$$

(3) For each $e \in EA$ with $a = \partial_0 e$ an element $\delta_e \in \mathcal{A}'_{\phi(a)}$ such that the following two conditions hold:

(a) the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}_e \arrow[r, "\alpha_e"] \arrow[d, "\phi_e"'] & \mathcal{A}_a \arrow[d, "\phi_a"] \\
\mathcal{A}'_{\phi(e)} \arrow[r, "ad(\delta_e)\cdot\alpha'_{\phi(e)}"] & \mathcal{A}'_{\phi(a)}
\end{tikzcd}$$

where $ad(x)(s) = xsx^{-1}$.

(b) For $f \in EA'$, $a' = \partial_0 f$ and $a \in \phi^{-1}(a')$, the map

$$\Phi_{a/f} : \left(\coprod_{e \in \phi_{(a)}^{-1}(f)} \mathcal{A}_a / \alpha_e \mathcal{A}_e \right) \longrightarrow \mathcal{A}'_{\phi(a)} / \alpha'_f \mathcal{A}'_f$$

defined by

$$\Phi_{a/f}([s]_e) = [\phi_a(s)\delta_e]_f$$

is bijective (where $s \in \mathcal{A}_a$ and $[s]_e$ is the class of s in $\mathcal{A}_a/\alpha_e\mathcal{A}_e$).

To simplify matters greatly let us consider only the special case where the action $ad(\delta_e)$ is trivial. This must be the case in particular when $\mathcal{A}'_{\phi(a)'}$ is abelian. Since the maps ϕ_a and ϕ_e are monomorphisms we may identify the groups $\mathcal{A}_a$ and $\mathcal{A}_e$ with their images in $\mathcal{A}_{\phi(a)}$ and $\mathcal{A}_{\phi(e)}$ respectively. Condition $(3)(a)$ then becomes

$$(3)(a') \qquad \alpha_e = \alpha'_{\phi(e)}|_{\mathcal{A}_e}.$$

We have the following,

Theorem 3 *Let $a \in VA$ and let $f \in EA'$ be such that $\partial_0(f) = \phi(a)$. If $\mathcal{A}'_{\phi(a)}$ is abelian, then the subgroup $\alpha_e\mathcal{A}_e$ is independent of the choice of edge $e \in \phi_{(a)}^{-1}(f)$.*

Proof Let $s \in \alpha_e \mathcal{A}_e$ for some $e \in \phi_{(a)}^{-1}(f)$. Let e' be another edge in $\phi_{(a)}^{-1}(f)$. Since $\alpha_e = \alpha_f$ restricted to $\mathcal{A}_e$ we have that $s \in \alpha_f \mathcal{A}_f$. As $\mathcal{A}'_{\phi(a)}$ is abelian it follows that for $x \in \mathcal{A}'_{\phi(a)}$ we have $[sx]_f = [x]_f$, thus

$$\Phi_{a/f}([s]_{e'}) = [s\delta_{e'}]_f = [\delta_{e'}]_f = \Phi_{a/f}([1]_{e'}).$$

Since $\Phi_{a/f}$ is injective we must then have that $[s]_{e'} = [1]_{e'}$ and thus $s \in \alpha_{e'} \mathcal{A}_{e'}$. So we see that for any two edges e and e' in $\phi_{(a)}^{-1}(f)$ we have

$$\alpha_e \mathcal{A}_e \subset \alpha_{e'} \mathcal{A}_{e'} \subset \alpha_e \mathcal{A}_e \square$$

In terms of conditions on the edge-indexed graphs we get the following corollary.

Corollary 2 *If $\phi : (A, i) \to (A', i')$ is a covering of edge-indexed graphs and we wish to find finite abelian groupings of (A, i) and (A', i') in such a way as to extend ϕ to a covering morphism of graphs of groups then we must have the following necessary condition on ϕ which we will call locally constant fibers:*

(LCF) If for $e_1, e_2 \in EA$, $\phi(e_1) = \phi(e_2)$ and $\partial_0 e_1 = \partial_0 e_2$, then $i(e_1) = i(e_2)$.

We should point out that for ease and for our purposes we have spoken of the condition (LCF) as a property that holds throughout the covering and note that is necessary in order to extend ϕ using groupings that are abelian throughout. We may, however, desire only to use an abelian group at a specific vertex $a' \in VA'$ (which would in turn force abelian groups at all vertices in the fiber above a' and all edges in the star of any of those vertices). In this case we would still require the condition of (LCF) but only for those edges in the star of a vertex in the fiber of a'. That is we could speak of ϕ being (LCF) at a' for some vertex $a' \in VA'$ if

For $e_1, e_2 \in EA$ with $\phi(e_1) = \phi(e_2)$ and $\phi(\partial_0 e_1 = \partial_0 e_2) = a'$
we have $i(e_1) = i(e_2)$.

