

Abstract simplicity of complete Kac-Moody groups over finite fields

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Abstract

Let G be a Kac-Moody group over a finite field corresponding to a generalized Cartan matrix A , as constructed by Tits. It is known that G admits the structure of a BN-pair, and acts on its corresponding building. We study the complete Kac-Moody group \widehat{G} which is defined to be the closure of G in the automorphism group of its building. Our main goal is to determine when complete Kac-Moody groups are abstractly simple, that is have no proper non-trivial normal subgroups. Abstract simplicity of \widehat{G} was previously known to hold when A is of affine type. We extend this result to many indefinite cases, including all hyperbolic generalized Cartan matrices A of rank at least three. Our proof uses Tits' simplicity theorem for groups with a BN-pair and methods from the theory of pro- p groups.

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1 Introduction

Let k denote a finite field which will remain fixed throughout the paper. Let A be a generalized Cartan matrix, and let \mathbb{G}_A be the corresponding Kac-Moody group functor of simply-connected type, as constructed by Tits in [25]. The group $G(A) := \mathbb{G}_A(k)$ is usually called a *minimal* or *incomplete* Kac-Moody group.

Distinct constructions of *complete* Kac-Moody groups are given in the papers of Carbone and Garland [4] and Remy and Ronan [20]; in both cases the group constructed is the completion of Tits' group $G(A)$ with respect to a certain topology. In [20], the topology comes from the action of $G(A)$ on its associated positive building; we denote the corresponding completion by $\widehat{G}(A)$.² In [4], one starts with the integrable highest-weight module V^λ for the Kac-Moody algebra $\mathfrak{g}(A)$ corresponding to a regular dominant integral weight λ , then considers a certain \mathbb{Z} -form, $V_{\mathbb{Z}}^\lambda$ of V^λ , and the action of $G(A)$ on a k -form, V_k^λ of $V_{\mathbb{Z}}^\lambda$. The corresponding completion of G (which depends on λ) will be denoted by $\widehat{G}^\lambda(A)$. We describe these constructions in detail in Section 2.

Though constructed in different ways, the groups $\widehat{G}(A)$ and $\widehat{G}^\lambda(A)$ have very similar structure. In particular, in both cases the complete Kac-Moody group is locally compact and totally disconnected. Recently, Baumgartner and Remy showed³ that for any weight λ , the Remy-Ronan completion $\widehat{G}(A)$ is a homomorphic image of the Carbone-Garland completion $\widehat{G}^\lambda(A)$ (see Theorem 2.5 for a precise statement).

The main goal of this paper is to investigate when complete Kac-Moody groups are abstractly simple. Recall that a topological group is called *abstractly simple* if it has no proper non-trivial normal subgroups, and *topologically simple* if it has no proper non-trivial closed normal subgroups. In view of Theorem 2.5, the simplicity question should be asked for the smaller groups $\widehat{G}(A)$. An obvious necessary condition for the simplicity of $\widehat{G}(A)$ is that A should be indecomposable: if A_1, \dots, A_k are indecomposable blocks of A , then $\widehat{G}(A) \cong \prod_{i=1}^k \widehat{G}(A_i)$. Remy [18, Theorem 2.A.1] proved⁴ that if A is indecomposable, then the group $\widehat{G}(A)$ is topologically simple when $|k| > 3$, and asked whether abstract simplicity holds as well (see [19, Question 30]). We answer this question in the affirmative for a large class of Kac-Moody groups.

Theorem 1.1. *Let A be an indecomposable generalized Cartan matrix. Assume that one of the following holds:*

- (a) $|k| > 3$, $p = \text{char}(k) > 2$, and any 2×2 submatrix of A is of finite or affine type;
- (b) $|k| > 3$ and any 2×2 submatrix of A is of finite type.

Then the group $\widehat{G}(A)$ is abstractly simple.

Remark. In the hypotheses of Theorem 1.1, a submatrix is not necessarily proper.

Abstract simplicity of $\widehat{G}(A)$ was previously known only when A is of finite type (in which case $\widehat{G}(A)$ is a finite group) or A is of affine type, in which case $\widehat{G}(A)$ is isomorphic to the group of K -points of a simple algebraic group defined over $K = k((t))$ [18]. The class of matrices covered by Theorem 1.1 includes many indefinite examples, including all hyperbolic matrices of rank at least three.

Remark. Recently, Caprace and Remy [3] proved (abstract) simplicity of the incomplete group $G(A)$ modulo its finite center in the case when the associated Coxeter group is not affine and assuming that $|k|$ is sufficiently large. If A is affine, the incomplete group $G(A)$ (modulo its center) is infinite and residually finite, and hence cannot be simple.

Briefly, our approach to proving Theorem 1.1 will be as follows. A celebrated theorem of Tits [2] gives sufficient conditions (called ‘‘simplicity axioms’’) for a group with a BN-pair to be simple. The group $\widehat{G}(A)$ routinely satisfies most of these axioms if $|k| > 3$ (for arbitrary A), which already implies topological

²Such completions are called *topological Kac-Moody groups* in [20].

³Private communication.

⁴Remy considers a more general class of quasi-split Kac-Moody groups.

simplicity of $\widehat{G}(A)$. It is not clear if the remaining axioms hold in general; however, they do hold if the “positive unipotent” subgroup $\widehat{U}(A)$ is topologically finitely generated – see Theorem 4.1 ($\widehat{U}(A)$ is defined as the completion of the group $U(A)$ generated by all positive root subgroups). This follows from the fact that $\widehat{U}(A)$ is a pro- p group and basic properties of pro- p groups (see [7, Chapter 1]).

Thus our main task is to prove finite generation of $\widehat{U}(A)$ under the hypotheses of Theorem 1.1 (see Theorem 6.4). In Section 6, we show that $\widehat{U}(A)$ is finitely generated as long as for every 2×2 submatrix C of A , the group $\widehat{U}(C)$ satisfies a certain technical condition which is “almost” equivalent to generation by simple root subgroups (see Theorem 6.1). In Section 7, we show that this condition is satisfied when C is a 2×2 matrix of finite or affine type (assuming $p > 3$), using an explicit realization of the group $\widehat{U}(C)$. When C is a 2×2 hyperbolic matrix, we do not know if $\widehat{U}(C)$ satisfies the above condition or even if $\widehat{U}(C)$ is finitely generated. Thus the questions of abstract simplicity of $\widehat{G}(A)$ and finite generation of $\widehat{U}(A)$ remain open whenever A contains a hyperbolic 2×2 submatrix.

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Some conventions about topological groups. Let G be a topological group. By a base (resp. subbase) for the topology of G we will mean a base (resp. subbase) of neighborhoods of the identity. Recall that a topological group G is *topologically generated by a set* S if the subgroup abstractly generated by S is dense in G . When discussing finite generation of pro- p groups, we will always be interested in topological generating sets, so the word ‘topological’ will often be omitted. Since (infinite) pro- p groups are never finitely generated as abstract groups, this convention should not cause any confusion.

2 Kac-Moody algebras and groups

2.1 Generalized Cartan matrices

Let $I = \{1, 2, \dots, l\}$ be a finite set. A matrix $A = (a_{ij})_{i,j \in I}$ is called a *generalized Cartan matrix* if its entries satisfy the following conditions:

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|--|---|
| (a) $a_{ij} \in \mathbb{Z}$ for $i, j \in I$, | (b) $a_{ii} = 2$, for $i \in I$ |
| (c) $a_{ij} \leq 0$ if $i \neq j$ | (d) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. |

By a *submatrix* of A , we mean a matrix of the form

$$A_J = (a_{ij})_{i,j \in J},$$

where J is a subset of I . We say that the submatrix A_J is *proper* if $J \neq I$. The matrix A is called *indecomposable* if there is no partition of the set I into two non-empty subsets so that $a_{ij} = 0$ whenever i belongs to the first subset, while j belongs to the second. A submatrix A_J is called an *indecomposable block* of A if A_J is indecomposable, and J is maximal with this property.

A generalized Cartan matrix A is said to be of

finite (classical) type if A is positive definite; in this case, A is the Cartan matrix of a finite dimensional semisimple Lie algebra,

affine type if A is positive semi-definite, but not positive definite,

indefinite type if A is neither of finite nor affine type.

An indefinite matrix is called *hyperbolic* (in the sense of [9, 5.10, p.66]) if every proper submatrix is of finite or affine type.

2.2 Kac-Moody algebras

For the rest of this section, fix a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, and let $l = |I|$. A *realization* of A over \mathbb{Q} is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where \mathfrak{h} is a vector space over \mathbb{Q} of dimension $2l - \text{rank}(A)$, and $\Pi = \{\alpha_1, \dots, \alpha_l\} \subseteq \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_l^\vee\} \subseteq \mathfrak{h}$ are linearly independent sets, such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for $i, j \in I$. As usual, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between \mathfrak{h} and \mathfrak{h}^* . Elements of Π are called *simple roots* and elements of Π^\vee *simple coroots*.

The associated Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}_A$ is a Lie algebra over \mathbb{Q} , generated by \mathfrak{h} and elements $(e_i)_{i \in I}$, $(f_i)_{i \in I}$ subject to the Serre-Kac relations:

- | | |
|--|--|
| 1) $[\mathfrak{h}, \mathfrak{h}] = 0$ | 5) $[e_i, f_j] = 0, i \neq j$ |
| 2) $[h, e_i] = \langle \alpha_i, h \rangle e_i, h \in \mathfrak{h}$ | 6) $(\text{ad } e_i)^{-a_{ij}+1}(e_j) = 0, i \neq j$ |
| 3) $[h, f_i] = -\langle \alpha_i, h \rangle f_i, h \in \mathfrak{h}$ | 7) $(\text{ad } f_i)^{-a_{ij}+1}(f_j) = 0, i \neq j$ |
| 4) $[e_i, f_i] = \alpha_i^\vee$ | |

It is easy to see that \mathfrak{g}_A depends only on A and not on its realization (see [9, 1.1]).

Relative to \mathfrak{h} , \mathfrak{g} has decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigsqcup_{\alpha \in \Delta} \mathfrak{g}^\alpha$, where

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x, h \in \mathfrak{h}\},$$

and $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}^\alpha \neq 0\}$. Elements of Δ are called the *roots* of \mathfrak{g} . Each root has the form $\sum_{i \in I} n_i \alpha_i$ where $n_i \in \mathbb{Z}$ and either $n_i \geq 0$ for all i , or $n_i \leq 0$ for all i . The roots are called positive or negative accordingly; the set of positive (resp. negative) roots will be denoted by Δ^+ (resp. Δ^-). The *height* of a root $\alpha = \sum_{i=1}^l n_i \alpha_i$ is defined to be the integer $\sum_i n_i$.

