MULTIVARIBLE CALCULUS

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Contents

Chapter 2

THE DESCRIPTION AND PREDICTION OF MOTION

2.1 Functions from $\mathbb{R}$ to $\mathbb{R}^n$ and the Description of Motion

In many ways, the simplest multivariable functions are functions from $\mathbb{R}$ to $\mathbb{R}^n$ for some $n \geq 1$. These are functions that have one input variable (one independent variable), and n output variables (n dependent variables).

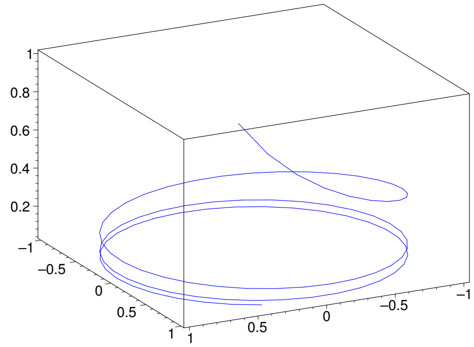
If $n = 2$ or if $n = 3$, we can think of the output variables as giving the coordinates of a point moving in $\mathbb{R}^2$ or $\mathbb{R}^3$, and we can think of the input variable as the time so that the function gives us the location of a moving point at time t. Functions like this describe curves in $\mathbb{R}^2$ or $\mathbb{R}^3$.

With this context in mind, it is natural to use $\mathbf{x}$ for the dependent variables, and t for the independent variable, and to write the function as $\mathbf{x}(t)$. Such functions are also called *vector valued functions of a real variable.*

Example 1. *Consider the function* $\mathbf{x}(t)$ *of the real variable* t *with values in* $\mathbb{R}^3$ *given by*

$$\mathbf{x}(t) = \big(\cos(t)\ , \sin(t)\ , 1/t\big)\ . \tag{2.1.1}$$

Here is a three dimensional plot of the curve traced out by $\mathbf{x}(t)$ *as* t *varies from* $t = 1$ *to* $t = 20$*. (Since* $x_3(t) = 1/t$ *is a decreasing function of* t*, the* $\mathbf{x}(0)$ *end of the curve is at the top.*



2.1.1 Continuity of functions from $\mathbb{R}$ to $\mathbb{R}^n$

Vector valued functions of one real variable that describe particle motion usually have certain *regularity properties*: For example, particle motions are usually at least *continuous*:

Definition 1 (Convergence and continuity in $\mathbb{R}^n$). *A sequence of vectors* $\{\mathbf{x}_j\}$ *in* $\mathbb{R}^n$ converges to $\mathbf{z} \in \mathbb{R}^n$ *in case for each* $\epsilon > 0$*, there is a natural number* N_ϵ *so that*

$$j \geq N_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}_j - \mathbf{z}\| \leq \epsilon \ , \tag{2.1.2}$$

in which case we say that $\mathbf{z}$ *is the limit of the sequence and write*

$$\lim_{n\to\infty} \mathbf{x}_j = \mathbf{z} \ . \tag{2.1.3}$$

A function $\mathbf{x}(t)$ *defined on an open interval* $]a,b[\subset \mathbb{R}$ *with values in* $\mathbb{R}^n$ *is* continuous at $t_0 \in]a,b[$ *in case* $\epsilon > 0$*, there is a real number* δ_ϵ *so that*

$$|t - t_0| \leq \delta_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{x}(t_0)\| \leq \epsilon \ , \tag{2.1.4}$$

in which case we write

$$\lim_{t\to t_0} \mathbf{x}(t) = \mathbf{x}(t_0) \ . \tag{2.1.5}$$

The function $\mathbf{x}(t)$ *is said to be* continuous *is it is continuous at each point in its domain. Such a function is often called a* curve *in* $\mathbb{R}^n$*.*

We begin with several observations. First, a sequence $\{\mathbf{x}_j\}$ cannot have more than one limit, and so our reference to "the limit" in the definition above does make sense. To see this, suppose that

$$\lim_{n\to\infty} \mathbf{x}_n = \mathbf{y} \qquad \text{and} \qquad \lim_{n\to\infty} \mathbf{x}_n = \mathbf{z} \ .$$

Fix any $\epsilon > 0$. Then there is a natural number N so that for all $j \geq N$, $\|\mathbf{x}_j - \mathbf{y}\| \leq \epsilon/2$ and $\|\mathbf{x}_j - \mathbf{z}\| \leq \epsilon/2$. But then by the triangle inequality, for any such j,

$$\|\mathbf{y} - \mathbf{z}\| \leq \|\mathbf{y} - \mathbf{x}_j\| + \|\mathbf{x}_j - \mathbf{z}\| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

Thus, for every $\epsilon > 0$, $\|\mathbf{y} - \mathbf{z}\| \leq \epsilon$. This is only possible in case $\mathbf{y} = \mathbf{z}$.

Next, observe that

$$\|\mathbf{x}(t) - \mathbf{x}(t_0)\|^2 = \sum_{j=1}^{n} |x_j(t) - x_j(t_0)|^2 \ . \tag{2.1.6}$$

Hence, for any single j

$$|x_j(t) - x_j(t_0)| \leq \|\mathbf{x}(t) - \mathbf{x}(t_0)\| \ . \tag{2.1.7}$$

Therefore, whenever (2.1.4) is true,

$$|t - t_0| \leq \delta_\epsilon \qquad \Rightarrow \qquad |x_j(t) - x_j(t_0)| \leq \epsilon \ , \tag{2.1.8}$$

meaning that whenever $\mathbf{x}(t)$ is continuous, each of its entries is a continuous real valued function.

Conversely, suppose that (2.1.8) is true for each j. Then for $|t - t_0| \leq \delta_\epsilon$,

$$\sqrt{\sum_{j=1}^{n} |x_j(t) - x_j(t_0)|^2} \leq \sqrt{n\epsilon^2} = \sqrt{n}\epsilon \ .$$

Then by (2.1.6),

$$|t - t_0| \leq \delta_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{x}(t_0)\| \leq \sqrt{n}\epsilon \ ,$$

which means that $\mathbf{x}(t)$ is continuous at t_0. Therefore:

- *A vector valued function* $\mathbf{x}(t)$ *of a real variable* t *is continuous at* t_0 *if and only if each of its entry functions* $x_j(t)$ *is a continuous at* t_0.

Therefore, everything we already know about continuity for single variable functions applies straight-away to vector valued functions of a real variable: Since $\sin(t)$ $\cos(t)$ and $1/t$ are all continuous functions on the interval $]1, 20[$, the vector valued function in Example 1 is continuous.

Finally, there is a close connection between convergence along a curve, as in $\lim_{t \to t_0} \mathbf{x}(t) = \mathbf{x}(t_0)$, an convergence of sequences, as in $\lim_{j \to \infty} \mathbf{x}_j = \mathbf{z}$.

Theorem 1 (Convergence along curves and sequences)**.** *Let* $\mathbf{x}(t)$ *be a function from* $]a,b[\subset \mathbb{R}$ *to* $\mathbb{R}^n$*. Then, for any* $t_0 \in]a,b[$*,* $\mathbf{x}(t)$ *is continuous at* t_0 *is and only if for each sequence* $\{t_j\}$ *of numbers in* $]a,b[$ *such that*

$$\lim_{j\to\infty} t_j = t_0 \ ,$$

it is the case that

$$\lim_{j\to\infty} \mathbf{x}(t_j) = \mathbf{x}(t_0) \ . \tag{2.1.9}$$

Proof: Suppose that $\mathbf{x}(t)$ is continuous at t_0, and pick any $\epsilon > 0$. Since $\mathbf{x}(t)$ is continuous, there is a $\delta_\epsilon > 0$ such that (2.1.4) is true. Let $\{t_j\}$ be any sequence of numbers in $]a,b[$ such that $\lim_{j\to\infty} t_j = t_0$. Then there is an N so that for all $j \geq N$, $|t_j - t_0| < \delta_\epsilon$. Hence, for all $j \geq N$,

$$\|\mathbf{x}(t_j) - \mathbf{x}(t_0)\| \leq \epsilon \ .$$

Since ϵ is arbitrary, this proves (2.1.9).

Conversely, suppose that $\mathbf{x}(t)$ is not continuous at t_0. Then for some $\epsilon > 0$, and each $j > 0$, there is a $t_j \in]a,b[$ so that $|t_j - t_0| \geq 1/j$, but $\|\mathbf{x}(t_j) - \mathbf{x}(t_0)\| > \epsilon$. But then (2.1.9) is impossible, and so whenever $\mathbf{x}(t)$ is not continuous at t_0, there exists a sequence $\{t_j\}$ of numbers in $]a,b[$ such that

$$\lim_{j\to\infty} t_j = t_0 \ ,$$

and for which (2.1.9) fails. $\square$

2.1.2 Differentiability of functions from $\mathbb{R}$ to $\mathbb{R}^n$

Physical motions of particles are usually not only continuous, but differentiable. In fact, as we shall explain later, as a consequence of Newton's second law, as long as no infinite forces act on a particle, its motion will be described by a curve that is at least twice differentiable.

Before we define differentiability, let us explain the idea of the derivative. Here is the intuitive idea:

- *To say that* $\mathbf{x}(t)$ *is differentiable at* $t = t_0$ *means that if we look at the graph of this function, and "zoom in" close enough on what the graph looks like near* $\mathbf{x}(t_0)$*, it will look indistinguishable from some line through* $\mathbf{x}(t_0)$*. That is, under "sufficiently high* *magnification" every segment of the graph of a differentiable vector valued function looks like a straight line segment.*

Now consider a vector valued function $\mathbf{x}(t) = \big(x(t), y(t), z(t)\big)$ with values in $\mathbb{R}^3$. Suppose that each of the coordinate functions $x(t)$, $y(t)$ and $z(t)$ is differentiable at $t = t_0$ in the familiar single-variable sense. Then we have, for $t \approx t_0$,

$$\begin{aligned} x(t) &\approx x(t_0) + (t - t_0)x'(t) \\ y(t) &\approx y(t_0) + (t - t_0)y'(t) \\ z(t) &\approx z(t_0) + (t - t_0)z'(t)\ . \end{aligned} \tag{2.1.10}$$

Define $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{v} = \big(x'(t_0), y'(t_0), z'(t_0)\big)$. We can express these approximations in vector notation as

$$\mathbf{x}(t) \approx \mathbf{x}_0 + (t - t_0)\mathbf{v}\ ,$$

telling us *which* straight line segment the graph of $\mathbf{x}(t)$ should look like near $\mathbf{x}_0$. This line, $\mathbf{x}_0 + t\mathbf{v}$, is called the *tangent line* to the graph of $\mathbf{x}(t)$ at $t = t_0$.