In these terms a covering ϕ is (LCF) if it is (LCF) at a' for all $a' \in VA'$. So (LCF) is a necessary condition for the edge-indexed graph covering ϕ to extend to a covering morphism of finite abelian graphs of groups. We also have the following sufficient condition on the finite abelian graphs of groups themselves.

Theorem 4 *Let $\phi : (A, i) \to (A', i')$ be a covering of edge-indexed graphs. Let $\mathbb{A}$ and $\mathbb{A}'$ be finite abelian groupings of (A, i) and (A', i') respectively. Suppose further that*

(i) $\mathcal{A}_a \leq \mathcal{A}'_{\phi(a)}$, $a \in VA$ and $\mathcal{A}_e \leq \mathcal{A}'_{\phi(e)}$, $e \in EA$ and $\mathcal{A}_a$ and $\mathcal{A}_e$ are identified with their images in $\mathcal{A}'_{\phi(a)}$ and $\mathcal{A}'_{\phi(e)}$ respectively.

(ii) $\alpha_e = \alpha'_{\phi(e)}|_{\mathcal{A}_e}$, $e \in EA$.

(iii) $\mathcal{A}_a \cap \alpha'_f \mathcal{A}'_f = \alpha_e \mathcal{A}_e$, where $a \in VA$ and $f \in EA'$ is such that $\partial_0(f) = \phi(a)$,

Then ϕ extends to a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{A} \longrightarrow \mathbb{A}'$.

Proof The monomorphisms ϕ_a and ϕ_e are naturally just the inclusion maps. We have seen that since we are using abelian groups condition (3)(a) of a covering morphism reduces to condition (3)(a') which is what we require by (ii). It remains only to define the elements δ_e for each $e \in EA$ and show that condition (3)(b) of a covering morphism is satisfied. So let $a \in VA$ and let $f \in EA'$ be such that $\partial_0(f) = \phi(a)$. We take $\{\delta_e\}_{e \in \phi_{(a)}^{-1}(f)}$ to be the distinct coset representatives of

$$\left(\mathcal{A}'_{\phi(a)}/\alpha'_f\mathcal{A}'_f\right) / \left(\mathcal{A}_a/\alpha_e\mathcal{A}_e\right) = \left(\mathcal{A}'_{\phi(a)}/\alpha'_f\mathcal{A}'_f\right) / \left(\mathcal{A}_a/\mathcal{A}_a \cap \alpha'_f\mathcal{A}'_f\right).$$

We first note that this defintion makes sense. $\mathcal{A}_a/\alpha_e\mathcal{A}_e$ is naturally a subgroup of $\mathcal{A}'_{\phi(a)}/\alpha'_f\mathcal{A}'_f$ by the map which takes $[s]_e$ to $[s]_f$. That map is well-defined since $\alpha_e\mathcal{A}_e$ is a subgroup of $\alpha'_f\mathcal{A}'_f$. It is clearly a homomorphism and it is in fact a monomorphism by (iii).

We also must show that the number of cosets above is precisely the number of edges $e \in p_{(a)}^{-1}(f)$. Recall that if K and L are subgroups of H then the index $[H/K : L/K \cap L] = [H : K]/[L : K \cap L]$. Hence the number of cosets we have is $i'(f)/i(e)$. However $i(e)$ is constant on $\phi_{(a)}^{-1}(f)$ and ϕ is a covering, so we have that the number of edges in $\phi_{(a)}^{-1}(f)$ must be exactly $i'(f)/i(e)$ as well.

The map $\Phi_{a/f}$ is then just the map that for each $e \in \phi_{(a)}^{-1}(f)$ sends the subgroup $\mathcal{A}_a/\alpha_e\mathcal{A}_e$ to the coset $(\mathcal{A}_a/\alpha_e\mathcal{A}_e)\delta_e$ which is clearly bijective as we have exactly one representative from each coset.$\square$

7 Elementary abelian groupings of the denominator clearing covering

In the last section we noted that a necessary condition for a covering ϕ to extend to a covering morphism of graphs of finite abelian groups is that ϕ satisfy (LCF). One special case of coverings which satisfy this condition are the 'denominator clearing' covers constructed in [BCR]. This was a construction used to show that any unimodular edge-indexed graph (A, i) is covered by an unimodular edge-indexed graph (B, j) which has bounded denominators. Of course in our situation where we hope to group both edge-indexed graphs with finite groups it is necessary that (A, i) has bounded denominators already. While the [BCR] covering in this case is no longer useful in its original purpose of bounding denominators it is still well-defined and an examination of how we might go about grouping each edge-indexed graph in this particular example will be useful to us in our search for how to extend more general edge-indexed graph coverings to covering morphims of graphs of fintite groups.