2.3 Real roots and the Weyl group

For $i \in I$ define $w_i \in \text{Aut}(\mathfrak{h}^*)$ by setting $w_i(\alpha) = \alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i$. The group $W = \langle \{w_i\} \rangle$ generated by the w_i is called the *Weyl group* associated to A . The set $\Phi = W(\Pi)$ is a subset of Δ , called the set of *real roots*. The remaining roots $\Phi^I = \Delta \setminus \Phi$ are called *imaginary roots*.

The Weyl group W has a faithful action on \mathfrak{h} defined by $w_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee$. Moreover, the pairing $\langle \cdot, \cdot \rangle$ is W -invariant, that is, $\langle w\alpha, wh \rangle = \langle \alpha, h \rangle$ for $\alpha \in \mathfrak{h}^*$, $h \in \mathfrak{h}$ and $w \in W$.

For each real root α , define the corresponding coroot α^\vee as follows: write α in the form $w\alpha_i$ for some $w \in W$ and $i \in I$ and set $\alpha^\vee = w\alpha_i^\vee$. One can show (see [9]) that α^\vee is independent of the above choice. The correspondence $\alpha \mapsto \alpha^\vee$ is not linear; however, it does satisfy some nice properties:

Proposition 2.1. *The following hold:*

- (a) *For each $\alpha \in \Phi$, the coroot α^\vee is an integral linear combination of $\{\alpha_i^\vee\}$, and the coefficients are all non-negative (resp. non-positive) if $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$). Furthermore, $(-\alpha)^\vee = -\alpha^\vee$.*
- (b) *Given $\alpha, \beta \in \Phi$, we have $\langle \alpha, \beta^\vee \rangle > 0$ (resp. $= 0, < 0$) $\Leftrightarrow \langle \beta, \alpha^\vee \rangle > 0$ (resp. $= 0, < 0$).*
- (c) *For every $\alpha \in \Phi$, we have $\langle \alpha, \alpha^\vee \rangle = 2$.*

In order to define Kac-Moody groups, we introduce a related group $W^* \subseteq \text{Aut}(\mathfrak{g})$. By definition, W^* is generated by elements $\{w_i^*\}_{i \in I}$, where

$$w_i^* = \exp(\text{ad } e_i) \exp(-\text{ad } f_i) \exp(\text{ad } e_i) = \exp(-\text{ad } f_i) \exp(\text{ad } e_i) \exp(-\text{ad } f_i).$$

The group W^* is a central extension of W . More specifically, there is a surjective homomorphism $\varepsilon : W^* \rightarrow W$ which sends w_i^* to w_i for all i ; the kernel of ε is an elementary abelian group of exponent 2 generated by $\{(w_i^*)^2\}$, as follows immediately from [25, 3.3].

Finally, we define certain elements $\{e_\alpha \in \mathfrak{g}\}_{\alpha \in \Phi}$. Given $\alpha \in \Phi$, write α in the form $w\alpha_j$ for some $j \in I$ and $w \in W$, choose $w^* \in W^*$ which maps onto w , and set $e_\alpha = w^*e_{\alpha_j}$. It is clear from [25, (3.3.2)] that e_α belongs to \mathfrak{g}^α , e_α is uniquely determined up to sign, and for all $i \in I$, $w_i^*e_\alpha = \eta_{\alpha,i}e_{w_i\alpha}$ for some constants $\eta_{\alpha,i} \in \{\pm 1\}$. These constants $\{\eta_{\alpha,i}\}$ will appear in the definition of Kac-Moody groups.

2.4 Kac-Moody groups and Tits' presentation

The construction of (incomplete) Kac-Moody groups over arbitrary fields is due to Tits [25]. One may define these groups by generators and relators. While not explicitly stated in Tits' paper, such a presentation appears in the papers of Carter [5] and (in a slightly different form) Morita and Rehmann [13].

The group $G = G(A)$ defined below is called the *incomplete simply-connected Kac-Moody group* corresponding to A . The presentation we use is "almost canonical" except for the choice of elements $\{e_\alpha\}$ which determine the constants $\{\eta_{\alpha,i}\}$.

By definition, $G(A)$ is generated by the set of symbols $\{\chi_\alpha(u) \mid \alpha \in \Phi, u \in k\}$ satisfying relations (R1)-(R7) below. In all the relations i, j are elements of I , u, v are elements of k (arbitrary, unless mentioned otherwise) and α and β are real roots.

$$(R1) \quad \chi_\alpha(u+v) = \chi_\alpha(u)\chi_\alpha(v);$$

(R2) Let (α, β) be a *prenilpotent pair*, that is, there exist $w, w' \in W$ such that

$$w\alpha, w\beta \in \Phi^+ \text{ and } w'\alpha, w'\beta \in \Phi^-.$$

Then

$$[\chi_\alpha(u), \chi_\beta(v)] = \prod_{m,n \geq 1} \chi_{m\alpha+n\beta}(C_{mn\alpha\beta}u^m v^n)$$

where the product on the right hand side is taken over all real roots of the form $m\alpha + n\beta$, $m, n \geq 1$, in some fixed order, and $C_{mn\alpha\beta}$ are integers independent of k (but depending on the order).

For each $i \in I$ and $u \in k^*$ set

$$\chi_{\pm i}(u) = \chi_{\pm \alpha_i}(u),$$

$$\tilde{w}_i(u) = \chi_i(u)\chi_{-i}(-u^{-1})\chi_i(u),$$

$$\tilde{w}_i = \tilde{w}_i(1) \text{ and } h_i(u) = \tilde{w}_i(u)\tilde{w}_i^{-1}.$$

The remaining relations are

$$(R3) \quad \tilde{w}_i\chi_\alpha(v)\tilde{w}_i^{-1} = \chi_{w_i\alpha}(\eta_{\alpha,i}u),$$

$$(R4) \quad h_i(u)\chi_\alpha(v)h_i(u)^{-1} = \chi_\alpha(vu^{-\langle\alpha, \alpha_i^\vee\rangle}) \text{ for } u \in k^*,$$

$$(R5) \quad \tilde{w}_ih_j(u)\tilde{w}_i^{-1} = h_j(u)h_i(u^{-a_{ji}}),$$

$$(R6) \quad h_i(uv) = h_i(u)h_i(v) \text{ for } u, v \in k^*, \text{ and}$$

$$(R7) \quad [h_i(u), h_j(v)] = 1 \text{ for } u, v \in k^*.$$

An immediate consequence of relations (R3) is that $G(A)$ is generated by $\{\chi_{\pm i}(u)\}$.

Remark. In [10, Proposition 2.3], it is shown that a pair (α, β) is prenilpotent if and only if $\alpha \neq -\beta$ and $|(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi| < \infty$. Thus the product on the right-hand side of (R2) is finite.

Intuitively, one should think of the above presentation as an analogue of the Steinberg presentation for classical groups with $\chi_\alpha(u)$ playing the role of $\exp(ue_\alpha)$. In the next subsection we give a representation-theoretic interpretation of Kac-Moody groups which makes the above analogy precise.

Next we introduce several subgroups of $G = G(A)$:

1. *Root subgroups* U_α . For each $\alpha \in \Phi$ let $U_\alpha = \{\chi_\alpha(u) \mid u \in k\}$. By relations (R1), each U_α is isomorphic to the additive group of k .
2. *The “extended” Weyl group* \widetilde{W} . Let \widetilde{W} be the subgroup of G generated by elements $\{\tilde{w}_i\}_{i \in I}$. One can show that \widetilde{W} is isomorphic to the group W^* introduced before, so there is a surjective homomorphism $\varepsilon : \widetilde{W} \rightarrow W$ such that $\varepsilon(\tilde{w}_i) = w_i$ for $i \in I$. Given $\tilde{w} \in \widetilde{W}$ and $w \in W$, we will say that \tilde{w} is a representative of w if $\varepsilon(\tilde{w}) = w$. It will be convenient to identify (non-canonically) W with a subset (not a subgroup) of \widetilde{W} which contains exactly one representative of every element of W . By abuse of notation, the set of those representatives will also be denoted by W . It follows from relations (R3) that $wU_\alpha w^{-1} = U_{w\alpha}$ for any $\alpha \in \Phi$ and $w \in W$.
3. *“Unipotent” subgroups*. Let $U^+ = \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$, and $U^- = \langle U_\alpha \mid \alpha \in \Phi^- \rangle$.
4. *“Torus” (“diagonal” subgroup)*. Let $H = \langle \{h_i(u) \mid i \in I, u \in k\} \rangle$. One can show that relations (R6)-(R7) are defining relations for H , so H is isomorphic to the direct sum of l copies of k^* .
5. *“Borel” subgroups*. Let $B^+ = \langle U^+, H \rangle$ and $B^- = \langle U^-, H \rangle$. By relations (R4), H normalizes both U^+ and U^- , so we have $B^+ = HU^+ = U^+H$ and $B^- = HU^- = U^-H$.
6. *“Normalizer.”* Let N be the subgroup generated by \widetilde{W} and H . Since \widetilde{W} normalizes H , we have $N = \widetilde{W}H$. It is also easy to see that $N/H \cong W$.

Tits [25] proved that (B^+, N) and (B^-, N) are BN-pairs⁵ of G . In fact, G admits the stronger structure of a twin BN-pair, but we will not use it. Let X^+ and X^- be the buildings associated with (B^+, N) and (B^-, N) , respectively. Since the field k is finite, the buildings X^+ and X^- are locally finite as chamber complexes. In fact, X^+ and X^- have constant thickness $|k| + 1$ (see [6, Appendix KMT]).

Below we list some of the fundamental properties of these BN-pairs. We will work mostly with the positive BN-pair (B^+, N) , and from now on, write B for B^+ and U for U^+ .

⁵Recall that BN-pairs are also called Tits systems.

- (a) $B \cap N = H$, so the Coxeter group associated to (B, N) is isomorphic to the Weyl group $W = W(A)$;
- (b) Bruhat decomposition: $G = BWB$;
- (c) Birkhoff decomposition: $G = U^-WB = B^-WU = UWB^- = BWU^-$.

Of course, (b) follows directly from (B, N) being a BN-pair, and the proof of (c) uses the twin BN-pair structure (see [10]).

Finally, we shall need a presentation by generators and relators for the group U established by Tits [24, Proposition 5].

Theorem 2.2. *The group U is generated by the elements $\{\chi_\alpha(u) \mid \alpha \in \Phi^+, u \in k\}$ subject to relations (R1) and (R2) defined earlier in this section.*

2.5 Representation-theoretic interpretation of Kac-Moody groups

The following interpretation of Kac-Moody groups was given by Carbone and Garland [4] (see also [26]). This construction generalizes that of Chevalley groups [22]. Let \mathcal{U} be the universal enveloping algebra of \mathfrak{g} . Let $\Lambda \subseteq \mathfrak{h}^*$ be the linear span of α_i , for $i \in I$, and $\Lambda^\vee \subseteq \mathfrak{h}$ be the linear span of α_i^\vee , for $i \in I$. Let $\mathcal{U}_{\mathbb{Z}} \subseteq \mathcal{U}$ be the \mathbb{Z} -subalgebra generated by $e_i^m/m!$, $f_i^m/m!$, and $\binom{h}{m}$, for $i \in I$, $h \in \Lambda^\vee$ and $m \geq 0$. Then $\mathcal{U}_{\mathbb{Z}}$ is a \mathbb{Z} -form of \mathcal{U} , i.e. $\mathcal{U}_{\mathbb{Z}}$ is a subring and the canonical map $\mathcal{U}_{\mathbb{Z}} \otimes \mathbb{Q} \rightarrow \mathcal{U}$ is bijective. For a field K , let $\mathcal{U}_K = \mathcal{U}_{\mathbb{Z}} \otimes K$, and $\mathfrak{g}_K = \mathfrak{g}_{\mathbb{Z}} \otimes K$.