Example 2 (A tangent line). *Consider the vector valued function*

$$\mathbf{x}(t) = (x(t), y(t), z(t)) = \big(t\ ,\ 2^{3/2}t^{3/2}/3\ ,\ t^2/2\big)\ .$$

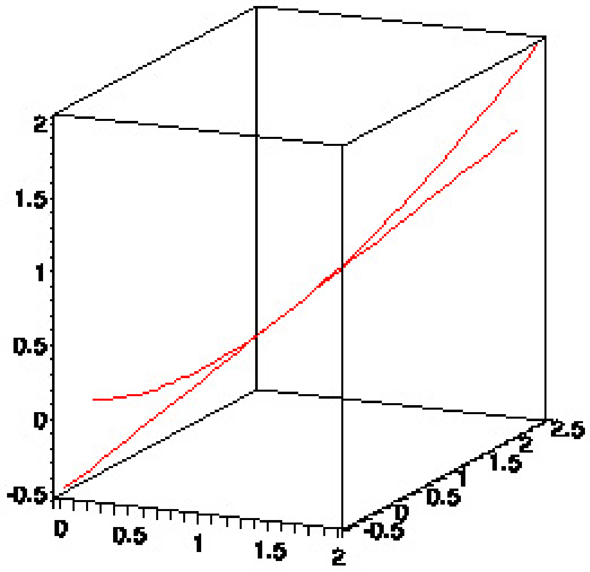
Then one computes, using single variable calculus,

$$\begin{aligned} x'(t) &= 1 \\ y'(t) &= 2^{1/2}t^{1/2} \\ z'(t) &= t\ . \end{aligned}$$

Then, taking $t_0 = 1$ we have from (2.1.10) that for $t \approx t_0$,

$$\mathbf{x}(t) \approx \mathbf{x}_0 + (t - t_0)\mathbf{v} = \big(1, 2^{3/2}/3, 1/2\big) + (t - 1)\big(1, 2^{1/2}, 1\big)\ . \tag{2.1.11}$$

Here is a graph showing both the curve $\mathbf{x}(t)$ and the tangent line $\mathbf{x}_0 + t\mathbf{v}$ for $0 \le t \le 2$:



As you can see, the straight line is a very *close match to the curve for* $t \approx t_0 = 1$*: Had we "zoomed in more", and shown the two graphs only for* $0.9 \leq t \leq 1.1$*, the two graphs would have been pretty much indistinguishable. On such an interval,* $\|\mathbf{x}(t) - [\mathbf{x}_0 + (t - t_0)\mathbf{v}]\|$ *would be very small compared to* $|t - t_0|$*. That is for some very small* $\epsilon > 0$*, we would have*

$$\|\mathbf{x}(t) - [\mathbf{x}_0 + (t - t_0)\mathbf{v}]\| \leq \epsilon|t - t_0| \ , \tag{2.1.12}$$

at least for those values of t *visible in our "zoomed in" graph.*

There are two important "sizes" displayed in the graph in the previous example:

(1) The "zoom factor", or size of piece of segment that is viewed. The traditional notation for this is δ; we "zoom in" so that the piece of the curve for valued of t with $t_0 - \delta \leq t \leq t_0 + \delta$ fills the viewing space.

(2) The "accuracy of fit factor". The traditional notation for this is ϵ. By construction, the line and the curve agree at $t = t_0$. However, as t moves away from t_0, there will generally be deviation. The accuracy of fit factor ϵ measures the worst case deviation, as a fraction of $|t - t_0|$. The smaller this is, the more accurate the fit is.

Roughly speaking, a curve is differentiable at t_0 if, for any desired accuracy of fit ϵ, if we zoom in to a small enough δ, the pieces of the curve and the line that are in the viewing area will fit to the desired degree of accuracy.

Definition 2 (Derivatives of Vector Valued Functions)**.** *Let* $\mathbf{x}(t)$ *be a vector valued function of the variable* t*. We say that* $\mathbf{x}(t)$ *is differentiable at* $t = t_0$ *with derivative* $\mathbf{v}$ *in case for every* $\epsilon > 0$*, there is a* $\delta_\epsilon > 0$ *so that*

$$|t - t_0| \le \delta_\epsilon \qquad \Rightarrow \qquad \|\mathbf{x}(t) - \mathbf{x}(t_0) - (t - t_0)\mathbf{v}\| \le \epsilon |t - t_0| \ . \tag{2.1.13}$$

In this case we say that $\mathbf{v}$ *is the* derivative *of* $\mathbf{x}(t)$ *at* $t = t_0$*, and shall often denote it by writing* $\mathbf{x}'(t_0)$*.*

A vector valued function is differentiable in some interval $]a, b[$ *if it is differentiable for each* t_0 *in* $]a, b[$*, and then we shall write* $\mathbf{x}'(t)$ *to denote the function associating to each* t *the derivative of* $\mathbf{x}(t)$ *at* t*.*

Now define $h = t - t_0$, and note that we can rewrite (2.1.13) as

$$|h| \le \delta_\epsilon \qquad \Rightarrow \qquad \left|\frac{1}{h}(\mathbf{x}(t + h) - \mathbf{x}(t_0)) - \mathbf{v}\right| \le \epsilon \ . \tag{2.1.14}$$

By what we have explained about convergence and limits in $\mathbb{R}^n$, this is equivalent to

$$\lim_{h \to 0} \frac{1}{h}\left(\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)\right) = \mathbf{v} \ , \tag{2.1.15}$$

and in particular, since limits are unique when they exist, there can be at most one vector $\mathbf{v}$ for which (2.1.13) is true. (Of course (2.1.13) refers to a particular choice of t_0. If this is changed, $\mathbf{v}$ may change too, but what is uniquely determined is the vector $\mathbf{v}$ at each fixed t_0.)

Moreover, when it comes to actually computing this vector $\mathbf{v}$, we can make ready use of what we know about single variable calculus: By what we have explained about convergence and limits in $\mathbb{R}^n$, (2.1.15) is equivalent to

$$\lim_{h \to 0} \frac{1}{h}\left(x_j(t_0 + h) - x_j(t_0)\right) = v_j \qquad \text{for} \qquad j = 1, \ldots, n \ .$$

So as far as doing the computations necessary to evaluate the derivative of a vector valued function, we only need to differentiate the entries of $\mathbf{x}(t)$ one at a time.

- *To compute the derivative of* $\mathbf{x}(t)$*, you simply differentiate it entry by entry.*

Example 3 (Computing the derivative of a vector valued function of t))**.** *Let* $\mathbf{x}(t)$ *be given by (2.1.1). Then for any* $t \neq 0$*,*

$$\mathbf{x}'(t) = (-\sin(t), \cos(t), -1/t^2) \ .$$

Because we just differentiate vectors entry by entry without mixing the entries up in any way, familiar rules for differentiating scalar valued functions hold for vector valued functions as well. In particular, the derivative of a sum is still the sum of the derivatives, etc.:

$$(\mathbf{x}(t)+\mathbf{y}(t))' = \mathbf{x}'(t)+\mathbf{y}'(t)\ . \tag{2.1.16}$$

Here is another example of how we can leverage our knowledge of single variable calculus:

- *If* $\mathbf{x}(t)$ *is differentiable at* $t = t_0$*, then it is continuous at* $t = t_0$*.*

Indeed, we know that for each j, $x_j(t)$ is continuous at $t = t_0$ provided it is differentiable at $t = t_0$. But since both continuity and differentiability can both be checked at the level of the individual entry functions, this is all we need to know.

We now turn to product rules. There are now several types of products to consider: product rules for scalar-vector multiplication and product rules for both the dot and cross products.

Theorem 2 (Differentiating dot and cross products)**.** *Suppose that* $\mathbf{v}(t)$ *and* $\mathbf{w}(t)$ *are differentiable vector valued functions for* t *in* $]a,b[$ *with values in* R^n*, and that both of these functions are differentiable at* $t_0 \in [a,b[$*. Then* $\mathbf{v}(t)\cdot\mathbf{w}(t)$ *and* $\mathbf{v}(t)\times\mathbf{w}(t)$*are differentiable at* t_0

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{w}(t))\bigg|_{t=t_0} = \mathbf{v}'(t_0)\cdot\mathbf{w}(t_0)+\mathbf{v}(t_0)\cdot\mathbf{w}'(t_0) \tag{2.1.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\times\mathbf{w}(t))\bigg|_{t=t_0} = \mathbf{v}'(t_0)\times\mathbf{w}(t_0)+\mathbf{v}(t_0)\times\mathbf{w}'(t_0)\ . \tag{2.1.18}$$

Proof: By definition we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(t)\cdot\mathbf{w}(t))\bigg|_{t=t_0} = \lim_{h\to 0}\frac{1}{h}(\mathbf{v}(t_0+h)\cdot\mathbf{w}(t_0+h)-\mathbf{v}(t_0)\cdot\mathbf{w}(t_0))\ . \tag{2.1.19}$$

Now we use the device of “adding and subtracting” that is used to prove the single variable product rule to write

$$\begin{aligned}\mathbf{v}(t_0+h)\cdot\mathbf{w}(t_0+h)-\mathbf{v}(t_0)\cdot\mathbf{w}(t_0) &= [\mathbf{v}(t_0+h)-\mathbf{v}(t_0)]\cdot\mathbf{w}(t_0+h)\\ &= \mathbf{v}(t_0)\cdot[\mathbf{w}(t_0+h)-\mathbf{w}(t_0)]\end{aligned} \tag{2.1.20}$$

Note that this identity is true because we have simply added $\mathbf{v}(t_0)\cdot\mathbf{w}(t_0+h)$ in the first line on the right, and subtracted it back out in the second. The advantage is that now in each term, only one of $\mathbf{v}$ and $\mathbf{w}$ is changing.