At this point we must refer the reader to [BCR] for the detailed construction and notation of the denominator clearing cover $p : (B, j) \longrightarrow (A, i)$. We assume that (A, i) is unimodular and has bounded denominators and we will try to faithfully group both (B, j) and (A, i) with finite abelian groups in such a way as to satisfy the conditions of Theorem 4 and thereby extend p to a covering morphism of graphs of groups.

First let us examine what the orders of these groups should be. Let a_0 and b_0 be the vertices as in [BCR]. That is a_0 is a chosen vertex in VA and b_0 is the unique vertex above it in VB. Recall the notation

$$N_a(b) := \frac{\Delta b}{\Delta a} \qquad N_a(e) := \frac{N_a(\partial_0 e)}{i(e)}.$$

Then we saw in [BCR] that $N_{b_0}(x)$ is an integer for all $x \in VB \cup EB$. Using the terminology of [BK] the ordering is integral and since $N_{b_0}(b_0) = 1$ it is the unique minimal integral vertex ordering. A finite grouping respecting this ordering is guaranteed to be faithful. For (A, i) the rational numbers $N_{a_0}(x)$ are generally not integers, but we have assumed that their denominators are bounded and thus as in [BK] we may find a number D which is the least common multiple of these denominators. The function $N(x) = D \cdot N_{a_0}(x)$ is then the minimal integral vertex ordering on this edge-indexed graph and a grouping respecting this ordering will be faithful. Let us try then to obtain finite groupings respecting these orderings. That is the order of a group $\mathcal{B}_b$ will be $N_{b_0}(b)$ and the order of a group $\mathcal{A}_a$ will be $N(a) = D \cdot N_{a_0}(a)$.

Let us look at the order of our desired groups in the local situation:

$$\begin{tikzcd}
\mathcal{B}_e \arrow[r] \arrow[d] & \mathcal{B}_b \arrow[d] \\
\mathcal{A}_f \arrow[r] & \mathcal{A}_a.
\end{tikzcd}$$

If $\partial_0 f = a$ and $\partial_1 f = a'$ then using the notation and results of [BCR, Section 3.2] the corresponding group orders should be:

$$\begin{tikzcd}
c(a,a') \arrow[r] \arrow[d] & c(a,a')c_{a'}(a) \arrow[d] \\
\frac{c(a,a')}{d(a)d_a(a')\delta(f)} \cdot D \arrow[r] & \frac{c(a,a')c_{a'}(a)}{d(a)} \cdot D.
\end{tikzcd} \tag{1}$$

We introduce the new notation that for an edge $f \in EA$ with $\partial_0 f = a$ and $\partial_1 f = a'$

$$r(f) := \frac{D}{d(a)d_a(a')\delta(f)}.$$

Thus diagram 1 becomes:

$$\begin{tikzcd}
c(a,a') \arrow[r] \arrow[d] & c(a,a')c_{a'}(a) \arrow[d] \\
c(a,a')r(f) \arrow[r] & c(a,a')c_{a'}(a)r(f)d_a(a')\delta(f).
\end{tikzcd} \tag{2}$$

We note that we may not be able to successfully exploit cyclic groupings in this situation. The difficulty is illustrated in the following example which arises from the covering as shown in figure 1 of [BCR]. The orders of our respective groups at a certain local situation is shown below.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \\ 6 \arrow[r] & 12 \end{tikzcd} \tag{3}$$

This does not work for us as $\mathbb{Z}/6\mathbb{Z} \cap \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ whereas our conditions would require that the intersection be trivial (the edge group above).

If, however, we use instead elementary abelian groups we are in the clear. Let $\mathcal{E}_n$ denote the elementary abelian group of order n. The following property is key:

$$\mathcal{E}_n \times \mathcal{E}_m = \mathcal{E}_{n \cdot m} \tag{4}$$

Thus when using elementary abelian groups the local picture becomes:

$$\begin{tikzcd} \mathcal{E}_{c(a,a')} \arrow[r] \arrow[d] & \mathcal{E}_{c(a,a')} \times \mathcal{E}_{c_{a'}(a)} \arrow[d] \\ \mathcal{E}_{c(a,a')} \times \mathcal{E}_{r(f)} \arrow[r] & \mathcal{E}_{c(a,a')} \times \mathcal{E}_{c_{a'}(a)} \times \mathcal{E}_{r(f)} \times \mathcal{E}_{d_a(a')\delta(f)}. \end{tikzcd} \tag{5}$$

The inclusion maps (which are not uniquely defined when using elementary abelian groupings) are here taken to be the natural maps that arise from the above factorization. It is not immediatley apparent that the vertex group inclusion can be taken to depend only on the vertex a and not a particular neighbor a', but we will see in the next section when we generalize that this is the case. For the time being we will focus just on the above "local" situation. Not only do we have the property that the edge maps upstairs are restrictions of those below, but we also now have the property that

$$\mathcal{E}_{c(a,a')} \times \mathcal{E}_{r(f)} \cap \mathcal{E}_{c(a,a')} \times \mathcal{E}_{c_{a'}(a)} = \mathcal{E}_{c(a,a')} \tag{6}$$

which is precisely what was needed.