Now let $\lambda \in \mathfrak{h}^*$ be a regular dominant integral weight, that is, $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{>0}$ for every $i \in I$. Let V^λ be the corresponding irreducible highest weight module. Choose a highest-weight vector $v_\lambda \in V^\lambda$, and let $V_{\mathbb{Z}}^\lambda \subset V^\lambda$ be the orbit of v_λ under the action of $\mathcal{U}_{\mathbb{Z}}$. Then $V_{\mathbb{Z}}^\lambda$ is a \mathbb{Z} -form of V^λ as well as a $\mathcal{U}_{\mathbb{Z}}$ -module. Similarly, $V_k^\lambda := k \otimes_{\mathbb{Z}} V_{\mathbb{Z}}^\lambda$ is a \mathcal{U}_k -module.

It is straightforward to establish the following (see [26, Proposition 3]).

Proposition 2.3. *There is a (unique) homomorphism $\pi_\lambda : G \rightarrow \text{Aut}(V_k^\lambda)$ such that*

$$\begin{aligned} \pi_\lambda(\chi_{\alpha_i}(u)) &= \sum_{m=0}^{\infty} u^m \frac{e_i^m}{m!} \quad \text{for } i \in I \text{ and } u \in k, \\ \pi_\lambda(\chi_{-\alpha_i}(u)) &= \sum_{m=0}^{\infty} u^m \frac{f_i^m}{m!} \quad \text{for } i \in I \text{ and } u \in k. \end{aligned}$$

The expressions on the right-hand side are well-defined automorphisms of V_k^λ since e_i and f_i are locally nilpotent on V_k^λ . Let $G^\lambda = \pi_\lambda(G)$. As we will see later in this section, the kernel of π_λ is finite, central and contained in H .

2.6 Complete Kac-Moody groups

As mentioned in the introduction, distinct completions of $G(A)$ were given in the papers of Carbone and Garland [4] and Remy and Ronan [20]. We now briefly review these constructions, starting with the Remy-Ronan completion. As above, let X^+ be the building associated with the positive BN-pair (B, N) , and consider the action of G on X^+ . Recall that X^+ is locally finite as a chamber complex. Define the topology on G by the subbase⁶ consisting of stabilizers of vertices of X^+ or, equivalently, fixators (pointwise stabilizers)

⁶Recall that by ‘subbase’ for a topology on a group, we mean a ‘subbase of neighborhoods of the identity’.

of chambers of X^+ . We shall call this topology the *building topology*. The completion of G in its building topology will be referred to as the *Remy-Ronan completion* and denoted by \widehat{G} .

We will make few references to the action of G on its building in this work. All we will need is the description of the building topology in purely group-theoretic terms. Since (B, N) is a BN-pair, we know that

- (a) The subgroup B of G is a chamber fixator,
- (b) G acts transitively on the set of chambers of X^+ .

Therefore, the family $\{gBg^{-1}\}_{g \in G}$ is a subbase for the building topology.

Let Z be the kernel of the natural map $G \rightarrow \widehat{G}$ (or, equivalently, the kernel of the action of G on X^+). Using results of Kac and Peterson [10], Remy and Ronan [20, 1.B] showed that Z is a subgroup of H (and hence finite); furthermore, Z coincides with the center of G .

Now let \widehat{B} (resp. \widehat{U}) be the closure of B (resp. U) in \widehat{G} . The natural images of N and H in \widehat{G} are discrete, and therefore we will denote them by the same symbols (without hats). This involves some abuse of notation since the image of H in \widehat{G} is isomorphic to H/Z .

The following theorem is a collection of results from [17] and [20]:

Theorem 2.4. *Let \widehat{G} , \widehat{B} and N be as above. The following hold:*

- (a) *The pair (\widehat{B}, N) is a BN-pair of \widehat{G} . Moreover, if \widehat{X}^+ is the associated building, there exists a \widehat{G} -equivariant isomorphism between X^+ and \widehat{X}^+ . In particular, the Coxeter group associated to (\widehat{B}, N) is isomorphic to $W = W(A)$.*
- (b) *The group \widehat{B} is an open profinite subgroup of \widehat{G} . Furthermore, \widehat{U} is an open pro- p subgroup of \widehat{B} .*

Now we turn to the Carbone-Garland completion. Let λ be a regular weight, and let $G^\lambda = G^\lambda(A)$ and V_k^λ be defined as in the previous subsection. Now we define the *weight topology* on G^λ by taking stabilizers of elements of V_k^λ as a subbase of neighborhoods of the identity. The completion of G^λ in this topology will be referred to as the *Carbone-Garland completion* and denoted by $\widehat{G}^\lambda(A)$. Since $G^\lambda(A)$ is a homomorphic image of $G(A)$, we can think of $\widehat{G}^\lambda(A)$ as a completion of G (and not G^λ). Let \widehat{B}^λ (resp. \widehat{U}^λ) be the closures of B (resp. U) in $\widehat{G}^\lambda(A)$. Then the obvious analogue of Theorem 2.4 holds; the fact that \widehat{U}^λ is a pro- p group will be proved at the end of this section (see Proposition 2.6); for all other assertions see [4, Section 6].

The following relationship between the Remy-Ronan and Carbone-Garland completions was established⁷ by Baumgartner and Remy:

Theorem 2.5. *For any regular weight λ , there exists a (canonical) continuous surjective homomorphism $\varepsilon_\lambda : \widehat{G}^\lambda \rightarrow \widehat{G}$. The kernel K_λ of ε_λ is equal to $\bigcap_{g \in \widehat{G}} \widehat{B}^\lambda$.*

It follows from Theorem 2.5 that the kernel of the map $\pi_\lambda : G \rightarrow G^\lambda$ is finite and central. Indeed, consider the sequence of homomorphisms

$$G \xrightarrow{\pi_\lambda} G^\lambda \longrightarrow \widehat{G}^\lambda \xrightarrow{\varepsilon_\lambda} \widehat{G}$$

Clearly, the composition of these three maps is the natural map from G to \widehat{G} . We know that the kernel of the latter map is finite and central, hence the same should be true for π_λ .

⁷Private communication.

In the case when A is an affine matrix, Garland [8] showed that K_λ is a central subgroup of $H \subseteq \widehat{B}^\lambda$ (and hence finite). It is not clear to us how large K_λ can be in general. Since \widehat{B}^λ is a profinite group, so is K_λ ; furthermore, K_λ has a finite index pro- p subgroup, which follows from Proposition 2.6 below.

Proposition 2.6. *Let \widehat{U}^λ be the closure of U in \widehat{G}^λ . The group \widehat{U}^λ is a pro- p group.*

Proof. In [4], it is shown that the k -vector space V_k^λ admits a basis $\Psi = \{v_1, v_2, v_3, \dots\}$ consisting of weight vectors, that is, for each $i \in \mathbb{N}$ there exists a weight μ_i of V^λ such that v_i lies in the weight component $V_{\mu_i}^\lambda$. Each weight μ of V^λ is of the form $\mu = \lambda - \sum_{i=1}^l k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$. Define the *depth* of μ to be $\text{depth}(\mu) = \sum_{i=1}^l k_i$. For convenience, we order the elements of Ψ such that $\text{depth}(\mu_i) \leq \text{depth}(\mu_j)$ if $i < j$.

For each $n \geq 1$ let V_n be the k -span of the set $\{v_1, v_2, \dots, v_n\}$. The group U stabilizes V_n ; moreover, it acts by upper-unitriangular matrices (with respect to the above basis). Therefore, we have a homomorphism $\pi_n : U \rightarrow GL_n(k)$ whose image is a finite p -group (since k has characteristic p). Then $U_n := \text{Ker } \pi_n$ consists of elements of U which fix V_n pointwise. Since $\bigcup_{n \geq 1} V_n = V_k^\lambda$, the groups $\{U_n\}_{n=1}^\infty$ form a base for the weight topology on U . Since each U_n is a normal subgroup of U of p -power index, the completion of U with respect to the weight topology is a pro- p group. \square

Remark. Remy and Ronan [20] prove that \widehat{U} is a pro- p group using its action on the building of \widehat{G} . This fact can also be deduced from Proposition 2.6 by applying Theorem 2.5.

We finish this section by describing explicitly the groups $G(A)$, $\widehat{G}(A)$ and $\widehat{G}^\lambda(A)$ in the special case $A = A_{d-1}^{(1)}$ for some $d \geq 2$ (in the notation of [9, Chapter 4]); an analogous result holds for any affine matrix A – see [8]. In this case, the incomplete group $G(A)$ is isomorphic to a central extension of the group $SL_d(k[t, t^{-1}])$ by k^* . The Remy-Ronan completion $\widehat{G}(A)$ is isomorphic to $PSL_d(k((t)))$. It is easy to see that the building topology on $\widehat{G}(A)$ coincides with the topology on $PSL_d(k((t)))$ induced from the local field $k((t))$.

Recall (see above) that the center of $G(A)$ always lies in the kernel of the natural map $G(A) \rightarrow \widehat{G}(A)$. On the other hand, the center of $G(A)$ does usually have non-trivial image in the Carbone-Garland completion $\widehat{G}^\lambda(A)$. For any λ , there is a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & k^* & \longrightarrow & \widehat{SL}_d(k((t))) & \longrightarrow & SL_d(k((t))) & \longrightarrow & 1 \\ & & & & \downarrow \rho_\lambda & & \downarrow \rho & & \\ & & & & \widehat{G}^\lambda(A) & \xrightarrow{\varepsilon_\lambda} & \widehat{G}(A) & & \end{array}$$

where $\widehat{SL}_d(k((t)))$ is the universal central extension of $SL_d(k((t)))$, the top row of the diagram is exact, and the homomorphisms ρ , ρ_λ and ε_λ are surjective (ρ is composition of the natural map from $SL_d(k((t)))$ to $PSL_d(k((t)))$ and an isomorphism between $PSL_d(k((t)))$ and $\widehat{G}(A)$). The map ρ_λ may or may not be an isomorphism depending on λ (see [8, Chapter 12]).