Combining (2.1.19) and (2.1.20), we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}(t)\cdot\mathbf{w}(t)\right)\right|_{t=t_0} &= \lim_{h\to 0}\left(\frac{\mathbf{v}(t_0+h)-\mathbf{v}(t_0)}{h}\cdot\mathbf{w}(t_0+h)\right)\\ &+ \lim_{h\to 0}\left(\mathbf{v}(t_0)\cdot\frac{\mathbf{w}(t_0+h)-\mathbf{w}(t_0)}{h}\right)\\ &= \mathbf{v}'(t_0)\cdot\mathbf{w}(t_0)+\mathbf{v}(t_0)\cdot\mathbf{w}'(t_0)\end{aligned}$$

The proof for cross products is exactly the same; simple replace each dot product with a cross product in the lines above. □

Finally there is the case of the product rule for scalar vector multiplication. If $g(t)$ is a real valued function defined on $]a,b[$, and $\mathbf{x}(t)$ is an $\mathbb{R}^n$ valued function defined on $]a,b[$, and if both are differentiable at $t_0\in]a,b[$, then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\mathbf{x}(t)\right)\right|_{t=t_0} = g'(t_0)\mathbf{x}(t_0)+g(t_0)\mathbf{x}'(t_0)\ . \tag{2.1.21}$$

We leave the proof of this, which can be patterned on the proof of Theorem 2, to the reader.

We close this subsection by returning to the notion of a tangent line, now that we have given a careful and precise definition of differentiability.

Definition 3 (Tangent line approximation)**.** *Suppose that* $\mathbf{x}(t)$ *is a function with values in* $\mathbb{R}^n$ *that is defined on some open interval* $]a,b[$ *with* $a<t_0<b$*, and* $\mathbf{x}(t)$ *is differentiable at* $t=t_0$*.*

Then the approximation

$$\mathbf{x}(t)\approx\mathbf{x}(t_0)+(t-t_0)\mathbf{x}'(t_0)$$

is called the tangent line approximation, and the parameterized line on the right hand side is called the tangent line to $\mathbf{x}(t)$ *at* t_0*. The vector* $\mathbf{x}'(t_0)$ *is called the* velocity vector at $t=t_0$.

We have already computed an explicit tangent line approximation (2.1.11) in Example 2. The following theorem specifies the sense in which the tangent line, among all possible lines, gives the "best fit" to the curve near $t=t_0$:

Theorem 3 (Best linear approximation)**.** *Let* $\mathbf{x}(t)$ *be differentiable at* $t=t_0$*. Then*

$$\lim_{h\to 0}\frac{1}{h}\|\mathbf{x}(t_0+h)-[\mathbf{y}_0+h\mathbf{w}]\|=0 \tag{2.1.22}$$

if and only if $\mathbf{y}_0=\mathbf{x}(t_0)$ *and* $\mathbf{w}=\mathbf{x}'(t_0)$*.*

Proof: It follows from (2.1.14) that when $\mathbf{x}(t)$ be differentiable at $t = t_0$, and $\mathbf{y}_0 = \mathbf{x}(t_0)$ and $\mathbf{w} = \mathbf{x}'(t_0)$, then (2.1.22) is true.

Next, observe that by the continuity of $\mathbf{x}(t)$ at $t = t_0$,

$$\lim_{h\to 0} \|\mathbf{x}(t_0 + h) - [\mathbf{y}_0 + h\mathbf{w}]\| = \|\mathbf{x}(0) - \mathbf{y}_0\| \ ,$$

so unless $\mathbf{y}_0 = \mathbf{x}(0)$, the limit in (2.1.22) would even diverge to $+\infty$. Therefore, suppose that $\mathbf{y}_0 = \mathbf{x}(0)$. Then multiplying and dividing by h, and adding and subtracting $\mathbf{x}(t+h) - \mathbf{x}(0) = \mathbf{x}(t+h) - \mathbf{y}_0$, and then using the triangle inequality,

$$\begin{aligned}
\|\mathbf{w} - \mathbf{x}'(t_0)\| &= \frac{1}{|h|}\|h\mathbf{w} - h\mathbf{x}'(t_0)\| \\
&= \frac{1}{|h|}\|(\mathbf{x}(t+h) - \mathbf{x}(0) - h\mathbf{x}'(t_0)) - (\mathbf{x}(t+h) - \mathbf{y}_0 - h\mathbf{w})\| \\
&= \frac{1}{|h|}\|\mathbf{x}(t+h) - \mathbf{x}(0) - h\mathbf{x}'(t_0)\| + \frac{1}{|h|}\|\mathbf{x}(t+h) - \mathbf{y}_0 - h\mathbf{w})\| \ .
\end{aligned}$$

As h tends to zero, the first term in the last line goes to zero by the differentiability of $\mathbf{x}(t)$ at $t = 0$, and the second term in the last line goes to zero by (2.1.22). Therefore, $\|\mathbf{w} - \mathbf{x}'(t_0)\| = 0$, and so $\mathbf{w} = \mathbf{x}'(0)$ is necessary for (2.1.22) to be true. □

2.1.3 Velocity and acceleration

Let $\mathbf{x}(t)$ represent the *position* of a point particle at time t. Then, as explained just above, $\mathbf{x}'(t)$ is called the velocity. It is often denoted by $\mathbf{v}(t)$. The velocity gives the rate of change of the position.

If the function $\mathbf{v}(t) = \mathbf{x}'(t)$ is differentiable, then $\mathbf{v}'(t)$ is called the *acceleration*, and is often denoted by $\mathbf{a}(t)$, so that $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{x}''(t)$. The acceleration is the second time derivative of the position and gives the rate of change of the velocity vector.

Consider $x_j(t)$, the jth entry of $\mathbf{x}(t)$. This is a garden variety real valued function of a single real variable. By Taylor's Theorem, given any t_0,

$$x_j(t) = x_j(t_0) + x_j'(t_0)(t - t_0) + \frac{1}{2}x_j''(t_0)(t - t_0)^2 + \mathcal{O}((t - t_0)^3) \ . \tag{2.1.23}$$

Doing the same for the other coordinates, and combining the results in vector form, we have

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{x}(t_0) + (t - t_0)\mathbf{x}'(t_0) + \frac{(t - t_0)^2}{2}\mathbf{x}''(t_0) + \mathcal{O}((t - t_0)^3) \\
&= \mathbf{x}(t_0) + (t - t_0)\mathbf{v}(t_0) + \frac{(t - t_0)^2}{2}\mathbf{a}(t_0) + \mathcal{O}((t - t_0)^3) \ . \qquad (2.1.24)
\end{aligned}$$

In the vector form, we have written the scalar multiples in front of the vectors, as we usually do.

Here you see what the velocity and acceleration tell you: If $h = t - t_0$ is small, then the leading order approximation to $\mathbf{x}(t_0 + h)$ is given by

$$\mathbf{x}(t_0) + h\mathbf{v}(t_0) + \frac{h^2}{2}\mathbf{a}(t_0) \ ,$$

and the errors in this approximation are $\mathcal{O}(|h|^3)$.

Example 4 (Velocity and acceleration). *Let* $\mathbf{x}(t)$ *be given by* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$. *Then, simply differentiating entry by entry, we find that for all* $t > 0$,

$$\mathbf{v}(t) = \mathbf{x}'(t) = (1, 2^{1/2}t^{1/2}, t)$$

and

$$\mathbf{a}(t) = \mathbf{x}''(t) = (0, 2^{-1/2}t^{-1/2}, 1).$$

Taking $t = t_0 = 1$, *for example, we find the instantaneous position, velocity and acceleration at* $t = t_0 = 1$*:*

$$\mathbf{x}(1) = (1, 2^{3/2}/3, 1/2) \qquad \mathbf{v}(1) = (1, 2^{1/2}, 1) \qquad \text{and} \qquad \mathbf{a}(1) = (0, 2^{-1/2}, 1) \ .$$

We now turn to an investigation of the magnitudes and directions of the velocity and the acceleration, with the goal of understanding what knowledge of these quantities tells us about the motion.

Definition 4 (Speed and unit the tangent vector). *The magnitude of the velocity vector is called the* speed. *We denote it by* $v(t)$. *That is,*

$$v(t) = |\mathbf{v}(t)| \ .$$

Provided that $v(t) \neq 0$, *we can define a unit vector valued function* $\mathbf{T}(t)$ *by*

$$\mathbf{T}(t) = \frac{1}{v(t)}\mathbf{v}(t) \ . \tag{2.1.25}$$

Then clearly

$$\mathbf{v}(t) = v(t)\mathbf{T}(t) \ . \tag{2.1.26}$$

The vector $\mathbf{T}(t)$ *is called the unit tangent vector at time* t. *It specifies the instantaneous direction of motion.*

Example 5 (Speed and the unit tangent vector)**.** *Let us continue with the curve* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ *from Example 3. There we found that* $\mathbf{v}(t) = (1, 2^{1/2}t^{1/2}, t)$*, and so the speed* $v(t)$ *is given by*

$$v(t) = \sqrt{1 + 2t + t^2} = |1 + t| \ .$$

For $t \neq -1$*, this is strictly positive, and then we have*

$$\mathbf{T}(t) = \frac{1}{1+t}(1, 2^{1/2}t^{1/2}, t) \ .$$

Theorem 4. *Let* $\mathbf{x}(t)$ *be a twice differentiable curve, and suppose that the speed* $v(t)$ *is nonzero on some open interval* $]b, c[$ *so that* $\mathbf{T}(t)$ *is defined for all* t *in this interval. Then, the rate of change of the speed,* $v'(t)$*, is given by*

$$v'(t) = \mathbf{a}(t) \cdot \mathbf{T}(t)$$

for all $b \leq t \leq c$*.*

Proof: Since $v^2(t) = \mathbf{v}(t) \cdot \mathbf{v}(t)$, we can apply Theorem 2 to conclude

$$2v(t)v'(t) = 2\mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\mathbf{v}(t) \cdot \mathbf{a}(t) = 2v(t)\mathbf{T}(t) \cdot \mathbf{a}(t) \ ,$$

and since $v(t) \neq 0$, we may divide through on both sides by $v(t)$. Finally, remember that the dot product is commutative. □

Now consider the decomposition of the acceleration $\mathbf{a}(t)$ into its components $\mathbf{a}_{\|}(t)$ and $\mathbf{a}_{\perp}(t)$, parallel and perpendicular to $\mathbf{T}(t)$. By definition, $\mathbf{a}_{\|}(t) = (\mathbf{a}(t) \cdot \mathbf{T}(t))\mathbf{T}(t)$. Thus, by Theorem 4, $\mathbf{a}_{\|}(t) = v'(t)\mathbf{T}(t)$. By (2.1.21),

$$\mathbf{a}(t) = (v(t)\mathbf{T}(t))' = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \ ,$$

and since $\mathbf{a}_{\|}(t) = v'(t)\mathbf{T}(t)$, it follows that $\mathbf{a}_{\perp}(t) = v(t)\mathbf{T}'(t)$. We summarize this important consequence of Theorem 4:

$$\mathbf{a}_{\|}(t) = v'(t)\mathbf{T}(t) \qquad \text{and} \qquad \mathbf{a}_{\perp}(t) = v(t)\mathbf{T}'(t) \ . \tag{2.1.27}$$

Thus the tangential component of the acceleration has to do with the rate of change of the speed, while the normal component has to do with the rate of change of the direction of the velocity vector, $\mathbf{T}(t)$.