8 Lattice subgroups with abelian stabilizers

So now let use the lessons taught in the last section to get some sufficient conditions on a covering $\phi : (B, j) \longrightarrow (A, i)$ of unimodular bounded denominator edge-indexed graphs so that it will extend to a covering morphism of finite faithful abelian graphs of groups. We have already seen one necessary condition in Section 6 which we called locally constant fibers (LCF). We now observe a further necessary condition.

Assume that $\phi : (B, j) \longrightarrow (A, i)$ is a covering of unimodular bounded denominator edge-indexed graphs satisfying (LCF). It follows that $j(e)$ divide $i(\phi(e))$ and we may thus form a new edge-indexed graph (B, k) with underlying graph B and indexing $k(e) = \frac{i(\phi(e))}{j(e)}$. We say that ϕ satisfies the relatively bounded denominator condition (RBD) if the edge-indexed graph (B, k) satisfies the bounded denominator condition. See example 2 of section 9 for an example of the graph (B, k) and a covering which does not sayify (RBD).

Theorem 5 *Let $\phi : (B, j) \longrightarrow (A, i)$ be a covering of edge-indexed graphs satisfying (LCF). Then a necessary condition for ϕ to extend to a covering morphism of finite graphs of groups is that ϕ satisfy (RBD).*

Proof Suppose there exist finite groupings $\mathbb{A} = (A, \mathcal{A})$ of (A, i) and $\mathbb{B} = (B, \mathcal{B})$ of (B, j) and a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{B} \longrightarrow \mathbb{A}$. Then it follows that for $b \in VB$ we have $\phi_b(\mathcal{B}_b) \leq \mathcal{A}_{\phi(b)}$ and so $|\mathcal{B}_b|$ divides $|\mathcal{A}_{\phi(b)}|$. We then have that

$$N(b) = \frac{|\mathcal{A}_{\phi(b)}|}{|\mathcal{B}_b|}$$

is an integral vertex ordering of (B, k) and thus necessarily (B, k) satisfies the bounded denominator condition and by definition ϕ satisfies (RBD). □

So we've seen that a necessary condition to extend ϕ to a covering morphism of finite faithful abelian graphs of groups is that ϕ satisfy (LCF) and (RBD). We would like to say that the conditions are also sufficient but we are unable to do so. Even if we drop the requirement that the groupings be abelian, (RBD) alone is not sufficient, as can be seen in example 3. If we strengthen the (LCF) condition, though, we are able to obtain sufficient conditions.

We say that a covering $\phi : (B, j) \longrightarrow (A, i)$ satisfies the globally constant fibers condition if

(GCF) For $e_1, e_2 \in EB$ with $\phi(e_1) = \phi(e_2)$ we have $j(e_1) = j(e_2)$.

Lemma 5 *Let $\phi : B \longrightarrow A$ be a graph morphism and let (B, j) be a unimodular edge-indexed graph with bounded denominators. Suppose further the indexing j is constant on the fibers of ϕ. That is $j(e_1) = j(e_2)$ if $\phi(e_1) = \phi(e_2)$. Note that such is the case for both (B, j) and (B, k) if ϕ extends to a covering of edge-indexed graphs satisfying (GCF) and (RBD). Then any vertex ordering $N : (B, j) \longrightarrow \mathbb{Q}^x_{>0}$ is constant on the fibers of ϕ. That is if $\phi(b_1) = \phi(b_2)$ then $N(b_1) = N(b_2)$*

Proof Let $\gamma = (e_1, e_2, \ldots, e_n)$ be a path in (B, j) from b_1 to b_2. Then by assumption we have that $j(e_i)$ and hence $\Delta(e_i)$ and hence $\Delta(\gamma)$ depends only on its image by ϕ in A. If $\Delta(\gamma) = 1$ then it follows that $N(b_1) = N(b_2)$. So assume instead that $\Delta(\gamma) \neq 1$. If necessary switch b_1 and b_2 and replace γ by γ^{-1} to insure that $\Delta(\gamma) = q$ for some rational q which is not an integer.

