SPHERE PACKINGS, LATTICES, GROUPS, AND INFINITE DIMENSIONAL ALGEBRA

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These notes arose from a graduate course taught at Rutgers University in the Fall of 2003. They may provide an introduction to, though not a comprehensive survey of, this vast subject. The author apologizes for any omissions, which are unintentional.

INTRODUCTION

The problem of packing identical spheres as densely as possible in Euclidean space has a 400 year history, having been initiated by Johannes Kepler in 1611 [Ke]. Though the problem is unsolved in general today, attempts to solve it have led to the discovery of a wealth of mathematics.

Most of the densest known sphere packings are *lattice packings*. A lattice in $\mathbb{R}^n$ is an additive subgroup $L \subset \mathbb{R}^n$ which is generated by some basis for the real vector space $\mathbb{R}^n$. A *lattice packing* in $\mathbb{R}^n$ is then a sphere packing where the centers of spheres are placed at the points of a lattice $L \subset \mathbb{R}^n$, the radius of each sphere being half the length of the shortest non-zero vectors in L.

The study of sphere packings and lattice packings fits naturally into a broader class of packing problems, including error-correcting codes in information transmission. In fact the '[24,12,8] extended binary code' of M. J. E. Golay ([Go]) led to the discovery by John Leech of one of the most remarkable lattices, the *Leech lattice* in $\mathbb{R}^{24}$ [Le].

The Leech lattice appears in several places in 'Moonshine' which is a term first coined by J. Conway and S. Norton in 1979 to describe the mysterious connections between finite sporadic simple groups and modular functions. The solution of the Monstrous moonshine conjectures by Frenkel, Lepowsky, Meurman [FLM1,2] and Borcherds [B4] expanded the Moonshine horizon to include interrelationships between lattices and hyperbolic reflection groups, generalized Kac-Moody Lie algebras, vertex (operator) algebras, automorphic forms and conformal field theory.

The development of Moonshine has been intertwined with the connections between mathematics and string theory. The solution of the Monstrous moonshine conjectures used ideas and tools from string theory, in particular vertex operators and the 'No-ghost theorem' which played an important role in early string theory.

Finding optimal sphere and lattice packings is an active area of current research. Many of the intricate connections between lattices, packings groups, automorphic forms and problems in Moonshine are not fully understood and give rise to substantial open problems.

(I) Lattice packings and sphere packings

The sphere packing problem is unsolved in general, as is the problem of finding the densest lattice packing in each dimension. In the known cases, particularly Hales' proof of the Kepler conjecture in $\mathbb{R}^3$ [Ha], the techniques used are particular to the given dimension. It is unlikely that there is any single construction which will solve these packing problems in every dimension. Recent approaches to studying the open sphere packing and lattice packing problems involve finding upper and lower bounds on the density of a packing. For example, H. Cohn and N. Elkies [CE] have adapted linear programming bounds for error-correcting codes to develop an analog for sphere packing, giving the the best known upper bounds in dimensions 4 through 36. Based on numerical computations of these bounds they conjecture that these methods can be used to solve the sphere packing problem in dimensions 8 and 24.

H. Cohn and A. Kumar have recently announced a proof that the 24-dimensional sphere packing using the Leech lattice is optimal among lattice packings in 24 dimensions [CK]. The proof combines the methods of [CE] with the geometry of the space of 24-dimensional lattices near the Leech lattice. They reduced the question to a computer calculation and checked it using exact rational arithmetic. They prove that the Leech lattice is the unique densest lattice in $\mathbb{R}^{24}$ but cannot rule out the possibility of denser non-lattice packings. They do prove that no sphere packing in $\mathbb{R}^{24}$ exceeds the density of the Leech lattice by a factor of more than $1 + 1.65 \times 10^{-30}$. It currently takes about six hours to check the computer parts of the calculation, plus human reasoning to deduce the result from the calculations.

Cohn and Kumar have also obtained a new proof of the corresponding 8-dimensional result using similar methods [CK]. The techniques of [CE] and [CK] are conjectured by the authors to give rise to a proof that the E_8 root lattice and the Leech lattice give the densest sphere packings in $\mathbb{R}^8$ and $\mathbb{R}^{24}$ respectively. H. Cohn and S. Miller have numerical evidence for this conjecture using orthogonal polynomials [CM]. The approach proposed is a refinement of the use of orthogonal polynomials in [CE].

(II) Moonshine

'Moonshine' is a term first coined by J. Conway and S. Norton in 1979 to describe the mysterious connections between the monster simple group, M, (and also other finite sporadic simple groups) and modular functions ([CN]).