In particular, when the speed is constant, so that $v'(t) = 0$, $\mathbf{a}_{\|}(t) = 0$, and then only the normal component of the acceleration can be non-zero. The following example is fundamentally important for understanding the meaning of the normal component of the acceleration.

Example 6 (Constant speed circular motion)**.** *Let* $\mathbf{x}(t)$ *be the* $\mathbb{R}^2$ *valued function given by*

$$\mathbf{x}(t) = \varrho\big(\cos(\Theta t), \sin(\Theta t)\big)$$

for some fixed $\varrho > 0$*. Since* $\|\mathbf{x}(t)\| = \sqrt{\varrho^2(\cos^2(\Theta t) + \sin^2(\Theta t)} = \varrho$*, the motion described by this curve runs over the circle of radius* ϱ *in the plane. Differentiating,*

$$\mathbf{v}(t) = \Theta\varrho\big(-\sin(\Theta t), \cos(\Theta t)\big) \qquad \text{and} \qquad \mathbf{a}(t) = -\Theta^2\varrho\big(\cos(\Theta t), \sin(\Theta t)\big) \ .$$

Thus,

$$v(t) = \|\mathbf{v}(t)\| = \Theta\varrho \qquad \text{and} \qquad \mathbf{T}(t) = \big(-\sin(\Theta t), \cos(\Theta t)\big) \ .$$

Since the speed is constant, or equivalently since $\mathbf{T}(t) \cdot \mathbf{a}(t) = 0$ *for all* t *,* $\mathbf{a}_{\|}(t) = 0$ *for all* t*. Therefore,*

$$\mathbf{a}_{\perp}(t) = \mathbf{a}(t) = -\Theta^2\varrho\big(\cos(\Theta t), \sin(\Theta t)\big) \ .$$

Therefore, the radius ϱ *of the circle is given by*

$$\varrho - \frac{v^2(t)}{\|\mathbf{a}_{\perp}(t)\|}$$

for all t*. Since for circular motion at constant speed,* $\mathbf{a}(t) = \mathbf{a}_{\perp}(t)$ *for all* t*, this can be rewritten as*

$$\|\mathbf{a}\| = \frac{v^2}{\varrho} \ .$$

Summarizing, for circular motion at constant speed, the magnitude of the acceleration is directly proportion to the square of the speed, and inversely proportional to the radius of the circle. The smaller the radius of the circle, the more "curved" the circle is.

The previous example motivates the following definition.

Definition 5 (Curvature and the unit normal vector)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^n$*, and suppose that the speed* $v(t)$ *is nonzero on some open interval* $]b, c[$ *so that* $\mathbf{T}(t)$ *is defined for all* t *in this interval.*

The curvature $\kappa(t)$ *at time* t *is defined by*

$$\kappa(t) = \frac{\|\mathbf{a}_{\perp}\|}{v^2(t)} \ . \tag{2.1.28}$$

Furthermore, if $\|\mathbf{a}_\perp\| \neq 0$, *we define the* unit normal vector $\mathbf{N}(t)$ *by*

$$\mathbf{N}(t) = \frac{1}{\|\mathbf{a}_\perp\|}\mathbf{a}_\perp \ , \tag{2.1.29}$$

and the radius of curvature $\varrho(t)$, *by*

$$\varrho(t) = \frac{1}{\kappa(t)} \ . \tag{2.1.30}$$

Comparing (2.1.27) and (2.1.29), we see that $\mathbf{N}(t)$ points in the same direction as $\mathbf{T}'(t)$. Thus, it points in the direction in which the curve is turning.

Since $\mathbf{T}(t)$ is the direction of $\mathbf{a}_{\|}(t)$, and $\mathbf{N}(t)$ is the direction of $\mathbf{a}_\perp(t)$, $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal. Here is another way to see this: Since $\mathbf{T}(t)$ is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1 \ .$$

Therefore, by Theorem 2,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}1 = 2\mathbf{T}(t) \cdot \mathbf{T}'(t) \ ,$$

which shows the orthogonality of $\mathbf{T}(t)$ and $\mathbf{T}'(t)$.

Example 7 (Normal and tangential acceleration)**.** *Let* $\mathbf{x}(t) = (t, (2t)^{3/2}/3, t^2/2)$. *We have computed in Example 5 that*

$$v(t) = 1 + t \qquad \text{and} \qquad \mathbf{T}(t) = \frac{1}{1+t}(1, (2t)^{1/2}, t) \ .$$

Therefore, $v'(t) = 1$, *and so* $\mathbf{a}_{\|}(t) = \mathbf{T}(t)$. *Thus,*

$$\mathbf{a}_{\|}(t) = \frac{1}{1+t}(1, (2t)^{1/2}, t) \ .$$

This is the tangential component of the acceleration. The normal component is $\mathbf{a}(t)$ *minus this. Since we have computed in Example 4 that*

$$\mathbf{a}(t) = \mathbf{x}''(t) = (0, (2t)^{-1/2}, 1) \ ,$$

the normal component is

$$(0, (2t)^{-1/2}, 1) - \frac{1}{1+t}(1, (2t)^{1/2}, t) = \frac{1}{1+t}(1, (1-t)(2t)^{-1/2}, 1) \ .$$

From here we compute

$$\|\mathbf{a}_\perp(t)\| = \frac{1}{\sqrt{2t}} \ .$$

Hence

$$\mathbf{N}(t) = \frac{\sqrt{2t}}{1+t}\big(-1, (1-t)(2t)^{-1/2}, 1\big)$$

and

$$\kappa(t) = \frac{\sqrt{2t}}{(1+t)^2} \qquad \text{and} \qquad \varrho(t) = \frac{(1+t)^2}{\sqrt{2t}} \ .$$

Next, notice that by combining (2.1.28) and (2.1.29), we obtain a simple formula for $\mathbf{a}_\perp(t)$, namely

$$\mathbf{a}_\perp(t) = v^2(t)\kappa(t)\mathbf{N}(t) \tag{2.1.31}$$

Comparing (2.1.27) and (2.1.31), we deduce two useful formulas, that deserve to be enshrined in a theorem:

Theorem 5. *Let $\mathbf{x}(t)$ be a twice differentiable curve in $\mathbb{R}^n$. Then*

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + v^2(t)\kappa(t)\mathbf{N}(t) \ , \tag{2.1.32}$$

and

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t) \ . \tag{2.1.33}$$

Proof: Simply compare (2.1.27) and (2.1.31). □

As a consequence of Theorem 5, we can deduce a simple and effective formula for computing curvature in $\mathbb{R}^3$:

Theorem 6 (Computing curvature in $\mathbb{R}^3$)**.** . *Let $\mathbf{x}(t)$ be a twice differentiable curve in $\mathbb{R}^3$ such that $v(t_0) > 0$. Let $\mathbf{v}$ and $\mathbf{a}$ denote the velocity and acceleration at time t_0. Let v and κ denote the speed and curvature at time t_0. Then*

$$\kappa = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v^3} \ . \tag{2.1.34}$$

Proof: We compute:

$$\begin{aligned} \mathbf{v}\times\mathbf{a} &= (v\mathbf{T})\times(v'\mathbf{T} + v^2\kappa\mathbf{N}) \\ &= vv'\mathbf{T}\times\mathbf{T} + v^3\kappa\mathbf{T}\times\mathbf{N} \\ &= v^3\kappa\mathbf{T}\times\mathbf{N} \end{aligned}$$

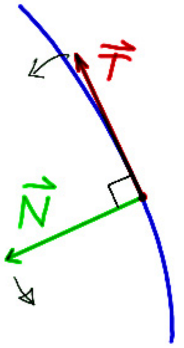
Since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal unit vectors, $\mathbf{T}\times\mathbf{N}$ is a unit vector, and hence $|\mathbf{v}\times\mathbf{a}| = v^3\kappa$. □

The aspect of the formula (2.1.34) that is three dimensional is that it involves the cross product. Nonetheless, it can also be applied in two dimensions: We can consider any planar curve in $\mathbb{R}^2$ as a curve in $\mathbb{R}^3$ for which $x_3(t) = 0$ for all t. Thus everything we have deduced about curves in $\mathbb{R}^3$ applies to curves in $\mathbb{R}^2$ as a special case.

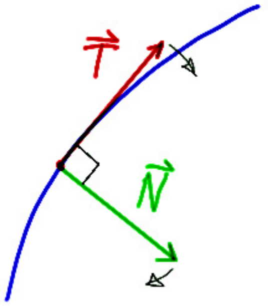
Let us further consider acceleration and curvature in two dimensions, for which it is easy to draw clear diagrams.