Notice that $\phi(\gamma)$ is a path in A from $a = \phi(b_1) = \phi(b_2)$ to itself. So let γ_2 be a lifting of $\phi(\gamma)$ starting at b_2 and ending at some vertex b_3 with $\phi(b_3) = a$. Inductively we define γ_n to be a lifting of $\phi(\gamma)$ starting at b_n and ending at some vertex b_{n+1}. Then $\Delta(\gamma_n) = q$ for all n and thus $\Delta(\gamma \circ \gamma_2 \circ \cdots \circ \gamma_n) = q^n$ thus contradicting our assumption that (B, j) has bounded denominators. □

Theorem 6 *Let $\phi : (B,j) \longrightarrow (A,i)$ be a covering of unimodular bounded denominator edge-indexed graphs satisfying (GCF) and (RBD). Then there exist finite faithful groupings $\mathbb{A} = (A, \mathcal{A})$ of (A,i) and $\mathbb{B} = (B, \mathcal{B})$ of (B,j) and a covering morphism $\Phi = (\phi, (\delta)) : \mathbb{B} \longrightarrow \mathbb{A}$.*

Proof Let (B,k) be the edge-indexed graph constructed as in the definition of (RBD). Let $\mathbb{B} = (B, \mathcal{B})$ be a faithful finite abelian grouping of (B,j) such that the groups are all isomorphic on each (global) fiber of ϕ. That is $\mathcal{B}_{b_1} = \mathcal{B}_{b_2}$ if $\phi(b_1) = \phi(b_2)$. We can find such a grouping since (B,j) is unimodular with bounded denominators and by the previous lemma any vertex ordering of (B,j) is constant on the fibers of ϕ. Likewise let $\mathbb{B}' = (B, \mathcal{B}')$ be a faithful finite abelian grouping of (B,k) with isomorphic groups along each fiber. Again the previous lemma and the (RBD) condition guarantee the existence of such a grouping. We will define a grouping $\mathbb{A} = (A, \mathcal{A})$ on (A,i) by taking the group at a vertex or edge $a \in VA \cup EA$ to be the direct product of the groups $\mathcal{B}_b$ and $\mathcal{B}'_b$ for some $b \in \phi^{-1}(a)$. Recall the choice of b in the fiber does not mattter as all groups in the fiber are isomorphic in both $\mathbb{B}$ and $\mathbb{B}'$.

The edge monomorphisms in $\mathbb{A}$ are those obtained naturally from the edge monomorphisms in $\mathbb{B}$ and $\mathbb{B}'$. First let us verify that $\mathbb{A}$ is consistant with the indexing on (A,i). If e is an edge in EA then

$$[\mathcal{A}_{\partial_0 e} : \alpha_e \mathcal{A}_e] = j(e) \cdot k(e) = i(e).$$

Since each grouping $\mathbb{B}$ and $\mathbb{B}'$ is faithful it follows that $\mathbb{A}$ is faithful (see for example section 5.5 of [R]). The groups of $\mathbb{B}$ are naturally subgroups of the groups of $\mathbb{A}$ and the edge monomorphisms of $\mathbb{A}$ were defined to be extensions of the edge monomorphisms of $\mathbb{B}$. All that remains is to check condition *(iii)* of Theorem 4 to establish the existence of our desired covering morphism. We see that if $\phi(e) = f$ with $\partial_0 e = b$ and $\partial_0 f = a$ then

$$\mathcal{B}_b \cap \alpha_f \mathcal{A}_f = \mathcal{B}_e$$

which is exactly what is desired. □

9 Examples

In this section we provide a number of examples of the construction of lattice subgroup pairs using the methods introduced in the previous sections.

Example 1 Consider the covering $p : (B,j) \longrightarrow (A,i)$ shown in figure 1.

Then $p : (B,j) \longrightarrow (A,i)$ is a denominator clearing covering (as in [BCR] and section 7). Hence there are abelian groupings $\mathbb{B}$ of (B,j) and $\mathbb{A}$ of (A,i), and a covering morphism $\Phi : \mathbb{B} \longrightarrow \mathbb{A}$ as shown in figure 2.

Note also that $p : (B,j) \longrightarrow (A,i)$ satisfies (GCF) and clearly also (RBD). Let $\Lambda = C_{q+1} * \cdots * C_{q+1}$ ($(q+1)$-factors), and let $\Gamma = C_{q+1} * C_{q+1}$, where C_{q+1} denotes the cyclic group of order $q+1$.

We obtain the inclusion $\Lambda \leq \Gamma$ of discrete subgroups of $G = Aut(X_{q+1})$. Furthermore Λ and Γ are both G-lattices and H-lattices, for $H = SL_2(\mathbb{F}_q((t^{-1})))$.