Notational remark. If A is a generalized Cartan matrix, the notations $G(A)$, $\widehat{G}(A)$, $U(A)$ etc. introduced in this section will have the same meaning throughout the paper. The reference to A will be omitted when clear from the context. The last remark does not apply to Section 3 where G stands for an arbitrary group.

3 Tits' abstract simplicity theorem

The following is a statement of the Tits simplicity theorem for groups with a BN-pair (see [2, Ch. IV, No. 2.7]).

Theorem 3.1. Consider a quadruple (G, B, N, U) where G is a group, (B, N) is a BN-pair of G whose associated Coxeter system is irreducible, and $U \leq B$ is a subgroup whose G -conjugates generate the entire group G . Assume the following:

- (a) U is normal in B and $B = UH$, where $H = B \cap N$.
- (b) $[G, G] = G$.
- (c) If Λ is a proper normal subgroup of U , then $[U/\Lambda, U/\Lambda] \neq U/\Lambda$.

Let $Z = \bigcap_{g \in G} gBg^{-1}$. Then the group G/Z is abstractly simple.

We remark that Tits' theorem has the following "topological" version whose proof is identical to the "abstract" version.

Theorem 3.2. Let G, B, N, Z be as above. Assume that G is a topological group, and B is a closed subgroup of G . Let U be a closed subgroup of B , and assume that G is topologically generated by the conjugates of U in G . Assume condition (a) above and replace (b) and (c) by conditions (b') and (c') below:

- (b') $[G, G]$ is dense in G .
- (c') If Λ is a proper normal **closed** subgroup of U , then $[U/\Lambda, U/\Lambda] \neq U/\Lambda$.

Then G/Z is topologically simple.

4 Simplicity of complete Kac-Moody groups via Tits' theorem

In this section G will denote an incomplete Kac-Moody group constructed from an **indecomposable** generalized Cartan matrix. Let B, N, U be as in Section 2, let \widehat{G} be the Remy-Ronan completion of G , and let \widehat{B} (resp. \widehat{U}) be the closures of B and (resp. U) in \widehat{G} . We shall analyze the conclusion of Tits' theorem applied to the quadruple $(\widehat{G}, \widehat{B}, N, \widehat{U})$.

The group $Z := \bigcap_{g \in \widehat{G}} g\widehat{B}g^{-1}$ is easily seen to be trivial (see Lemma 4.3e) and therefore, \widehat{G} is abstractly (resp. topologically) simple provided the hypotheses of Theorem 3.1 (resp. Theorem 3.2) are satisfied.

We will show that the hypotheses of Theorem 3.2 are satisfied provided $|k| > 3$, thus giving a slightly different proof of Remy's theorem on topological simplicity of \widehat{G} [18, Theorem 2.A.1]. We will also prove that hypotheses (b) and (c) of Theorem 3.1 are satisfied as long as \widehat{U} is (topologically) finitely generated. These results will follow from Lemma 4.3 and Lemma 4.4 below. Thus we will obtain the desired sufficient condition for abstract simplicity of \widehat{G} :

Theorem 4.1. Assume that $|k| > 3$. If \widehat{U} is topologically finitely generated, then \widehat{G} is abstractly simple.

Before verifying Tits' simplicity axioms for the quadruple $(\widehat{G}, \widehat{B}, N, \widehat{U})$, we obtain an auxiliary result about incomplete groups.

Lemma 4.2. The following hold:

- (a) The group G is generated by conjugates of U .
- (b) If $|k| > 3$, then $[G, G] = G$.

Proof. (a) We know that G is generated by root subgroups $\{U_{\pm\alpha_i}\}_{i \in I}$ (recall that $\{\alpha_i\}$ are simple roots). Since $U_{\alpha_i} \subset U$ and $w_i U_{\alpha_i} w_i^{-1} = U_{w_i \alpha_i} = U_{-\alpha_i}$, conjugates of U generate G .

(b) Let $i \in I$ and $u \in k$, and let $g = \chi_i(u)$. Choose $t \in k^*$ such that $t^2 \neq 1$ (this is possible since $|k| > 3$), and let $v = u/(t^2 - 1)$. We have

$$\chi_i(u) = \chi_i((t^{\langle \alpha_i, \alpha_i^\vee \rangle} - 1)v) = \chi_i(t^{\langle \alpha_i, \alpha_i^\vee \rangle} v) \chi_i(-v) = h_i(t) \chi_i(v) h_i(t)^{-1} \chi_i(v)^{-1} = [h_i(t), \chi_i(v)].$$

So, $\chi_i(u) \in [G, G]$, and similarly one shows that $\chi_{-i}(u) \in [G, G]$. Therefore, $[G, G]$ contains a generating set for G . \square

Now we are ready to establish Theorem 4.1 and the corresponding statement about topological simplicity of \widehat{G} . By Theorems 3.1 and 3.2, it suffices to prove the following two results: ⁸

Lemma 4.3. *The following hold:*

- (a) \widehat{G} is generated by conjugates of \widehat{U} .
- (b) $[\widehat{G}, \widehat{G}]$ is dense in \widehat{G} . Moreover, $[\widehat{G}, \widehat{G}] = \widehat{G}$ if \widehat{U} is finitely generated.
- (c) \widehat{U} is normal in \widehat{B} and $\widehat{B} = \widehat{U}(\widehat{B} \cap N)$.
- (d) The Coxeter system of the BN-pair (\widehat{B}, N) is irreducible.
- (e) The group Z is trivial.

Lemma 4.4. *Let Λ be a proper normal subgroup of \widehat{U} . If*

- (a) Λ is closed, or
- (b) \widehat{U} is finitely generated,

then

$$[\widehat{U}/\Lambda, \widehat{U}/\Lambda] \neq \widehat{U}/\Lambda.$$

The proofs of Lemmas 4.3 and 4.4 are based on the following properties of pro- p groups.

Proposition 4.5. *Let K be a pro- p group, and let K^* be the closure of $[K, K]K^p$ in K (where K^p is the subgroup generated by p^{th} -powers of elements of K).*

- (a) *A subset X of K generates K (topologically) if and only if the image of X in K/K^* generates K/K^* .*
- (b) *Suppose that K is finitely generated. Then any two minimal generating sets of K have the same cardinality.*

Proof. (a) is proved in [21, Proposition 25]; it also follows from [7, Proposition 1.9(iii)] and [7, Proposition 1.13]. To prove (b) note that if K is finitely generated, then K/K^* can be viewed as a finite-dimensional space over \mathbb{F}_p ; let d be the dimension of this space. Let X be a generating set of K . Clearly, $|X| \geq d$. On the other hand, there exists a subset Y of X , with $|Y| = d$ such that the image of Y in K/K^* is a basis of K/K^* . By (a), Y generates K . Thus any minimal generating set of K has cardinality d . \square

Proposition 4.6. *Let K be a pro- p group generated by a finite set $\{a_1, \dots, a_d\}$. Then any element of $[K, K]$ can be written in the form $[a_1, g_1][a_2, g_2] \dots [a_d, g_d]$ for some $g_1, \dots, g_d \in K$. In particular, $[K, K]$ is closed.*

⁸The first assertion of Lemma 4.3(b) and Lemma 4.4 are not needed for the proof of Theorem 4.1.

Proof. The second assertion of Proposition 4.6 is the statement of [7, Proposition 1.19]. The first assertion is established in the course of the proof of [7, Proposition 1.19]. \square

Now we are ready to establish Lemmas 4.3 and 4.4. It will be convenient to prove Lemma 4.4 first.

Proof of Lemma 4.4. Suppose that $[\widehat{U}/\Lambda, \widehat{U}/\Lambda] = \widehat{U}/\Lambda$ or, equivalently, that $\widehat{U} = \Lambda[\widehat{U}, \widehat{U}]$. Then Λ generates \widehat{U} modulo $[\widehat{U}, \widehat{U}]$. Since \widehat{U} is a pro- p group, Λ generates \widehat{U} (topologically) by Proposition 4.5(a). If Λ is closed, then $\Lambda = \widehat{U}$, so Λ is not proper. Thus we proved the desired result, assuming (a).

Now assume (b) that \widehat{U} is finitely generated. Let $\{a_1, a_2, \dots, a_d\}$ be a finite generating set of \widehat{U} contained in Λ (such a set exists by Proposition 4.5(b)). Applying Proposition 4.6 with $K = \widehat{U}$, we see that $\Lambda \supseteq [\widehat{U}, \widehat{U}]$ (since Λ is normal in \widehat{U}). But $\widehat{U} = \Lambda[\widehat{U}, \widehat{U}]$, so we conclude that $\widehat{U} = \Lambda$, a contradiction. \square

Proof of Lemma 4.3. (a) Let G_1 be the subgroup of \widehat{G} generated by conjugates of \widehat{U} . By Lemma 4.2, G_1 contains G , whence G_1 is dense in \widehat{G} . But G_1 is also open (hence closed) in \widehat{G} since $G_1 \supseteq \widehat{U}$. Therefore, G_1 must be equal to \widehat{G} .

(b) The density of $[\widehat{G}, \widehat{G}]$ is clear since $[\widehat{G}, \widehat{G}] \supseteq [G, G] = G$ and G is dense in \widehat{G} . Now assuming that \widehat{U} is finitely generated, we shall prove that $[\widehat{G}, \widehat{G}]$ is also open in \widehat{G} . Since \widehat{U} is a finitely generated pro- p group, its commutator subgroup $[\widehat{U}, \widehat{U}]$ is closed by Proposition 4.6. So, $\widehat{U}/[\widehat{U}, \widehat{U}]$ is also a finitely generated (abelian) pro- p group. On the other hand, \widehat{U} is generated by elements of order p since each root subgroup is isomorphic to the additive group of k . So, $\widehat{U}/[\widehat{U}, \widehat{U}]$ must be finite, and $[\widehat{U}, \widehat{U}]$ must be open in \widehat{U} and hence in \widehat{G} . Since $\widehat{G} \supset \widehat{U}$, we have shown that $[\widehat{G}, \widehat{G}]$ is open in \widehat{G} .

(c) We know (by construction) that the corresponding results hold for incomplete groups, that is, U is normal in B , $B = UH$ and $H = N \cap B$. Taking the completions of both sides of the last two equalities, and using the fact that (the images of) H and N in \widehat{G} are discrete, we get $\widehat{B} = \widehat{U}H$ and $H = N \cap \widehat{B}$. The normality of \widehat{U} in \widehat{B} is clear.

(d) The Coxeter group associated to (\widehat{B}, N) is isomorphic to W , and W is irreducible since the matrix A is indecomposable.

(e) Recall that X^+ denotes the building associated with the incomplete BN-pair (B, N) , and let \mathcal{C} be the chamber of X^+ whose stabilizer in G is B . It follows directly from definitions that the stabilizer of \mathcal{C} in \widehat{G} is \widehat{B} . So, $Z = \bigcap_{g \in \widehat{G}} g\widehat{B}g^{-1}$ consists of elements which stabilize all chambers in X^+ (recall that G acts transitively on the set of chambers). Therefore $Z = \{1\}$. \square

5 “Relative” Kac-Moody groups

Let $A = (a_{ij})_{i \in I}$ be a generalized Cartan matrix. As before, let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots. Recall that given a subset J of I , we denote by A_J the $|J| \times |J|$ matrix $(a_{ij})_{i, j \in J}$.