In $\mathbb{R}^2$, there are just two directions perpendicular to $\mathbf{T}$, namely $\mathbf{T}^\perp$ and $-\mathbf{T}^\perp$. Since $\mathbf{N}$ is perpendicular to $\mathbf{T}$, either $\mathbf{N} = \mathbf{T}^\perp$ or else $\mathbf{N} = -\mathbf{T}^\perp$. Which it is depends on whether the turning is clockwise or counterclockwise. The formula (2.1.33) shows that $\mathbf{N}(t)$ points in the same direction as $\mathbf{T}'(t)$: i.e., $\mathbf{N}$ points in the "direction of turning". If, as you drive along a track, you are turning counter-clockwise, then $\mathbf{N} = \mathbf{T}^\perp$, and if you are turning clockwise, then $\mathbf{N} = -\mathbf{T}^\perp$.

Continuing with the driving theme, the acceleration you feel when you step down on either the brake of the accelerator is tangential acceleration. The acceleration you feel when you turn the steering wheel, keeping the speed steady, is normal acceleration.



Turning counter-clockwise: $\vec{N} = \vec{T}^\perp$



Turning clockwise: $\vec{N} = -\vec{T}^\perp$

2.1.4 Torsion and the Frenet–Seret formulae for a curve in $\mathbb{R}^3$

Curves in $\mathbb{R}^3$ are especially important since we live in a three dimensional world. In this case, we can compute a unit normal to the plane through $\mathbf{x}(t_0)$ that contains both $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ by taking the cross product of $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$.

Definition 6 (Binormal vector and osculating plane). *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$. *Then at each* t_0 *for which* $v(t_0) \neq 0$ *and* $\kappa(t_0) \neq 0$, *so that* $\mathbf{T}(t_0)$ *and*

$\mathbf{N}(t_0)$ *are well defined, the* binormal vector $\mathbf{B}(t_0)$ *is defined by*

$$\mathbf{B}(t_0) = \mathbf{T}(t_0) \times \mathbf{N}(t_0) \ , \tag{2.1.35}$$

and the osculating plane *at* t_0 *is the plane specified by the equation*

$$\mathbf{B}(t_0) \cdot (\mathbf{x} - \mathbf{x}(t_0)) = 0 \ . \tag{2.1.36}$$

Since $\mathbf{B}(t_0)$ is orthogonal to $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$, (2.1.36) is the equation of the plane through $\mathbf{x}(t_0)$ that contains both $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$. Since $\mathbf{v} = v\mathbf{T}$ and $\mathbf{a} = v'\mathbf{T} + v^2\kappa\mathbf{N}$, $\mathbf{v} \times \mathbf{a} = v^3\kappa\mathbf{B}$, which yields the useful formulas

$$\mathbf{B}(t_0) = \frac{1}{v^3(t_0)\kappa(t_0)}\mathbf{v}(t_0) \times \mathbf{a}(t_0) = \frac{1}{\|\mathbf{x}'(t_0) \times \mathbf{x}''(t_0)\|}\mathbf{x}'(t_0) \times \mathbf{x}''(t_0) \ . \tag{2.1.37}$$

It follows that the direction of $\mathbf{B}$ is the same as that of $\mathbf{v} \times \mathbf{a}$. Therefore, the osculating plane at time t_0 is the plane through $\mathbf{x}(t_0)$ that contains $\mathbf{v}(t_0)$ and $\mathbf{a}(t_0)$. For this reason, the osculating plane is sometimes called the *instantaneous plane of motion*, and another equation for the osculating plane at $t = t_0$ is

$$(\mathbf{v}(t_0) \times \mathbf{a}(t_0)) \cdot (\mathbf{x} - \mathbf{x}(t_0)) = 0 \ .$$

In particular, it is not necessary to go through all the work of computing $\mathbf{T}$, $\mathbf{N}$ and then $\mathbf{B}$ if all you wanted to find was an equation for the osculation plane. You can find the equation directly from a computation of $\mathbf{v}$ and $\mathbf{a}$.

We emphasize that we are assuming throughout these paragraphs, as in Definition 6, that $v(t_0) \neq 0$ and $\kappa(t_0) \neq 0$, so that $\mathbf{T}(t_0)$ and $\mathbf{N}(t_0)$ are well defined. Otherwise, it does not make sense to refer to "the" plane through $\mathbf{x}(t_0)$ containing $\mathbf{v}(t_0)$ and $\mathbf{a}(t_0)$.

Example 8 (An osculating plane)**.** *Let* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$. *In Example 4, we have computed that*

$$\mathbf{x}(1) = (1, 2^{3/2}/3, 1/2) \qquad \mathbf{v}(1) = (1, 2^{1/2}, 1) \qquad \text{and} \qquad \mathbf{a}(1) = (0, 2^{-1/2}, 1) \ .$$

We now compute

$$\mathbf{v}(1) \times \mathbf{a}(1) = (2^{-1/2}, -1, 2^{-1/2}) \ .$$

The equation for the osculating plane then is

$$(2^{-1/2}, -1, 2^{-1/2}) \cdot (x - 1, y - 2^{3/2}/3, z - 1/2) = 0$$

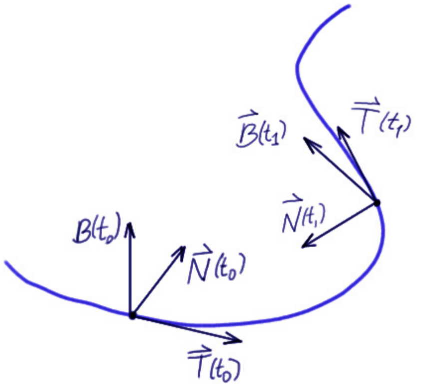
which reduces to

$$x - 2^{1/2}y + z = 6 \ .$$

Let $\mathbf{x}(t)$ be a twice differentiable curve in $\mathbb{R}^3$ such that $v(t) \neq 0$ and $\kappa(t) \neq 0$ for $t \in]a,b[$. Then for each time $t \in]a,b[$, the vectors

$$\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$$

are a right handed orthonormal basis of $\mathbb{R}^3$. That is, a curve in $\mathbb{R}^3$ carries around with itself a special orthonormal basis. How does this basis change with time?



Since $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ is a basis, we can express each of $\mathbf{T}'(t)$, $\mathbf{N}'(t)$ and $\mathbf{B}'(t)$ as linear combinations of these basis elements. Indeed, we have already seen that

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t) \ . \tag{2.1.38}$$

Next, consider $\mathbf{B}'(t)$. Since for each t, $\mathbf{B}(t) \cdot \mathbf{T}(t) = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{B}(t) \cdot \mathbf{T}(t)) = \mathbf{B}'(t) \cdot \mathbf{T}(t) + \mathbf{B}(t) \cdot \mathbf{T}'(t) \ .$$

But

$$\mathbf{B}(t) \cdot \mathbf{T}'(t) = \mathbf{B}(t) \cdot (v(t)\kappa(t)\mathbf{N}(t)) = 0 \ ,$$

and so

$$\mathbf{B}'(t) \cdot \mathbf{T}(t) = 0 \ .$$

Also, since for each t, $\mathbf{B}(t) \cdot \mathbf{B}(t) = 1$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{B}(t) \cdot \mathbf{B}(t)) = 2\mathbf{B}'(t) \cdot \mathbf{B}(t) \ ,$$

so that

$$\mathbf{B}'(t) \cdot \mathbf{B}(t) = 0 \ .$$

Since $\mathbf{B}'(t)$ has no component in the directions of $\mathbf{T}(t)$ or $\mathbf{B}(t)$, it follows that $\mathbf{B}'(t)$ is a multiple of $\mathbf{N}(t)$. This multiple deserve a name. Therefore, in analogy with (2.1.38), we define the *torsion* $\tau(t)$ by $\mathbf{B}'(t) = v(t)\tau(t)\mathbf{N}(t)$.

Definition 7 (Torsion)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$*. Then at each* t_0 *for which* $\{\mathbf{v}(t_0), \mathbf{a}(t_0)\}$ *is a linearly independent set of vectors, so that* $\mathbf{T}(t)$*,* $\mathbf{N}(t)$ *and* $\mathbf{B}(t)$ *are well defined for* t *in a neighborhood of* $t = t_0$*. Then the torsion at* $t = t_0$ *is the quantity* $\tau(t_0)$ *defined by*

$$\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t) \ . \tag{2.1.39}$$

The torsion describes the rate at which the instantaneous osculating plane rotates about the tangent line: If one looks down along the tangent line in the direction of $\mathbf{T}(t_0)$, and there is positive torsion, one sees the binormal vector turning counterclockwise.

Theorem 7 (Computing torsion in $\mathbb{R}^3$)**.** *Let* $\mathbf{x}(t)$ *be a thrice differentiable curve in* $\mathbb{R}^3$ *such that* $v(t_0) > 0$ *and* $\kappa(t_0) > 0$*. Let* $\mathbf{v}$ *and* $\mathbf{a}$ *denote the velocity and acceleration at time* t_0*. Let* v *and* κ *denote the speed and curvature at time* t_0*. Then*

$$\tau = -\frac{\mathbf{a} \cdot (\mathbf{v} \times \mathbf{a})'}{v^6 \kappa^2} \ . \tag{2.1.40}$$

Proof We start from the formula

$$\mathbf{v} \times \mathbf{a} = v^3 \kappa \mathbf{B}$$

that was derived in (2.1.35). Differentiation both sides we find

$$\begin{aligned} (\mathbf{v} \times \mathbf{a})' &= (v^3\kappa)'\mathbf{B} + (v^3\kappa)\mathbf{B}' \\ &= (v^3\kappa)'\mathbf{B} - \tau\kappa v^4 \mathbf{N} \ . \end{aligned} \tag{2.1.41}$$

Taking the dot product of both sides with $\mathbf{a} = v'\mathbf{T} + v^2\kappa\mathbf{N}$ yields

$$\mathbf{a} \cdot (\mathbf{v} \times \mathbf{a})' = -\tau\kappa^2 v^6 \ .$$

solving for τ, we obtain (2.1.40). □

Finally, let us derive a formula for $\mathbf{N}'(t)$:

$$\mathbf{N}'(t) = -v(t)\kappa(t)\mathbf{T}(t) + v(t)\tau(t)\mathbf{B}(t) \ . \tag{2.1.42}$$

Here are the computations that lead to this formula.