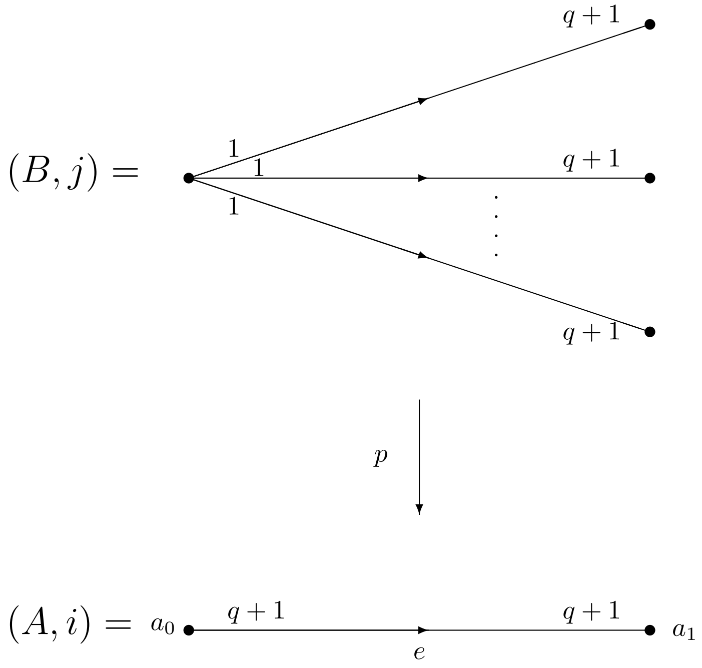

Figure 1: Covering $p : (B, j) \longrightarrow (A, i)$ of Example 1

Since p has finite fibers, Γ is cocompact in both G and H, and thus Λ is cocompact in both G and H.

Example 2 Consider the covering $p : (B, j) \longrightarrow (A, i)$ shown in figure 3.

This is an example of a covering of edge-indexed graphs where both (A, i) and (B, j) are unimodular and have bounded denominator. However no covering morphism of faithful graphs of finite groups exists. To see this we can look at the orders of the groups at each vertex relative to the orders of the groups at the base vertices a_0 and b_0. In the graph (A, i) a group at a vertex a_n must be of order $2^{\frac{n+1}{2}}$ times the order of the group at a_0. The order of a group at a vertex b_n lying above a_n must be of the order 2^n times the order of the group at b_0. Since the order of the group a_i must be at least as large as the order of the group at b_i we must have the order of the group at a_0 is infinite. Note that the covering here satisfies (GCF) but does not satisfy (RBD) as can be seen by looking at the edge-indexed graph (B, k) shown in figure 4.

Example 3 For each $n \geq 1$ consider the covering $p_n : (B_n, j_n) \longrightarrow (A, i)$ shown in figure 5.

For each $n \geq 1$ we have a covering of edge-indexed graphs where both (A, i) and (B, j) are unimodular and have bounded denominator. Yet when $n \geq 4$ no covering of faithful graphs of finite groups exist. Unlike the last example, both graphs here are finite. The problem, though, can still be seen by examing forced requirements for the orders of groups. Since the order of the group at b_0 is at least 2, the order of the group at b_k is at least 2^{2k+1} for $k < n$ and the order of

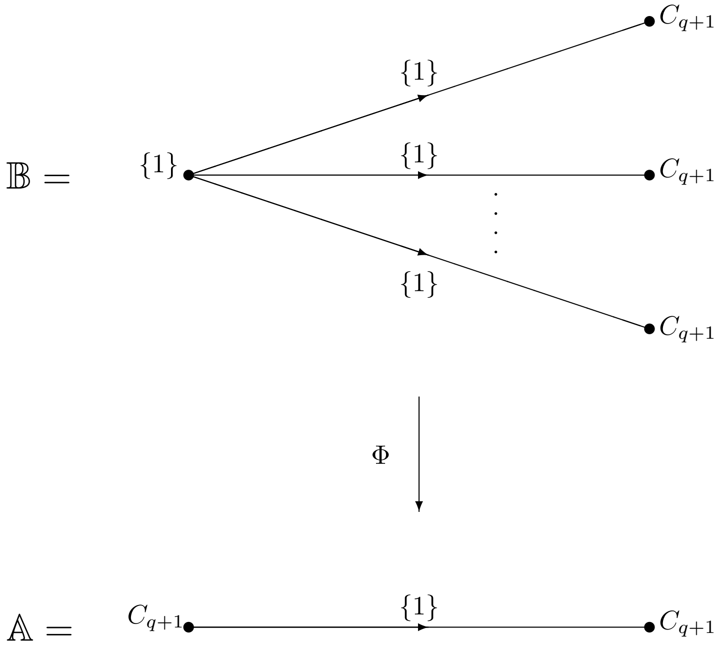

Figure 2: Covering morphism $\Phi : \mathbb{B} \longrightarrow \mathbb{A}$ of Example 1