One can associate two *a priori* different groups to the matrix A_J : the usual Kac-Moody group $G(A_J)$, and the “relative” Kac-Moody group G_J which is defined as the subgroup of $G(A)$ generated by $\{U_{\pm\alpha_i}\}_{i \in J}$. The first result of this section asserts that these two groups are canonically isomorphic (Proposition 5.1). Next, one can consider two different topologies on $G(A_J)$ – the usual building topology and the topology induced from the building topology on $G(A)$ via the above isomorphism $G(A_J) \cong G_J$. While the two topologies may not be the same, the main result of this section is that their restrictions to the subgroup $U(A_J)$ are the same (Theorem 5.2). We believe this theorem is of independent interest; however, its main

application in this paper is given in the next section; it allows us to reduce the problem of finite generation of $\widehat{U}(A)$ to a certain question about the groups $\widehat{U}(A_J)$ where J runs over all subsets of cardinality 2.

In order to state our results precisely, we introduce the following notation. We set $G = G(A)$, $\Phi = \Phi(A)$, $W = W(A)$, $U = U(A)$ and $U^- = U^-(A)$. Fix a subset $J \subseteq I$, and let $\Pi_J = \{\alpha_i\}_{i \in J}$, let W_J be the subgroup of W generated by $\{w_i\}_{i \in J}$ and $\Phi_J = W_J(\Pi_J)$. Also let $\Phi_J^\pm = \Phi_J \cap \Phi^\pm$. Clearly, W_J (resp. Φ_J) can be canonically identified with $W(A_J)$ (resp. $\Phi(A_J)$).

We have already defined the group G_J . Note that by relations (R3), $G_J \supset U_\alpha$ for every $\alpha \in \Phi_J$. Also introduce the subgroups

$$U_J := \{U_\alpha \mid \alpha \in \Phi_J^+\} \quad \text{and} \quad U_J^- := \{U_\alpha \mid \alpha \in \Phi_J^-\}$$

of G_J . We will see shortly that $U_J = G_J \cap U$ and $U_J^- = G_J \cap U^-$.

Proposition 5.1. *There is a natural isomorphism $\varphi : G(A_J) \rightarrow G_J$. Moreover, φ maps $U(A_J)$ onto U_J and $B(A_J)$ onto B_J .*

In view of this proposition, we can identify U_J (resp. G_J) with $U(A_J)$ (resp. $G(A_J)$). Let \widehat{U}_J be the closure of $U(A_J)$ in $\widehat{G}(A_J)$, as before, and let \overline{U}_J be the closure of U_J in \widehat{G} .

Theorem 5.2. *The groups \widehat{U}_J and \overline{U}_J are (topologically) isomorphic.*

Proof of Proposition 5.1. As before, identify $\Phi(A_J)$ with the subset Φ_J of Φ . To distinguish between generators of $G(A_J)$ and $G(A)$, we use the symbols $\{x_\alpha(u) \mid \alpha \in \Phi(A_J), u \in k\}$ for the generators of $G(A_J)$ (the generators of $G(A)$ are denoted $\{\chi_\alpha(u)\}$ as usual). From the defining presentation of Kac-Moody groups, it is clear that there exists a map $\varphi : G(A_J) \rightarrow G$ such that $\varphi(x_\alpha(u)) = \chi_\alpha(u)$ for $\alpha \in \Phi(A_J), u \in k$. Clearly, $\varphi(G(A_J)) = G_J$, $\varphi(U(A_J)) = U_J$ and $\varphi(B(A_J)) = B_J$, so we only need to show that φ is injective. We proceed in several steps.

Step 1: φ is injective on $U(A_J)$. Let $\psi : U(A) \rightarrow U(A_J)$ be the unique homomorphism such that

$$\psi(\chi_\alpha(u)) = \begin{cases} x_\alpha(u) & \text{if } \alpha \in \Phi_J, \\ 1 & \text{if } \alpha \notin \Phi_J. \end{cases}$$

The fact that such a homomorphism exists follows immediately from Theorem 2.2. It is also clear that $\psi\varphi(g) = g$ for any $g \in U(A_J)$, whence the restriction of φ to $U(A_J)$ must be injective.

Step 2: φ is injective on $H(A_J)$. This follows directly from the fact that relations (R6)-(R7) are defining relations for the tori $H(A_J)$ and $H(A)$.

Step 3: $\varphi^{-1}(B_J) = B(A_J)$ where $B_J = B \cap G_J$. It is clear that $\varphi^{-1}(B_J) \supseteq B(A_J)$. Suppose that $\varphi^{-1}(B_J)$ is strictly larger than $B(A_J)$. Since $(B(A_J), N(A_J))$ is a BN-pair of $G(A_J)$, we conclude that $\varphi^{-1}(B_J)$ is of the form $B(A_J)W(A_K)B(A_J)$ for some non-empty subset $K \subseteq J$. This would mean that $B_J = \varphi(B(A_J)W(A_K)B(A_J))$ contains at least one of the generators of W , which is impossible since $B \cap N = H$.

Step 4: Conclusion. Let $K = \text{Ker } \varphi$. Note that $K \subset B(A_J)$ since $B(A_J)$ is the full preimage of B_J under φ . Take any $g \in K$ and write it as $g = uh$ where $u \in U(A_J)$ and $h \in H(A_J)$. Then $\varphi(u) = \varphi(h)^{-1}$. On the other hand, it is clear that $\varphi(u) \in U$ and $\varphi(h) \in H$. Since $H \cap U = 1$ and φ is injective on both $U(A_J)$ and $H(A_J)$, we conclude that $u = h = 1$. Therefore, K is trivial. \square

Proof of Theorem 5.2. By definition, \bar{U}_J and \widehat{U}_J are the completions of U_J with respect to the topologies (\mathcal{T}_1) and (\mathcal{T}_2) , respectively, where (\mathcal{T}_1) is given by the subbase $\{gBg^{-1} \cap U_J\}_{g \in G}$ and (\mathcal{T}_2) is given by the subbase $\{gB_Jg^{-1} \cap U_J\}_{g \in G_J}$. We have to show that (\mathcal{T}_1) and (\mathcal{T}_2) coincide.

The inequality $(\mathcal{T}_1) \geq (\mathcal{T}_2)$ is clear. Indeed, for any $g \in G_J$ we have $gBg^{-1} \cap U_J = gB_Jg^{-1} \cap U_J$ since $gB_Jg^{-1} = g(B \cap G_J)g^{-1} = gBg^{-1} \cap G_J$. To prove the reverse inequality, we need the following lemma, which is proved at the end of the section.

Lemma 5.3. *Let C be a generalized Cartan matrix. Then the building topology on $U(C)$ is given by the subbase $\{gU(C)g^{-1} \cap U(C)\}_{g \in G(C)}$.*

In view of the natural isomorphism $U(A_J) \cong U_J$, Lemma 5.3 applied with $C = A_J$ reduces the proof of the inequality $(\mathcal{T}_2) \geq (\mathcal{T}_1)$ to the following statement:

Claim 5.4. *Given $g \in G$, there exists a finite set $T \subset G_J$ such that*

$$gBg^{-1} \cap U_J \supseteq \bigcap_{t \in T} tU_Jt^{-1} \cap U_J.$$

Fix $g \in G$. By the Birkhoff decomposition, $g = g_-wg_+$ for some $g_+ \in B$, $w \in W$ and $g_- \in U^-$. We will show that there exist $g_1, \dots, g_k \in U_J^-$ and $v \in W_J$ such that

- (a) if $x \in U_J$ is such that $g_i^{-1}xg_i \in U_J$ for each i , then $g_-^{-1}xg_- \in U_J$,
- (b) if $y \in U_J$ is such that $v^{-1}yv \in U_J$, then $w^{-1}yw \in U$.

First, let us see why (a) and (b) will imply Claim 5.4. Indeed, let $\{g_i\}$ and v be as above, and set $T = \{g_i\} \cup \{g_-v\}$. Let $x \in \bigcap_{t \in T} tU_Jt^{-1} \cap U_J$. Then by (a), $g_-^{-1}xg_- \in U_J$. Applying (b) with $y = g_-^{-1}xg_-$, we get that $w^{-1}g_-^{-1}xg_-w \in U$. Finally, $g^{-1}xg = g_+^{-1}w^{-1}g_-^{-1}xg_-wg_+ \in U$ since $g_+ \in B$ and B normalizes U . So, $x \in gUg^{-1}$.

Proof of (a). Write g_- in the form $x_1 \dots x_k$ such that for each i , $x_i \in U_{-\alpha_{n_i}}$ (for some simple root α_{n_i}). Now define $g_0, \dots, g_k \in U_J^-$ and $y_0, \dots, y_k \in G$ inductively: set $y_0 = g_0 = 1$; for $i > 0$, set $y_i = y_{i-1}x_i$, $g_i = g_{i-1}x_i$ if $n_i \in J$ and $g_i = g_{i-1}$ if $n_i \notin J$. Note that $y_k = g_-$.

Suppose that x satisfies the hypotheses of (a). We shall prove by induction that $y_i^{-1}xy_i = g_i^{-1}xg_i$ for all i . The base case $i = 0$ is trivial. Now let $i > 0$, and suppose that $y_{i-1}^{-1}xy_{i-1} = g_{i-1}^{-1}xg_{i-1}$. If $n_i \in J$, $y_{i-1}^{-1}y_i = g_{i-1}^{-1}g_i = x_i$, so the induction step is clear. If $n_i \notin J$, then $U_{-\alpha_{n_i}}$ centralizes U_J . Indeed, U_J is generated by $\{U_\beta\}_{\beta \in \Phi_J^+}$. If $\beta \in \Phi_J^+$, the set $\mathbb{Z}_{>0}\beta + \mathbb{Z}_{>0}(-\alpha_{n_i})$ does not contain any roots since any element of this set has both positive and negative coefficients when expressed as a linear combination of simple roots. So, the pair $(\beta, -\alpha_{n_i})$ is prenilpotent and moreover, the corresponding roots groups commute by (R2).

By the induction hypotheses, $y_{i-1}^{-1}xy_{i-1} = g_{i-1}^{-1}xg_{i-1} \in U_J$. Therefore,

$$y_i^{-1}xy_i = x_i^{-1}(y_{i-1}^{-1}xy_{i-1})x_i = y_{i-1}^{-1}xy_{i-1} = g_{i-1}^{-1}xg_{i-1} = g_i^{-1}xg_i.$$

This completes the induction step and hence the proof of (a).