First, since for each t, $\mathbf{N}(t) \cdot \mathbf{N}(t) = 1$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{N}(t) \cdot \mathbf{N}(t)) = 2\mathbf{N}'(t) \cdot \mathbf{N}(t) \ ,$$

so that

$$\mathbf{N}'(t) \cdot \mathbf{N}(t) = 0 \ . \tag{2.1.43}$$

Second, since for each t, $\mathbf{N}(t) \cdot \mathbf{T}(t) = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{N}(t) \cdot \mathbf{T}(t)) = \mathbf{N}'(t) \cdot \mathbf{T}(t) + \mathbf{N}(t) \cdot \mathbf{T}'(t) \ .$$

But

$$\mathbf{N}(t) \cdot \mathbf{T}'(t) = \mathbf{N}(t) \cdot (v(t)\kappa(t)\mathbf{N}(t)) = v(t)\kappa(t) \ ,$$

and so

$$\mathbf{N}'(t) \cdot \mathbf{T}(t) = -v(t)\kappa(t) \ , \tag{2.1.44}$$

Third, since for each t, $\mathbf{N}(t) \cdot \mathbf{B}(t) = 0$,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{N}(t) \cdot \mathbf{B}(t)) = \mathbf{N}'(t) \cdot \mathbf{B}(t) + \mathbf{N}(t) \cdot \mathbf{B}'(t) \ .$$

But

$$\mathbf{N}(t) \cdot \mathbf{B}'(t) = -\mathbf{N}(t) \cdot (v(t)\tau(t)\mathbf{N}(t)) = -v(t)\tau(t) \ ,$$

and so

$$\mathbf{N}'(t) \cdot \mathbf{B}(t) = v(t)\tau(t) \ , \tag{2.1.45}$$

Combining (2.1.43), (2.1.44) and (2.1.45), we see that (2.1.42) is true.

Summarizing the results, we have proved the following:

Theorem 8 (Frenet–Seret formulae)**.** *Let* $\mathbf{x}(t)$ *be a thrice differentiable curve in* $\mathbb{R}^3$ *with non-zero speed and curvature at each* t *in some open interval so that* $\mathbf{T}(t)$*,* $\mathbf{N}(t)$ *and* $\mathbf{B}(t)$ *are all defined and differentiable on this interval. Then for all* t *in this interval,*

$$\begin{aligned} \mathbf{T}'(t) &= \quad v(t)\kappa(t)\mathbf{N}(t) \\ \mathbf{N}'(t) &= -v(t)\kappa(t)\mathbf{T}(t) + v(t)\tau(t)\mathbf{B}(t) \\ \mathbf{B}'(t) &= -v(t)\tau(t)\mathbf{N}(t) \ . \end{aligned}$$

There is a convenient way to combine these three formulae into one.

Definition 8 (Darboux vector)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve with non-zero speed and curvature at each* t *in some open interval so that* $\mathbf{T}(t)$*,* $\mathbf{N}(t)$ *and* $\mathbf{B}(t)$ *are all defined on this interval. The* Darboux vector $\boldsymbol{\omega}$ *is defined on this interval by by*

$$\boldsymbol{\omega} = \tau\mathbf{T} + \kappa\mathbf{B} \ .$$

The point of the definition is that since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is constructed to be a right-handed orthonormal basis of $\mathbb{R}^3$, it follows that

$$\mathbf{T}\times\mathbf{N} = \mathbf{B} \qquad \mathbf{N}\times\mathbf{B} = \mathbf{T} \qquad \text{and} \qquad \mathbf{B}\times\mathbf{T} = \mathbf{N} \ ,$$

where the first of these identities is just the definition of $\mathbf{B}$. Thus one has

$$\begin{aligned} \boldsymbol{\omega}\times\mathbf{T} &= (\tau\mathbf{T} + \kappa\mathbf{B})\times\mathbf{T} = \quad \kappa\mathbf{N} \\ \boldsymbol{\omega}\times\mathbf{N} &= (\tau\mathbf{T} + \kappa\mathbf{B})\times\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B} \\ \boldsymbol{\omega}\times\mathbf{B} &= (\tau\mathbf{T} + \kappa\mathbf{B})\times\mathbf{B} = -\tau\mathbf{N} \ . \end{aligned}$$

Comparing with Theorem 2.1.46, we see that

$$\begin{aligned} \mathbf{T}'(t) &= v(t)\boldsymbol{\omega}(t)\times\mathbf{T}(t) \\ \mathbf{N}'(t) &= v(t)\boldsymbol{\omega}(t)\times\mathbf{N}(t) \\ \mathbf{B}'(t) &= v(t)\boldsymbol{\omega}(t)\times\mathbf{B}(t) \ . \end{aligned} \tag{2.1.46}$$

As we shall see later in this chapter, this means that for small $h > 0$, the orthonormal basis $\{\mathbf{T}(t+h), \mathbf{B}(t+h), \mathbf{B}(t+h)\}$ is, up to errors of size h^2, what one would get by applying a rotation of angle $v(t)\|\boldsymbol{\omega}(t)\|$ about the axis of rotation in the direction of $\boldsymbol{\omega}(t)$. That is, the Darboux vector describes the instantaneous rate and direction of rotation of the orthonormal basis $\{\mathbf{T}(t), \mathbf{B}(t), \mathbf{B}(t)\}$.

2.1.5 Curvature and torsion are independent of parameterization.

The same path can be parameterized many ways. For example, consider

$$\mathbf{x}(t) = \big(\cos(t), \sin(t)\big) \qquad \text{and} \qquad \mathbf{y}(u) = \big(\cos(-u^3), \sin(-u^3)\big) \ .$$

As t and u vary over $\mathbb{R}$, both of these curves trace out the unit circle in $\mathbb{R}^2$, but they trace it out in different speeds and directions.

Definition 9 (Reparameterization)**.** *Let* $\mathbf{x}(t)$ *be a curve in* $\mathbb{R}^n$ *defined on an open interval* $]a,b[\subset \mathbb{R}$*, and let* $\mathbf{y}(u)$ *be another curve in* $\mathbb{R}^n$ *defined on an open interval* $]c,d[\subset \mathbb{R}$*. Either* a *or* c *may be* $-\infty$*, and either* b *or* d *may be* $+\infty$*. Then* $\mathbf{y}(u)$ *is a* reparameterization of $\mathbf{x}(t)$ *in case there is a a continuous, strictly monotone increasing or decreasing function* $t(u)$ *from* $]c,d[$ *onto* $]a,b[$ *such that*

$$\mathbf{y}(u(t)) = \mathbf{x}(t) \qquad \text{for all} \quad t \in]a,b[\ .$$

Example 9. *Define* $t(u) = -u^3$ *and* $u(t) = -t^{1/3}$*. Then with* $\mathbf{x}(t) = \big(\cos(t), \sin(t)\big)$ *and* $\mathbf{y}(u) = \big(\cos(-u^3), \sin(-u^3)\big)$*, we have both*

$$\mathbf{y}(u) = \mathbf{x}(t(u)) \quad \text{for all} \quad u \in \mathbb{R}$$

and

$$\mathbf{x}(t) = \mathbf{y}(u(t)) \quad \text{for all} \quad u \in \mathbb{R}\ .$$

Thus the $\mathbf{x}(t)$ *and* $\mathbf{y}(u)$ *are reparameterizations of each other, and they both parameterize the unit circle.*

As in the example, whenever $\mathbf{y}(u)$ is a reparameterization of $\mathbf{x}(t)$, then $\mathbf{x}(t)$ is a reparameterization of $\mathbf{y}(u)$. Indeed, if $t(u)$ is any continuous, strictly monotone increasing function $t(u)$ from $]c,d[$ onto $]a,b[$, then it is both one-to-one and onto, and so it has an inverse function $u(t)$ from $]c,d[$ to $]a,b[$ which is also continuous and strictly monotone increasing.

• *It turns out that while any curve can be parameterized in inifinitely many ways, the curvature at a point on the path is a purely geometric property of the path traced out by the curve – it is independent of the parameterization. Not only that, so is the unit normal vector, and, up to a sign, so is the unit tangent vector.*

To see this suppose that $\mathbf{x}(t)$ and $\mathbf{y}(u)$ are two parameterizations of the same path in $\mathbb{R}^n$. Suppose that

$$\mathbf{x}(t_0) = \mathbf{y}(u_0)$$

so that when $t = t_0$ and $u = u_0$, both curves pass through the same point. Let us suppose also that the two parameterizations are related in a smooth way, so that $t(u)$ is twice continuously differentiable in u.

Then, by the chain rule,

$$\mathbf{y}'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{y}(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{x}(t(u)) = \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)\mathbf{x}'(t(u))\ .$$

Evaluating at $u = u_0$, and recalling that $t_0 = t(u_0)$, we get the following relation between the speed at which the two curve pass through the point in question:

$$\|\mathbf{y}'(u_0)\| = \left|\frac{\mathrm{d}t}{\mathrm{d}u}\right| \|\mathbf{x}'(t_0)\| \ .$$

Therefore,

$$\begin{aligned}\frac{1}{\|\mathbf{y}'(u_0)\|}\mathbf{y}'(u_0) &= \left(\left|\frac{\mathrm{d}t}{\mathrm{d}u}\right|^{-1}\frac{\mathrm{d}t}{\mathrm{d}u}\right)\frac{1}{\|\mathbf{x}(t_0)\|}\mathbf{x}'(t_0)\\ &= \pm\frac{1}{\|\mathbf{x}(t_0)\|}\mathbf{x}'(t_0) \ .\end{aligned}$$

The plus sign is correct is t is an increasing function of u, in which case the two parameterizations trace the path out in the same direction, and the minus sign is correct if t is a decreasing function of u.

This shows that up to a sign, the unit tangent vector $\mathbf{T}$ at the point in question comes out the same for the two parameterizations.