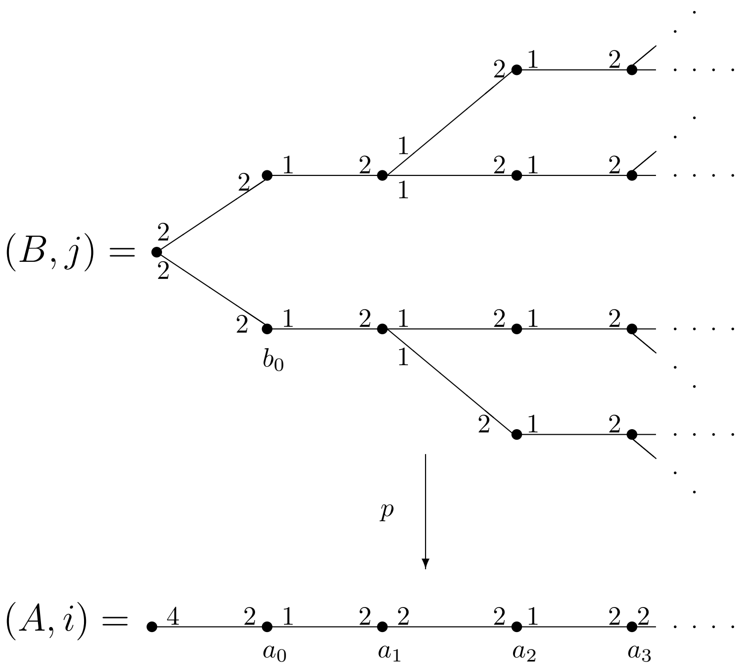

Figure 3: Covering $p : (B, j) \longrightarrow (A, i)$ of Example 2

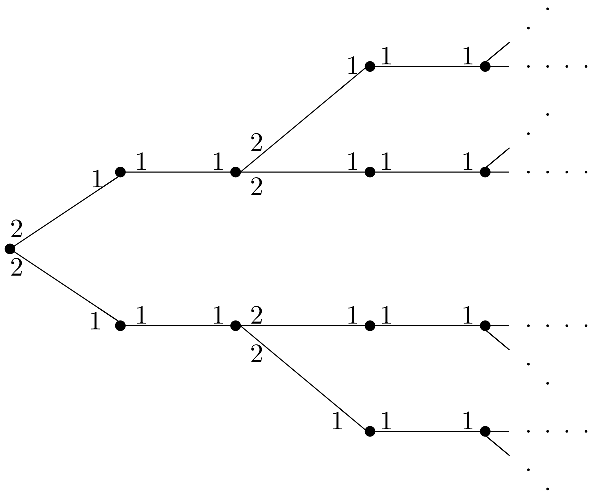

Figure 4: Edge-Indexed Graph (B, k)

the group at b_n is at least $3 \cdot 2^n$. Thus the order of the group at a_0 is also at least $3 \cdot 2^n$ which is not possible for a faithful grouping of (A, i) when $n \geq 4$ by a result of Goldschmidt [G].

Example 4 Consider the covering $p : (B, j) \longrightarrow (A, i)$ shown in figure 6.

In this last example we have a covering of edge-indexed graph which satisfies (LCF) but not (GCF). It turns out that this particular covering does extend to a covering of faithful graphs of finite abelian groups (in fact, it can be done using only 2-groups). We thus have the following corollary.

Corollary 3 *Although (RBD) and (GCF) together are sufficient for the existence of our desired graph of group coverings,(GCF) is not a necessary condition.*

This leads to an interesting question. While our proof of theorem 6 certainly made use of (GCF) (via lemma 5), perhaps it could be proven without such use. Perhaps something less is needed. We are thus led to conclude this paper with the following question.

Question 2 *If we are given a covering of unimodular edge-indexed graphs having bounded denominator, is (LCF) + (RBD) sufficient for the existence of an extension to a covering morphsim of faithful graphs of finite groups?*