There are fifteen primes p for which the normalizer $N_{\Gamma_0(p)}$ of the congruence subgroup $\Gamma_0(p)$ in $SL_2(\mathbb{R})$ has the 'genus-zero property', meaning that the compactification of the upper-half plane modulo $N_{\Gamma_0(p)}$ is a Riemann surface of genus zero. It follows that the field of modular functions invariant under $N_{\Gamma_0(p)}$ is generated by a single function (called a *Hauptmodul*). In the 1970's A. Ogg observed that these fifteen primes coincide exactly with the fifteen primes that divide the order of the monster simple group M, only conjectured (by Fischer and Griess) to exist at the time. With J. McKay and J. Thompson, Conway and

Norton conjectured the existence of an infinite dimensional $\mathbb{Z}$-graded representation V of M such that the dimensions of the graded pieces are given by the q-coefficients of the famous elliptic modular function $J(q)$. They also conjectured that the functions $J_g(q)$, obtained by replacing the traces on the identity by the traces on other elements g of M, are Hauptmoduls.

A near-proof of the Conway-Norton conjectures was given by A. Atkin, P. Fong and S. Smith. Their work however did not give a construction of a 'Moonshine module' V. An explicit construction of V was discovered by Frenkel, Lepowsky and Meurman [FLM1]. In addition, their construction provided a vertex operator realization of the 'Griess algebra'. This was constructed by Griess in order to show that its automorphism group is the conjectural Monster simple group. It was not however obvious that the Moonshine module V of Frenkel, Lepowsky and Meurman had the conjectured properties for *all* the elements g of the Monster M.

This missing link was provided by the work of R. Borcherds [B4] who showed that the construction of Frenkel, Lepowsky and Meurman satisfied all the conjectural properties of Conway and Norton. The work of Borcherds included the development of the theory of generalized Kac-Moody algebras, an axiomatization of vertex algebras and new automorphic forms.

The Leech lattice appears in several places in Moonshine. Let $\Pi_{25,1}$ be the unique 26-dimensional even unimodular Lorentzian lattice. Associated to this lattice is a reflection group W whose Dynkin diagram is the Leech lattice Λ in the sense that the simple roots of W form a subset of $\Pi_{25,1}$ isometric to Λ ([CS]). Given any Dynkin diagram, one can build a Kac-Moody Lie algebra. In [B5], Borcherds begins with the Kac-Moody algebra with Dynkin diagram the Leech lattice and adds the 'imaginary simple roots'. The result is Borcherds' fake Monster Lie algebra, a generalized Kac-Moody Lie algebra.

The Moonshine module V of Frenkel, Lepowsky and Meurman gives an intriguing connection between conformal field theory and the Leech lattice. They construct V as a bosonic string compactified by the Leech lattice which is then given a $\mathbb{Z}_2$ orbifold structure [FLM1,2]. This gives a meromorphic bosonic string theory on which the Monster simple group acts naturally [FLM1,2, Tu].

In [FLM2] Frenkel, Lepowsky and Meurman also conjecture that the Moonshine module vertex (operator) algebra has a uniqueness property analogous to the uniqueness of the Golay code and of the Leech lattice. Important tools for approaching this conjecture seem to be 'completely-extendable conformal intertwining algebras' introduced by Huang [Hu]. These are natural conformal field theoretic analogs of linear binary codes and nondegenerate rational lattices. Huang's work builds on earlier work of Frenkel, Lepowsky and Meurman which indicates that vertex operator algebras have many properties analogous to those of doubly-even codes and even lattices [FLM2]. Their viewpoint is that linear binary codes, nondegenerate rational lattices and conformal field theories are three stages in a hierarchy, and that the results in one stage have corresponding results in the other stages.

Deep connections between lattices, codes, conformal field theory and infinite dimensional algebra are still to be investigated. The far-reaching works of Borcherds, Huang and Frenkel, Lepowsky and Meurman seem to touch only the surface.

In [B2], Borcherds outlines 15 open problems relating to the Moonshine module. A number of these problems are still open, including the structure of and relationships between the Monster vertex algebra, its integral quadratic forms, generalized Kac-Moody algebras, hyperbolic and complex hyperbolic reflection groups.

(III) Automorphic theory

Borcherds proved the Conway-Norton Moonshine conjectures for the Moonshine module V of Frenkel, Lepowsky and Meurman by using the no-ghost theorem from string theory to construct a family of generalized Kac-Moody superalgebras of rank 2. These infinite dimensional superalgebras are closely related to the monster finite simple group and other sporadic simple groups.

Borcherds also constructed a second family of Kac-Moody superalgebras related to elements of Conway's sporadic simple group Co_1. These are similar to the first series of superalgebras he constructed, except they are related to the vertex algebra of the Leech lattice instead of the monster vertex algebra V.

The largest superalgebra in this series is the fake-monster Lie algebra. The root lattice of the fake monster is $\Pi_{25,1}$, the unique 26-dimensional even unimodular Lorentzian lattice.

Starting with Borcherds' fake Monster Lie algebra, one may consider its denominator function. This turns out to be an automorphic form for an orthogonal group which can be written as an infinite product ([B1]). There is in fact an infinite family of such automorphic forms, all of which have explicitly known zeros. Borcherds asks in [B2] if automorphic forms constructed in a similar way account for all automorphic forms whose zeros have a 'simple' construction.

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