Next, let us differentiate once more. We find

$$\begin{aligned}\mathbf{y}''(u) = \frac{\mathrm{d}}{\mathrm{d}u}\mathbf{y}'(u) &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)\mathbf{x}'(t(u))\right)\\ &= \left(\frac{\mathrm{d}^2t}{\mathrm{d}u^2}\right)\mathbf{x}'(t(u)) + \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2\mathbf{x}''(t(u)) \ .\end{aligned}$$

Evaluating at $u = u_0$, and recalling that $t_0 = t(u_0)$, we find the following formula relating the acceleration along the two curves as they pass though the point in question:

$$\mathbf{y}''(u_0) = \left(\frac{\mathrm{d}^2t}{\mathrm{d}u^2}\right)\mathbf{x}'(t_0) + \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2\mathbf{x}''(t_0) \ .$$

Notice that the first term on the right is a multiple of $\mathbf{T}$, and hence when we decompose $\mathbf{y}''(u_0)$ into its tangential and orthogonal components, this piece contributes only to the tangential component. Hence

$$\mathbf{y}''_\perp(u_0) = \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2\mathbf{x}''_\perp(t_0) \ .$$

Because of the square, $\mathbf{y}''_\perp(u_0)$ is a positive multiple of $\mathbf{x}''_\perp(t_0)$, and so these two vectors point in the exact same direction. That is,

$$\mathbf{N} = \frac{1}{\|\mathbf{y}''_\perp(u_0)\|}\mathbf{y}''_\perp(u_0) = \frac{1}{\|\mathbf{x}''_\perp(t_0)\|}\mathbf{x}''_\perp(t_0) \ ,$$

showing that the normal vector $\mathbf{N}$ is independent of the parameterization.

Next, we consider the curvature. Since

$$\begin{aligned}\frac{1}{\|\mathbf{y}'(u_0)\|^2}\|\mathbf{y}''_\perp(u_0))\| &= \left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^{-2}\frac{1}{|\mathbf{x}'(t_0)|^2}\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^{2}\|\mathbf{x}''_\perp(t_0)\| \\ &= \frac{1}{\|\mathbf{x}'(t_0)\|^2}\|\mathbf{x}''_\perp(t_0)\|\ ,\end{aligned}$$

we get the exact same value for the curvature at the same point, using either parameterization. This show that although in practice we use a particular parameterization to compute the curvature κ and the unit normal $\mathbf{N}$, the results do not depend on the choice of the parameterization, and are in fact an intrinsically geometric property of the path that the curve traces out.

So far what we have said about reparameterization is valid in $\mathbb{R}^n$ for all $n \geq 2$. In $\mathbb{R}^3$, there is more to say. In $\mathbb{R}^3$, we also have the binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and the torsion τ.

Let us go on to consider $\mathbf{B}(t)$ and $\tau(t)$. Since $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, it follows that $\mathbf{B}(t)$ is well defined, independent of the parameterization, up to a sign. Then, consideration of the formula

$$\mathbf{B}'(t) = -v(t)\tau(t)\mathbf{N}(t)$$

under two parameterization shows that like the curvature, the torsion is independent of the parameterization. The calculations that show this are very similar to the calculations we have just made for κ, $\mathbf{T}$ and $\mathbf{N}$, and are left to the reader. The conclusion is that the torsion also is determined by the geometry of the path itself, and not how fast or slow we move along it.

2.1.6 Speed and arc length

The speed $v(t)$ represents the rate of change of the distance traveled with time. Given some reference time t_0, define

$$s(t) = \int_{t_0}^{t} v(u)\mathrm{d}u\ . \qquad (2.1.47)$$

Then by the Fundamental Theorem of Calculus,

$$\frac{\mathrm{d}}{\mathrm{d}t}s(t) = v(t)$$

and clearly $s(t_0) = 0$. Hence the rate of change of $s(t)$ is $v(t)$, which is the rate of change of the distance traveled with time, as one has moved along the path traced out by $\mathbf{x}(t)$.

Definition 10 (Arc length)**.** *The function $s(t)$ defined by (2.1.47) is called the arc length along the path traced out by $\mathbf{x}(t)$ since time t_0.*

Example 10 (Computation of arc length)**.** *Let $\mathbf{x}(t)$ be given by*

$$\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$$

as in Example 4. Then, as we have seen, for all $t > 0$, $v(t) = 1 + t$. Therefore,

$$s(t) = \int_0^t (1 + u)\mathrm{d}u = t + \frac{t^2}{2} \ .$$

If you took a piece of string, and cut it so it can be run along the path from the starting point to the position at time t, the length of the string would be $t + t^2/2$ units of distance.

By definition, $v(t) \geq 0$, and so $s(t)$ has a non negative derivative. This means that it is an increasing function. As long as $v(t) > 0$; i.e., as long as the particle never comes to even an instantaneous rest, $s(t)$ is strictly monotone increasing.

Suppose also that $s(t)$ increases without bound, so that

$$\lim_{t\to\infty} s(t) = \infty \ .$$

Then for any $s \geq 0$, there is exactly one value of $t \geq 0$ so that

$$s(t) = s \ . \tag{2.1.48}$$

This value of t, considered as a function of s, is the inverse function to the arc length function:

$$t(s) = t \ . \tag{2.1.49}$$

It answers a very simple question, namely: *How much time will have gone by when the distance travelled is s units of length?*

If you can compute an explicit expression for $s(t)$, such as the result $s(t) = t + t^2/2$ that we found in Example 9, what you then need to do to answer the question is to find the inverse function $t(s)$; i.e., to solve (2.1.48) to find t in terms of s:

Example 11 (Time as a function of arc length)**.** *Let* $\mathbf{x}(t)$ *be given by* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ *as in Example 10. Then, as we have seen, for all* $t > 0$, $s(t) = t + (t^2/2)$. *To find* t *as a function of* s*, write this as*

$$s = t + \frac{t^2}{2}$$

and solve for t *in terms of* s*. In this case,*

$$t + \frac{t^2}{2} = \frac{1}{2}((t+1)^2 - 1)$$

so $t = \sqrt{2s+1} - 1$. *That is,*

$$t(s) = \sqrt{2s+1} - 1 \ .$$

This function tells you how long it took to travel a given distance s *when moving along the curve.*

We can then get a new parameterization of our curve by defining $\mathbf{x}(s)$ by

$$\mathbf{x}(s) = \mathbf{x}(t(s)) \ .$$

This is called the *arc length parameterization.* We have changed our habits of notation somewhat: Now we use the sme letter $\mathbf{x}$ for both parameterizations to emphasize that they are two parameterizations of the same curve.

Example 12 (Arc length parameterization)**.** *Let* $\mathbf{x}(t) = (t, 2^{3/2}t^{3/2}/3, t^2/2)$ *as in Example 11. Then, as we have seen, for all* $t > 0$, $t(s) = \sqrt{2s+1} - 1$ *Therefore,*

$$\mathbf{x}(s) = \mathbf{x}(t(s)) = (\sqrt{2s+1} - 1, 2^{3/2}(\sqrt{2s+1} - 1)^{3/2}/3, (\sqrt{2s+1} - 1)^2/2) \ .$$

The arc length parameterization generally is complicated to work out explicitly. Even when you can work it out, it often looks a lot more complicated than whatever t parameterization you started with. So what is it good for?

The point about the arc length parameterization is that it is purely geometric, so that it helps us to understand the geometry of a curvethe path that a parameterized curve traces out.. If we compute the rate of change of the unit tangent vector $\mathbf{T}$ as a function of s, we are computing the rate of turning per unit distance along the curve. This is an intrinsic property of the curve itself. If we compute rate of change of the unit tangent vector $\mathbf{T}$ as a function of t, we are computing something that depends on how fast we are moving on the curve, and not just on the curve itself. Indeed, if we use the arc length parameterization, $v(s) = 1$ for all s, and so the factors involving speed drop out of all of our formulas. For example,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}(s) = \mathbf{T}(s)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{T}(s) = \kappa(s)\mathbf{N}(s)\ .$$

Often, this last formula is taken as the definition of the normal vector $\mathbf{N}$ and curvature κ. The advantage of this definition is that it is manifestly geometric, so that the normal vector $\mathbf{N}$ and curvature κ do not depend on the parameterization of the curve. The disadvantage is that it is generally very difficult to explicitly work out the arc length parameterization. In order to more quickly arrive at computational examples, we have chosen the form of the definition that is convenient for computation. The next subsection will illustrate the utility of the arc length parameterization in studying geometric problems.

2.1.7 The osculating circle

Let $\mathbf{u}_1$ and and $\mathbf{u}_2$ be any given pair of orthogonal unit vectors in $\mathbb{R}^n$, let $\mathbf{c} \in \mathbb{R}^n$, and let ϱ be any positive number. Consider the parameterized curve $\mathbf{z}(s)$ given by

$$\mathbf{z}(t) = \mathbf{c} + \varrho(\cos(s/\varrho)\mathbf{u}_1 + \sin(s/\varrho)\mathbf{u}_2)\ .$$

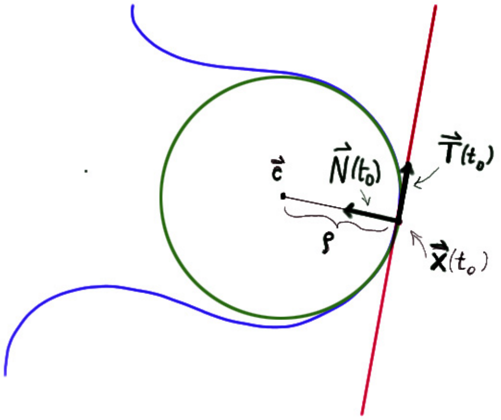
Then, since $\mathbf{u}_1$ and and $\mathbf{u}_2$ are orthogonal unit vectors, $\|\mathbf{z}(t) - \mathbf{c}\| = \varrho$ for all t. Also clearly, $\mathbf{z}(t)$ lies in the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ for all t. Thus, the curve $\mathbf{z}(t)$ describes circular motion on the circle of radius ϱ about the center $\mathbf{c}$ in the plane parameterized by $\mathbf{c} + r\mathbf{u}_1 + t\mathbf{u}_2$. Morover,

$$\mathbf{z}'(s) = (-\sin(s/\varrho)\mathbf{u}_1 + \cos(s/\varrho)\mathbf{u}_2)\ ,$$

which is a unit vector. Since $\mathbf{z}'(s) = 1$ for all s, this curve is already in its arc-length parameterization.