Proof of (b). The following argument was suggested to us (in a slightly different form) by Bertrand Remy and Pierre-Emmanuel Caprace. First of all, the statement of (b) is clearly implied by the following: for any $w \in W$ there exists $v \in W_J$ such that $wvU_J(wv)^{-1} \subseteq U$. We know that U_J is generated by the root subgroups $\{U_\alpha\}_{\alpha \in \Phi_J^+}$. Since $zU_\alpha z^{-1} = U_{z\alpha}$ for every $z \in W$, it suffices to prove the following statement about root systems:

For any $w \in W$ there exists $v \in W_J$ such that $wv(\alpha_i) > 0$ for all $i \in J$.

Recall that the simple roots $\{\alpha_i\}$ are linearly independent elements of the \mathbb{Q} -vector space \mathfrak{h}^* . Let $\mathfrak{h}_{\mathbb{R}}^* = \mathfrak{h}^* \otimes_{\mathbb{Q}} \mathbb{R}$. Let $R = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \beta, \alpha_i^\vee \rangle > 0 \text{ for all } i\}$ be the fundamental chamber and $R_J = \{\beta \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \beta, \alpha_i^\vee \rangle > 0 \text{ for all } i \in J\}$ be the J -relative fundamental chamber (in the sense of [11, 5.7]). Now pick any $x \in R$ (it is well-known that R is non-empty). By [11, 5.7, Proposition 5(i)], the union of W -translates of R is contained in the union of W_J -translates of R_J . Therefore, given $w \in W$, there exists $v \in W_J$ such that $v^{-1}w^{-1}x \in R_J$. For any $i \in J$ we have $\langle v^{-1}w^{-1}x, \alpha_i^\vee \rangle > 0$, whence $\langle x, wv\alpha_i^\vee \rangle > 0$. Since $x \in R$, the last inequality implies that $wv\alpha_i^\vee$ must be a positive coroot, whence $wv\alpha_i$ is a positive root. The proof is complete. \square

Proof of Lemma 5.3. It will suffice to show that $gU(C)g^{-1} \cap U(C) = gB(C)g^{-1} \cap U(C)$ for any $g \in G(C)$. So, fix $g \in G(C)$ and $x \in U(C)$ such that $g^{-1}xg \in B(C)$. Then $g^{-1}xg = hu$ where $u \in U(C)$ and $h \in H(C)$. We need to show that $h = 1$.

Assume that $h \neq 1$. Then we can choose a matrix D such that C is a submatrix of D and h does not lie in the center of $G(D)$ (as before, we identify $G(C)$ with a subgroup of $G(D)$). The existence of such a D follows easily from defining relations (R4) – we just have to ensure that h acts non-trivially by conjugation on one of the root subgroups.

Now let (\mathcal{T}) be the building topology on $G(D)$. Let $\overline{U}(C)$ (resp. $\widehat{G}(D)$, $\widehat{U}(D)$) be the completion of $U(C)$ (resp. $G(D)$, $U(D)$) with respect to (\mathcal{T}) . By Theorem 2.4b), $\widehat{U}(D)$ is a pro- p group, hence so is $\overline{U}(C)$. Therefore, $x^{p^n} \rightarrow 1$ in (\mathcal{T}) as $n \rightarrow \infty$, whence $(g^{-1}xg)^{p^n} \rightarrow 1$ in (\mathcal{T}) as well. On the other hand, $(g^{-1}xg)^{p^n} = (hu)^{p^n} = h^{p^n}u_n$ where $u_n \in U(C)$. Since the order of h is finite, prime to p and $\overline{U}(C)$ is compact, there exists a subsequence $\{n_k\}$ such that $h^{p^{n_k}} = h$ for all k and $\{u_{n_k}\}$ converges to some element $v \in \overline{U}(C)$. Thus $hv = 1$ in $\widehat{G}(D)$. Since $\overline{U}(C)$ is pro- p , we conclude that $h = 1$ in $\widehat{G}(D)$. So, h lies in the kernel of the natural map $G(D) \rightarrow \widehat{G}(D)$ which, as we know, coincides with the center of $G(D)$. This contradicts our initial assumption. \square

6 Finite generation of \widehat{U} : reduction to the rank 2 case

In this section we prove that finite generation of \widehat{U} is essentially determined by rank 2 subsystems of Φ .

Definition. Let A be a generalized Cartan matrix and $\widehat{U} = \widehat{U}(A)$. We say that \widehat{U} is *well-behaved* if for any non-simple root $\gamma \in \Phi(A)^+$ we have $U_\gamma \subseteq [\widehat{U}, \widehat{U}]$.

By Proposition 4.5(a), if \widehat{U} is well-behaved then \widehat{U} is topologically generated by simple root subgroups $\{U_\alpha\}_{\alpha \in \Pi(A)}$ (in particular, \widehat{U} is topologically finitely generated).

For the rest of this section we fix a matrix $A = (a_{ij})_{i,j \in I}$, and we write $\widehat{U} = \widehat{U}(A)$ and $\Phi = \Phi(A)$. For a subset J of I we define Φ_J as in the previous section. Recall that $\{\alpha_i\}_{i \in I}$ are simple roots, and w_i is the reflection associated with α_i .

Theorem 6.1. *Suppose that for any subset $J \subseteq I$ of cardinality 2, the group $\widehat{U}(A_J)$ is well-behaved. Then \widehat{U} is also well-behaved and hence (topologically) finitely generated.*

Theorem 6.1 is an easy consequence of the following lemma.

Lemma 6.2. *Let $\gamma \in \Phi^+$ be a non-simple root. There exist simple roots α_i, α_j and $w \in W$ such that $w\alpha_i > 0$, $w\alpha_j > 0$ and $\gamma = w\alpha$ for some non-simple root $\alpha \in \Phi_{\{i,j\}}^+$.*

Proof of Lemma 6.2. If γ lies in a subsystem generated by two simple roots, that is, $\gamma = n\alpha_i + m\alpha_j$ for some $n, m \in \mathbb{N}$ and $i, j \in I$, the assertion is obvious (we can take $w = 1$). From now on assume that $\gamma = \sum_{i=1}^l n_i \alpha_i$ where at least three n_i 's are nonzero. We will prove the lemma by induction on $\text{height}(\gamma)$.

We note that there exists $k \in I$ such that $\text{height}(w_k \gamma) < \text{height}(\gamma)$. Indeed, if $\text{height}(w_i \gamma) \geq \text{height}(\gamma)$ for all i , then $\langle \gamma, \alpha_i^\vee \rangle \leq 0$ for all i and hence $\langle \alpha_i, \gamma^\vee \rangle \leq 0$ for all i . But γ is a linear combination of α_i with nonnegative coefficients, so we must have $\langle \gamma, \gamma^\vee \rangle \leq 0$. The latter is impossible since $\langle \gamma, \gamma^\vee \rangle = 2$.

Since γ does not lie in a rank two subsystem, $w_k \gamma$ is not simple. By induction, there exist simple roots α_i and α_j , $w \in W$ and $\alpha \in \Phi_{\{i,j\}}^+$ such that $w\alpha_i > 0$, $w\alpha_j > 0$ and $w_k \gamma = w\alpha$.

Note that $\gamma = w_k w \alpha$. If both $w_k w \alpha_i$ and $w_k w \alpha_j$ are positive, we are done. Suppose, this is not the case. Then we must have $w\alpha_i = \alpha_k$ or $w\alpha_j = \alpha_k$, and without loss of generality we assume that $w\alpha_i = \alpha_k$. We have

$$\begin{aligned} \gamma &= w_k(w\alpha) = w\alpha - \langle w\alpha, \alpha_k^\vee \rangle \alpha_k = w\alpha - \langle w\alpha, (w\alpha_i)^\vee \rangle w\alpha_i = \\ &= w\alpha - \langle w\alpha, w(\alpha_i^\vee) \rangle w\alpha_i = w\alpha - \langle \alpha, \alpha_i^\vee \rangle w\alpha_i = w(\alpha - \langle \alpha, \alpha_i^\vee \rangle \alpha_i) = w(w_i \alpha). \end{aligned}$$

Since $\alpha \in \Phi_{\{i,j\}}^+$ and α is not simple, we have $w_i \alpha \in \Phi_{\{i,j\}}^+$. The proof will be complete if we show that $w_i \alpha$ is not simple. If $w_i \alpha$ is simple, then $\text{height}(w_i \alpha) < \text{height}(\alpha)$, so we must have $\langle \alpha, \alpha_i^\vee \rangle > 0$. Therefore, $\langle w\alpha, \alpha_k^\vee \rangle = \langle w\alpha, w\alpha_i^\vee \rangle > 0$, whence

$$\text{height}(\gamma) = \text{height}(w_k w \alpha) < \text{height}(w\alpha) = \text{height}(w_k \gamma),$$

contrary to our assumptions. □

Proof of Theorem 6.1. Fix a non-simple positive real root γ . We have to show that $U_\gamma \in [\widehat{U}, \widehat{U}]$. By Lemma 6.2, there exist $\alpha_i, \alpha_j \in \Pi$, $w \in W$ and a non-simple root $\alpha \in \Phi_{\{i,j\}}^+$ such that $w\alpha_i > 0$, $w\alpha_j > 0$ and $\gamma = w\alpha$. Let $J = \{i, j\}$ and $V = \widehat{U}_J$. Since $V \cong \widehat{U}(A_J)$ by Theorem 5.2, the hypotheses of Theorem 6.1 imply that V is topologically generated by $U_{\alpha_i} \cup U_{\alpha_j}$. Since $wU_\beta w^{-1} = U_{w\beta}$ for any $\beta \in \Phi$, we conclude that $wVw^{-1} \subseteq \widehat{U}$. By hypotheses, we also know that $U_\alpha \subset [V, V]$. Therefore, $U_\gamma = U_{w\alpha} = wU_\alpha w^{-1} \subset [wVw^{-1}, wVw^{-1}] \subseteq [\widehat{U}, \widehat{U}]$. □

In the next section we establish a sufficient condition for finite generation of \widehat{U} in the rank two case:

Theorem 6.3. *Let C be a 2×2 generalized Cartan matrix. Assume that either C is finite and $|k| > 3$, or that C is affine, $|k| > 3$ and $p > 2$. Then the group $\widehat{U}(C)$ is well-behaved.*

Before proving Theorem 6.3 we explain how to deduce Theorem 1.1. Theorem 6.4 below is a direct consequence of Theorem 6.1 and Theorem 6.3. Theorem 1.1 is obtained by combining Theorem 6.4 and Theorem 4.1.

Theorem 6.4. *Let A be an indecomposable generalized Cartan matrix, and suppose that $|k| > 3$. Assume that one the following holds:*

- (a) $p = \text{char}(k) > 2$, and any 2×2 submatrix of A is of finite or affine type;
- (b) any 2×2 submatrix of A is of finite type.

Then the group $\widehat{U}(A)$ is finitely generated.