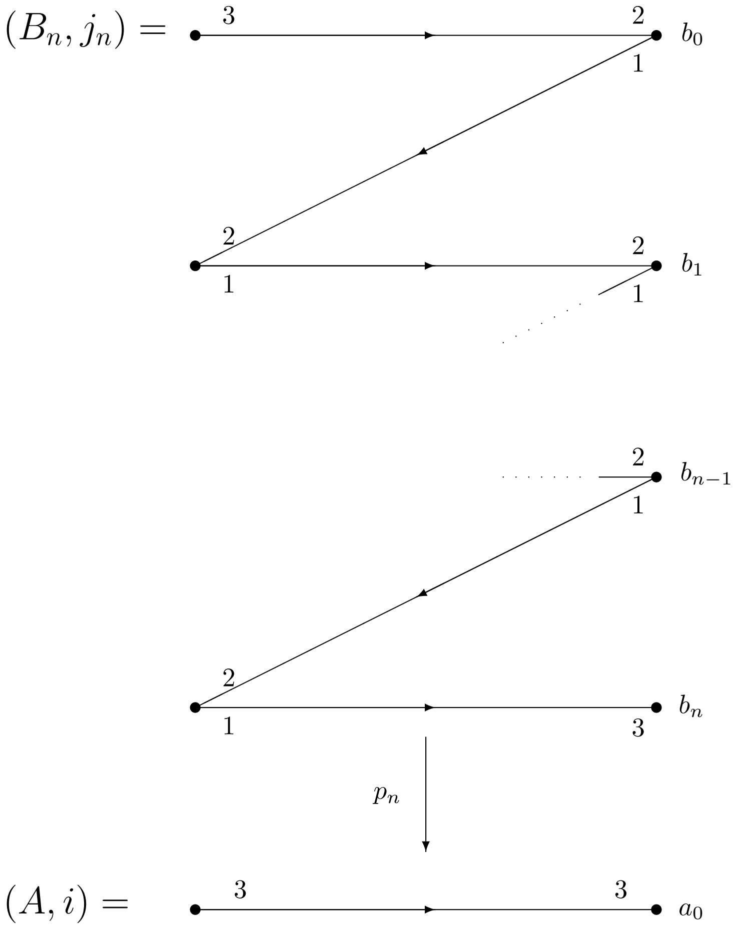

Figure 5: Covering $p_n : (B_n, j_n) \longrightarrow (A, i)$ of Example 3

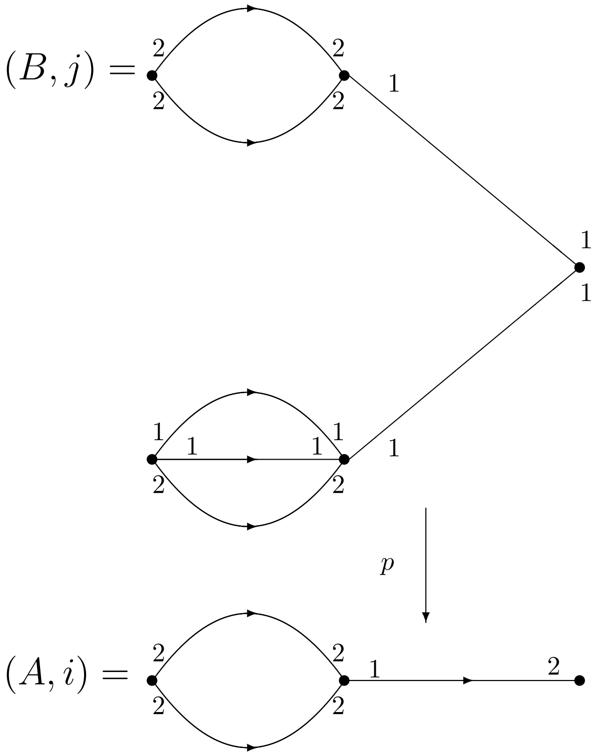

Figure 6: $p : (B, j) \longrightarrow (A, i)$ of Example 4

References

[B] Bass, H. Covering theory for graphs of groups. *Journal of Pure and Applied Algebra*, vol 89, pp. 3-47, 1993.

[BCR] Bass, H, Carbone L, and Rosenberg, G. The existence theorem for tree lattices. *Appendix [BCR], 'Tree Lattices' by Hyman Bass and Alex Lubotzky* Progress in Mathematics 176, Birkhauser, Boston, 2000.

[BK] Bass, H and Kulkarni, R. Uniform tree lattices, *Journal of the Amer Math Society* 3 (4), 1990.

[BL] BL Bass, H and Lubotzky, A. Tree lattices, *Progress in Mathematics 176*, Birkhauser, Boston, 2000.

[C1] Carbone, L. Non-uniform lattices on uniform trees. *Memoirs of the AMS*, vol 152, no 724, July 2001.

[C2] Carbone, L. The tree lattice existence theorems. *Comptes Rendus de l'Academie des Sciences*, Serie I, Mathematique, 335, pp. 223-228, 2002.

[CP] Carbone, L and Powell, L. Tree lattice subgroups.*A report on undergraduate research through VIGRE at Harvard*, 2000-2001.

[CR] Carbone, L and Rosenberg, G. Infinite towers of tree lattices. *Mathematical Research Letters* vol 8, pp. 1-10, 2001.

[G] Goldschmidt, D M. Automorphisms of trivalent graphs. *Annals of Mathematics* vol 2, no 111, pp. 377-40, 1980.

[L] Lim, Seonhee. Counting overlattices in automorphism groups of trees. Preprint 2003.

[R] Rosenberg, G. Towers and covolumes of tree lattices. *PhD. Thesis, Columbia University*, 2001

[S] Serre, J-P. Trees (Translated from the French by John Stilwell), Springer-Verlag, Berlin Heidelberg, 1980.