Remark. A natural way to establish topological finite generation of $\widehat{U}(A)$ would be to prove that the incomplete group $U(A)$ is finitely generated (as an abstract group). By a theorem of Abramenko [1], when $|k| > 3$, $U(A)$ is finitely generated if and only if every 2×2 submatrix of A is finite. It is easy to see that the ‘if’ part of this statement also follows from our proof of Theorem 6.4.

7 Finite generation: the rank 2 case

In this section we prove Theorem 6.3.

If C is of finite type, the group $G(C)$ is finite, so $\widehat{U}(C)$ is isomorphic to $U(C)$. Theorem 6.3 in this case follows easily from defining relations of type (R2) (see [12], where the coefficients $\{C_{mn\alpha\beta}\}$ are computed explicitly in terms of the root system generated by α and β). Before considering the affine case, we make some general remarks about affine Kac-Moody groups.

A good general reference for incomplete affine Kac-Moody groups is the thesis of Ramagge [14] (see also [15], [16]). In particular, using results of [14], one can obtain an explicit realization of incomplete twisted affine Kac-Moody groups. An explicit realization of complete twisted affine Kac-Moody groups is probably known, however we are unaware of a proof in the literature. For completeness, we shall demonstrate such realization in the case of 2×2 matrices - see Proposition 7.2.

Let C be an affine matrix (of arbitrary size). Then the incomplete group $G(C)$ modulo its finite center is isomorphic to the group of fixed points of a finite order (possibly trivial) automorphism ω_C of the group of $k[t, t^{-1}]$ -points of some simply-connected Chevalley group. The automorphism ω_C is non-trivial if and only if C is a *twisted* affine matrix.⁹ The complete group $\widehat{G}(C)$ has an analogous description where the ring $k[t, t^{-1}]$ is replaced by the field $k((t))$. Furthermore, $\widehat{G}(C)$ is isomorphic to the group of $k((t))$ -points of some simple algebraic group \mathbb{G}_C defined over $k((t))$: if C is non-twisted, \mathbb{G}_C is the Chevalley group mentioned above; if C is twisted, \mathbb{G}_C is non-split.

One may ask if every simple algebraic group over $k((t))$ is isogenous to one of the form $\widehat{G}(C)$ for some C . The answer is ‘no’ since the groups $\widehat{G}(C)$ are always residually split. Moreover, there is a bijective correspondence between isogeny classes of residually split simple algebraic groups over $k((t))$ and groups of the form $\widehat{G}(C)$, with C affine. If C is an affine matrix of type $X_n^{(r)}$ (in the notation of [9, Chapter 4]), then $\widehat{G}(C)$ is a residually split group whose isogeny class is given by Tits’ index of the form ${}^r X_{n,m}^{(d)}$ for some m, d in the notation of [23] (these conditions determine the isogeny class uniquely).

Now assume that C is a 2×2 affine matrix. Up to isomorphism of the corresponding Kac-Moody groups, there are only two possibilities:

$$C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad (\text{non-twisted case}), \quad \text{and} \quad C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad (\text{twisted case}).$$

In the first case C has type $A_1^{(1)}$, the Tits index is ${}^1 A_{1,1}^{(1)}$, and $\widehat{G}(C)$ is isomorphic to $PSL_2(k((t)))$. In the second case C has type $A_2^{(2)}$, the Tits index is ${}^2 A_{2,1}^{(1)}$, and $\widehat{G}(C)$ is isomorphic to $SU_3(k((t)), h, \sigma)$, where σ is a k -automorphism of $k((t))$ of order 2 and h is a hermitian form in three variables relative to σ (see [23]). If $p \neq 2$, we can (and will) assume that

$$\sigma(t) = -t \quad \text{and} \quad h((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1\sigma(y_3) - x_2\sigma(y_2) + x_3\sigma(y_1).$$

⁹For a classification of twisted affine matrices see [9, Chapter 4].

Then

$$SU_3(k((t)), h, \sigma) = \{g \in SL_3(k((t))) \mid J^{-1}g^*J = g^{-1}\} \quad \text{where } J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and $g \mapsto g^*$ is the map from $SL_3(k((t)))$ to $SL_3(k((t)))$ obtained by applying σ to each entry of g followed by transposition.

The proof of Theorem 6.3 in both cases follows the same strategy, but the twisted case requires more computations. Our starting point is the following obvious lemma:

Lemma 7.1. *Let P be a profinite group, and let $P_1 \supset P_2 \supset \dots$ be a descending chain of open normal subgroups of P which form a base of neighborhoods of identity. For each $i \in \mathbb{N}$ choose elements $g_{i,1}, \dots, g_{i,n_i} \in P_i$ which generate P_i modulo P_{i+1} . Let K be a closed subgroup of P such that $g_{i,j} \in KP_{i+1}$ for all i and j . Then K contains P_1 . \square*

We shall apply this lemma with $P = \widehat{U}$, $K = [P, P]$ and a certain filtration $\{P_i\}$ of P satisfying the above conditions and such that P_1 contains U_γ for every non-simple positive root γ . Clearly, this will prove that \widehat{U} is well-behaved, so we only need to show the existence of a filtration with required properties.

$$\text{Case 1: } C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

In this case, $\Phi^+ = \{(n+1)\alpha_1 + n\alpha_2, n\alpha_1 + (n+1)\alpha_2\}_{n \in \mathbb{Z}_{\geq 0}}$. The group \widehat{U} can be embedded into $SL_2(k[[t]])$ via the following map:

$$\chi_{(n+1)\alpha_1 + n\alpha_2}(u) \mapsto e_n(u) := \begin{pmatrix} 1 & ut^n \\ 0 & 1 \end{pmatrix}, \quad \chi_{n\alpha_1 + (n+1)\alpha_2}(u) \mapsto f_{n+1}(u) := \begin{pmatrix} 1 & 0 \\ ut^{n+1} & 1 \end{pmatrix},$$

for all $n \in \mathbb{Z}_{\geq 0}$ and $u \in k$. For each $n \geq 0$ let $E_n = \{e_n(u) \mid u \in k\}$ and $F_n = \{f_n(u) \mid u \in k\}$. We also define the elements $\{h_n(u) \mid n \geq 1, u \in k\}$ by

$$h_n(u) = \begin{pmatrix} 1 + ut^n & 0 \\ 0 & (1 + ut^n)^{-1} \end{pmatrix}.$$

Under the above identification, \widehat{U} consists of matrices in $SL_2(k[[t]])$ whose reduction mod t is upper-triangular. Now define the filtration $\{P_n\}$ as follows: $P_1 = P_2E_1$, and for $n \geq 2$ set

$$P_n = SL_2^n(k[[t]]) = \{g \in SL_2(k[[t]]) \mid g \equiv 1 \pmod{t^n}\}.$$

Clearly, P_1 contains all non-simple positive root subgroups.

For each $n \geq 2$, P_n is generated modulo P_{n+1} by the elements $\{e_n(u), f_n(u), h_n(u) \mid u \in k\}$, and P_1 is generated modulo P_2 by $\{e_1(u)\}$. Direct computation shows that $e_n(u) \equiv [h_1(1), e_{n-1}(u/2)] \pmod{P_{n+1}}$ for $n \geq 1$, $f_n(u) \equiv [h_1(1), f_{n-1}(-u/2)] \pmod{P_{n+1}}$ for $n \geq 2$ and $h_n(u) \equiv [e_1(1), f_{n-1}(u)] \pmod{P_{n+1}}$ for $n \geq 2$. So, all the hypotheses of Lemma 7.1 are satisfied, and we are done with Case 1.

$$\text{Case 2: } C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Let $\delta = 2\alpha_1 + \alpha_2$. Then $\Phi^+ = \{\pm\alpha_1 + n\delta \mid n \in \mathbb{Z}\} \cup \{\pm 2\alpha_1 + (2n+1)\delta \mid n \in \mathbb{Z}\}$ (see [9, Exercise 6.6]). For each $\alpha \in \Phi$ define an element $e_\alpha \in \mathfrak{sl}_3(k[[t, t^{-1}]])$ as follows:

α	e_α
$\alpha_1 + 2n\delta$	$(e_{12} + e_{23})t^{2n}$
$\alpha_1 + (2n + 1)\delta$	$2(e_{12} - e_{23})t^{2n+1}$
$2\alpha_1 + (2n + 1)\delta$	$e_{13}t^{2n+1}$
$-\alpha_1 + 2n\delta$	$2(e_{21} + e_{32})t^{2n}$
$-\alpha_1 + (2n + 1)\delta$	$(e_{21} - e_{32})t^{2n+1}$
$-2\alpha_1 + (2n + 1)\delta$	$e_{31}t^{2n+1}$

Proposition 7.2. *Let $\widehat{G} = \widehat{G}(C)$, $\widehat{U} = \widehat{U}(C)$ and $\mathcal{G} = \{g \in SL_3(k((t))) \mid J^{-1}g^*J = g^{-1}\}$. Then \widehat{G} is isomorphic to \mathcal{G} (as a topological group) via the map ι defined by*

$$\iota : \chi_\alpha(u) \mapsto 1 + (ue_\alpha) + (ue_\alpha)^2/2 \text{ for } \alpha \in \Phi \text{ and } u \in k.$$

Furthermore, $\iota(\widehat{U}) = \mathcal{U}$ where $\mathcal{U} = \{g \in \mathcal{G} \cap SL_3(k[[t]]) \mid g \text{ is upper-unitriangular mod } t\}$.

Remark. The expression $1 + (ue_\alpha) + (ue_\alpha)^2/2$ should really be “thought of” as $\exp(ue_\alpha)$ since $e_\alpha^3 = 0$ for every $\alpha \in \Phi$.

The proof of Proposition 7.2 will be given in Appendix A. Henceforth we identify \widehat{G} with \mathcal{G} and \widehat{U} with \mathcal{U} . Before proceeding, we introduce some terminology. Let $M_3(k)$ denote the space of 3×3 matrices over k .

Definition. Let $g \in GL_3^1(k[[t]])$. Write g in the form $1 + \sum_{i \geq 1} g_i t^i$ where $g_i \in M_3(k)$, and let n be the smallest integer such that $g_n \neq 0$. We will say that g has *degree* n and write $\deg(g) = n$. The matrix g_n will be called the *leading coefficient* of g ; we will write $\text{LC}(g) = g_n$.

Given a subgroup H of $GL_3^1(k[[t]])$ and $n \geq 1$, let

$$L_n(H) = \{\text{LC}(g) \mid g \in H \text{ and } \deg(g) = n\} \cup \{0\}.$$

Then it is easy to see that $L_n(H)$ is an \mathbb{F}_p -subspace of $M_3(k)$. The following result is also straightforward.

Lemma 7.3. *Let S be a subgroup of $GL_3^1(k[[t]])$. For each $i \geq 1$ let $S_i = S \cap GL_3^i(k[[t]])$. Fix $n \in \mathbb{N}$, and let $X \subset S$ be a set of elements of degree n . Then X generates S_n modulo S_{n+1} if and only if the set $\{\text{LC}(g) \mid g \in X\}$ spans $L_n(S)$. \square*

Now we return to the proof of Theorem 6.3. Let $P = \widehat{U}$. Define the filtration $\{P_n\}_{n=1}^\infty$ of P as follows: if $n \geq 2$, set $P_n = P \cap GL_3^n(k[[t]])$, and let P_1 be the set of matrices in $P \cap GL_3^1(k[[t]])$ whose $(3, 1)$ -entry lies in $t^2k[[t]]$.

Consider the following elements of \widehat{G} :

$$\{e_n^{(1)}(u), f_n^{(1)}(u), h_n(u) \mid n \in \mathbb{Z}, u \in k\} \text{ and } \{e_n^{(2)}(u), f_n^{(2)}(u) \mid n \text{ is odd}, u \in k\},$$

where

$$\begin{aligned} e_n^{(1)}(u) &= \chi_{\alpha_1+n\delta}(u), & f_n^{(1)}(u) &= \chi_{-\alpha_1+n\delta}(u), & \text{and } h_n(u) &= [e_0^{(1)}(u), f_n^{(1)}(1)] \\ e_n^{(2)}(u) &= \chi_{2\alpha_1+n\delta}(u), & f_n^{(2)}(u) &= \chi_{-2\alpha_1+n\delta}(u). \end{aligned}$$

Let $E_n^{(i)}$, $F_n^{(i)}$ and H_n be the subsets $\{e_n^{(i)}(u)\}$, $\{f_n^{(i)}(u)\}$ and $\{h_n(u)\}$, respectively.

Now consider the subsets $\{Z_n\}_{n=1}^\infty$ of P_1 defined as follows:

$$\begin{aligned} Z_1 &= E_1^{(2)} \cup E_1^{(1)} \cup H_1 \cup F_1^{(1)}, & Z_{2n} &= E_{2n}^{(1)} \cup F_{2n}^{(1)} \cup H_{2n}, & \text{and} \\ Z_{2n+1} &= E_{2n+1}^{(1)} \cup F_{2n+1}^{(1)} \cup H_{2n+1} \cup E_{2n+1}^{(2)} \cup F_{2n+1}^{(2)} & \text{for } n \geq 1. \end{aligned}$$

We claim that Z_n generates P_n modulo P_{n+1} for each $n \geq 1$. This result follows directly from Lemma 7.3 applied with $S = P_1$ and $\{S_n\} = \{P_n\}$. Indeed, for each $n \geq 1$, define $L_n \subset M_3(k)$ as follows:

$$L_n = \{g \in M_3(k) \mid gJ = (-1)^n Jg\}.$$

It is clear from the definitions that $L_n(P_1) \subseteq L_n(\widehat{U}) = L_n$. On the other hand, direct computation shows that for $n \geq 2$, all (non-identity) elements of Z_n have degree n and their leading coefficients span L_n , so $L_n(P_1) = L_n$. Similarly, one shows that the leading coefficients of elements of Z_1 span $L_1(P_1)$.

In order to finish the proof by using Lemma 7.1, we need suitable commutation relations between the elements $\{e_n^{(i)}(u), f_n^{(i)}(u), h_n(u)\}$. Once again, these are obtained by direct computation:

$$\begin{aligned} e_{2n}^{(1)}(u) &\equiv [e_1^{(2)}(1), f_{2n-1}^{(1)}(-u)] \pmod{P_{2n+1}} & e_{2n+1}^{(1)}(u) &\equiv [e_1^{(2)}(1), f_{2n}^{(1)}(u)] \pmod{P_{2n+2}} \\ f_{2n}^{(1)}(u) &\equiv [f_1^{(2)}(1), e_{2n-1}^{(1)}(u)] \pmod{P_{2n+1}} & f_{2n+1}^{(1)}(u) &\equiv [f_1^{(2)}(1), e_{2n}^{(1)}(-u)] \pmod{P_{2n+2}} \\ e_{2n+1}^{(2)}(u) &\equiv [e_0^{(1)}(1), e_{2n+1}^{(1)}(-u/4)] \pmod{P_{2n+2}} & f_{2n+1}^{(2)}(u) &\equiv [f_1^{(1)}(1), f_{2n}^{(1)}(-u/4)] \pmod{P_{2n+2}}. \end{aligned}$$

Finally, elements $\{h_n(u)\}_{n \geq 1}$ lie in $[P, P]$ by definition.

Appendix A: On explicit realization of twisted affine Kac-Moody groups

In this section we prove Proposition 7.2. Recall that

$$C = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (1)$$

Proposition 7.2. Let $\widehat{G} = \widehat{G}(C)$, $\widehat{U} = \widehat{U}(C)$ and $\mathcal{G} = \{g \in SL_3(k((t))) \mid J^{-1}g^*J = g^{-1}\}$. Then \widehat{G} is isomorphic to \mathcal{G} (as a topological group) via the map ι defined by

$$\iota : \chi_\alpha(u) \mapsto 1 + (ue_\alpha) + (ue_\alpha)^2/2 \quad \text{for } \alpha \in \Phi \quad \text{and } u \in k. \quad (2)$$

Furthermore, $\iota(\widehat{U}) = \mathcal{U}$ where $\mathcal{U} = \{g \in \mathcal{G} \cap SL_3(k[[t]]) \mid g \text{ is upper-unitriangular mod } t\}$.

Proof. We proceed in several steps.

Step 1: First, we claim that (2) defines a (unique) homomorphism ι_0 from the incomplete group $G = G(C)$ to $SL_3(k[t, t^{-1}])$. This follows directly from the presentation of G by generators and relators.

Step 2: Let $K = \text{Ker } \iota_0$ and let Z be the kernel of the natural map $G \rightarrow \widehat{G}$. At this step we show that $K \subseteq Z$. Recall that $(\mathcal{T}_{\text{build}})$ denotes the building topology on G . Let $(\mathcal{T}_{\text{aux}})$ be the topology on $\iota_0(G)$

given by the base $\{\iota(V)\}$ where V runs over subgroups of G open in (\mathcal{T}_{build}) . Let \tilde{G} be the completion of $\iota_0(G)$ with respect to (\mathcal{T}_{aux}) . Clearly, there exists a continuous homomorphism $\varepsilon : \hat{G} \rightarrow \tilde{G}$. By [18, Theorem 2.A.1], \hat{G} is topologically simple, whence ε is injective (since \tilde{G} is clearly non-trivial). Thus we conclude that $K \subseteq Z$. It follows immediately that the map $\iota_0 : G \rightarrow \iota_0(G)$ canonically extends to an isomorphism of topological groups $\iota : \hat{G} \rightarrow \tilde{G}$.

Step 3: Consider the topology (\mathcal{T}_{cong}) on $\iota_0(G)$ induced from the congruence topology on $SL_3(k((t)))$, and let \bar{G} be the completion of $\iota_0(G)$ with respect to (\mathcal{T}_{cong}) . At this step we show that \bar{G} coincides with \mathcal{G} . It is clear that $\bar{G} \subseteq \mathcal{G}$ since $\iota_0(G) \subset \mathcal{G}$ by construction and \mathcal{G} is closed in the congruence topology.

Now we prove the reverse inclusion $\mathcal{G} \subseteq \bar{G}$. The group \mathcal{G} is an isotropic simple algebraic group over the local field $k((t))$ and hence has a BN-pair $(\mathcal{B}, \mathcal{N})$ given by Bruhat-Tits theory. An explicit description of $(\mathcal{B}, \mathcal{N})$ is given in [23, 1.15]: $\mathcal{B} = \{g \in \mathcal{G} \cap SL_3(k[[t]]) \mid g \text{ is upper-triangular mod } t\}$ and \mathcal{N} is the semi-direct product of the group

$$D := \left\{ \left(\begin{array}{ccc} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma(x)^{-1} \end{array} \right) \mid x \in k((t)) \right\}$$

and the group of order 2 generated by the matrix J defined previously in (1). Note that $\mathcal{B} = \iota(H)\mathcal{U}$ where H is the diagonal subgroup of G .

Let \bar{U} be the closure of $\iota_0(U)$ in \mathcal{G} . Applying Lemma 7.1 with $P = \mathcal{U}$ and $\{P_i\} = \{P \cap SL_3^i(k[[t]])\}$ and arguing as in Section 7, we conclude that $\bar{U} = \mathcal{U}$. From the explicit description of \mathcal{N} , it is clear that $\mathcal{N} = (\mathcal{N} \cap \mathcal{U})\iota_0(N)$. Thus \bar{G} contains both \mathcal{U} and \mathcal{N} . Since $(\mathcal{B}, \mathcal{N})$ is a BN-pair and $\mathcal{B} \subset \iota_0(N)\mathcal{U}$, it follows that \mathcal{G} is generated by \mathcal{U} and $\iota_0(N)$. Since \bar{G} contains $\bar{U} = \mathcal{U}$ and $\iota_0(N)$, we conclude that $\bar{G} = \mathcal{G}$.

Step 4: Now we prove that the groups \bar{G} and \tilde{G} are topologically isomorphic. Equivalently, we will show that the topologies (\mathcal{T}_{cong}) and (\mathcal{T}_{aux}) on $\iota_0(G)$ coincide. The inequality $(\mathcal{T}_{aux}) \leq (\mathcal{T}_{cong})$ is clear. This inequality also implies that there is a natural homomorphism $\varepsilon_1 : \bar{G} \rightarrow \tilde{G}$. Since $\bar{G} = \mathcal{G}$ is simple, ε_1 is injective.

Now suppose that (\mathcal{T}_{cong}) is strictly stronger than (\mathcal{T}_{aux}) . Since both topologies (\mathcal{T}_{cong}) and (\mathcal{T}_{aux}) are countably based, it follows that there is a subgroup $V \subset \iota_0(G)$, open in (\mathcal{T}_{cong}) , and a sequence $\{g_n\}$ in $\iota_0(G)$ such that

- (a) g_n converges to 1 with respect to (\mathcal{T}_{aux}) ;
- (b) $g_n \notin V$ for all n .

Condition (a) implies that $g_n \in \iota_0(B)$ for all sufficiently large n . Let \bar{B} be the closure of $\iota_0(B)$ in \bar{G} . Clearly, \bar{B} is compact and countably based, so there exists a subsequence $\{g_{n_k}\}$ which converges to some $g \in \bar{B}$. Since $\{g_{n_k}\}$ converges to 1 with respect to (\mathcal{T}_{aux}) , it follows that g lies in the kernel of $\varepsilon_1 : \bar{G} \rightarrow \tilde{G}$. Since ε_1 is injective, we conclude that $g = 1$, contrary to condition (b).

Step 5: Combining steps 3 and 4, we get $\tilde{G} = \bar{G} = \mathcal{G}$, so the map $\iota : \hat{G} \rightarrow \tilde{G}$ defined at the end of step 2 has the desired properties. The equality $\iota(\hat{U}) = \mathcal{U}$ holds by construction. \square

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