Let $\mathbf{x}(s)$ be a twice differentiable curve parameterized by arc length. Suppose that $\kappa(s_0) > 0$. Then for s close to s_0, the curve $\mathbf{x}(s)$ is well approximated by such a circular motion, and the circle that “fits best” is uniquely determined. This will be the *osculating circle*. The name comes from the Latin *circulum osculans*, introduced by Leibniz, meaning "kissing circle".

Here is a graph showing a curve in the plane, together with its the tangent line, and its tangent circle at a particular point $\mathbf{x}(s_0)$.



As you can see, the tangent circle gives a much closer fit to the curve than does the tangent line, and hence its name. The term "tangent" comes from the Latin verb *tangere*, meaning to "to touch", so while the tangent line "shakes hands" with the curve at the point of tangency, the osculating circle kisses it. Leibniz had in mind that the closer fit of the osculating circle corresponded to the greater intimacy of kissing relative to shaking hands, and hence the terminology.

Here is one way to think about what the osculating circle represents: If $\mathbf{x}(s)$ is your position at time t as you drive along some curved track at constant speed you would "feel" at each instant of time t as if you were diving at the same constant speed on a circular track of radius $\varrho(s)$ with a center at a point $\mathbf{c}(s)$ in the plane spanned by $\mathbf{v}(s)$ and $\mathbf{a}(s)$. This "instantaneous circular track" is the osculating circle.

The osculating circle is only well defined when the curvature $\kappa(s_0)$ is not zero, so that $\mathbf{N}(s_0)$ is well defined. It will lie in the plane through $\mathbf{x}(s_0)$ spanned by $\mathbf{T}(s_0)$ and $\mathbf{N}(s_0)$, which we have already introduced as the *osculating plane*. Here is the formal definition:

Definition 11 (Osculating circle)**.** *Let* $\mathbf{x}(s)$ *be parameterized by arc length, and twice differentiable at* s_0*, with* $\kappa(s_0)$ *strictly positive. Then osculating circle to this curve at*

$s = s_0$ is the parameterized path given $\mathbf{z}(s)$ given by

$$\begin{aligned} \mathbf{z}(s) = \mathbf{c}(s_0) &- \varrho(s_0)\cos(\kappa(s_0)(s - s_0))\mathbf{N}(s_0) \\ &+ \varrho(s_0)\sin(\kappa(s_0)(s - s_0))\mathbf{T}(s_0) \end{aligned} \tag{2.1.50}$$

where

$$\mathbf{c}(s_0) = \mathbf{x}(s_0) + \varrho(s_0)\mathbf{N}(s_0) \tag{2.1.51}$$

The path traced out by $\mathbf{z}(s)$ as t varies is a circle centered at $\mathbf{c}(s_0)$ with radius $\varrho(s_0)$, and lying in the plane parameterized by $\mathbf{x}(s_0) + r\mathbf{T}(s_0) + s\mathbf{N}(s_0)$; i.e., the osculating plane at t_0.

The formula for $\mathbf{z}(s)$ may seem a bit complicated. Let us do a few simple computations to familiarize our selves with it. First, since $\{\mathbf{T}(s_0), \mathbf{N}(s_0)\}$ is orthonormal,

$$\|\mathbf{z}(s) - \mathbf{c}(s_0)\| = \varrho(s_0)$$

for all s, and since evidently $\mathbf{z}(s)$ is always in the osculating plane, the path traced out by $\mathbf{z}(s)$ is indeed a circle of radius $\varrho(s_0)$ in this plane with center at $\mathbf{c}(s_0)$.

Next, let us compute $\mathbf{z}'(s)$. Using the fact that $\varrho(s_0)\kappa(s_0) = 1$, we find that

$$\mathbf{z}'(s) = \sin(\kappa(s_0)(s - s_0))\mathbf{N}(s_0) + \cos(\kappa(s_0)(s - s_0))\mathbf{T}(s_0)\ .$$

Again since $\{\mathbf{T}(s_0), \mathbf{N}(s_0)\}$ is orthonormal,

$$\|\mathbf{z}'(s)\| = 1$$

for all t. That is, $\mathbf{z}(s)$ describes unit speed circular motion. Also, we see that

$$\mathbf{z}(s_0) = \mathbf{x}(s_0) \qquad \text{and that} \qquad \mathbf{z}'(s_0) = \mathbf{T}(s_0) = \mathbf{x}'(s_0)\ .$$

Therefore, the both $\mathbf{x}(s)$ and $\mathbf{z}(s)$ have the same tangent line at $s = s_0$:

$$\mathbf{x}(0) + (s - s_0)\mathbf{x}'(s_0) = \mathbf{z}(0) + (s - s_0)\mathbf{z}'(s_0)$$

for all s.

This already says that the osculating circle is at least a "first order fit" to the curve, but things are much better than that.The osculating circle is actually a "second order fit" in the sense made precise in the following theorem:

Theorem 9. *Let* $\mathbf{x}(s)$ *be parameterized by arc length and twice differentiable at* s_0*, with* $\kappa(s_0) > 0$*. Let* $\mathbf{z}(s)$ *be the osculating circle to* $\mathbf{x}(s)$ *at* $s = s_0$*. Then*

$$\lim_{h \to 0} \frac{\|\mathbf{x}(s_0+h) - \mathbf{z}(s_0+h)\|}{h^2} = 0 \ , \tag{2.1.52}$$

and among all constant speed circular curves in $\mathbb{R}^n$*, the osculating circle is the only one with this property.*

Before proving Theorem 9, we work out an example.

Example 13 (Computing the osculating circle)**.** *Let* $\mathbf{x}(s) = (t, (2t)^{3/2}/3, t^2/2)$*. We shall compute the osculating circle at the point* $\mathbf{x}(1)$*.*

In Example 7, we computed $\mathbf{N}(s)$*,* $\mathbf{T}(s)$*,* $v(s)$ *and* $\varrho(s)$*. Evaluating them at* $t = 1$*, we get*

$$\mathbf{T}(1) = \frac{1}{2}(1, 2^{1/2}, 1) \qquad \mathbf{N}(1) = \frac{1}{\sqrt{2}}(-1, 01) \qquad v(1) = \frac{1}{2} \qquad \text{and} \qquad \varrho(1) = 2^{3/2}.$$

Therefore, since $\mathbf{x}(1) = (1, 2^{3/2}/3, 1/2)$*, (2.1.51) gives*

$$\mathbf{c}(1) = (1, 2^{3/2}/3, 1/2) + 2(-1, 0, 1) = (-1, 2^{3/2}/3, 5/2) \ .$$

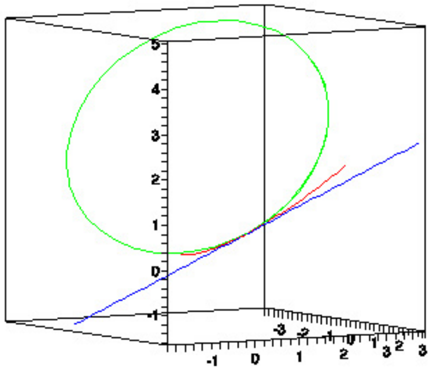
We will define $s(t)$ *so that* $s = 0$ *corresponds to* $t = 1$*; i.e.,* $s(t) = \int_0^t v(r)\mathrm{d}r$*. With this choice,* $s_0 = 0$*, and the osculating circle is parameterized by*

$$\mathbf{z}(s) = (-1, 2^{3/2}/3, 5/2) - \cos(2^{-3/2}s)2(-1, 0, 1) + \sin(2^{-5/2}s)\sqrt{2}(1, 2^{1/2}, 1) \ .$$

In Example 4, we have computed the tangent line at $t_0 = 1$*, and found it to be given by*

$$(1, 2^{3/2}/3, 1/2) + (t-1)(1, 2^{1/2}, 1) \ .$$

Here is a graph showing the curve itself, the tangent line, and the osculating circle at the point $(1, 2^{3/2}/3, 1/2)$*.*



As you can see, the osculating circle gives a much better fit than the tangent line at the point of tangency, which is the content of Theorem 9. Also, notice that to compute the osculating circle, we did not have to find the arc length parameterization of $\mathbf{x}(t)$*: We could compute everything we needed using the* t *parameterization.*

We remark that graphing the osculatory circle, and in the previous example, is a good way to visualize the osculating plane, since this is the plane in which the osculating circle lies. In fact, if you graph the curve, and two successive osculation circles, say at $\mathbf{x}(t_0)$ and $\mathbf{x}(t_0+h)$, $h>0$ you can see how the osculating plane turns: *This is a very good way to visualize the torsion along the curve.*

Proof of Theorem 9: We base the proof on (2.1.24). This tells us that

$$\mathbf{x}(s_0+h) = \mathbf{x}(s_0) + h\mathbf{x}'(s_0) + \frac{h^2}{2}\mathbf{x}''(s_0) + \mathcal{O}(|h|^3)$$

and

$$\mathbf{z}(s_0+h) = \mathbf{z}(s_0) + h\mathbf{z}'(s_0) + \frac{h^2}{2}\mathbf{z}''(s_0) + \mathcal{O}(|h|^3)$$

Therefore,

$$\lim_{h\to 0}\frac{\|\mathbf{x}(s_0+h)-\mathbf{z}(s_0+h)\|}{h^2} = \lim_{h\to 0}\left\|\frac{\mathbf{x}(s_0)-\mathbf{z}(s_0)}{h^2} + \frac{\mathbf{x}'(s_0)-\mathbf{z}'(s_0)}{h} + (\mathbf{x}''(s_0)-\mathbf{z}''(s_0))\right\| .$$

Thus, $\dfrac{\|\mathbf{x}(s_0+h)-\mathbf{z}(s_0+h)\|}{h^2} = 0$ if and only if

$$\begin{aligned} \mathbf{x}(s_0) &= \mathbf{z}(s_0) \\ \mathbf{x}'(s_0) &= \mathbf{z}'(s_0) \\ \mathbf{x}''(s_0) &= \mathbf{z}''(s_0) . \end{aligned} \tag{2.1.53}$$

It is then easily checked that the general arc length parameterized circle

$$\mathbf{z}(t) = \mathbf{c} + \varrho(\cos(s/\varrho)\mathbf{u}_1 + \sin(s/\varrho)\mathbf{u}_2)$$

satisfies this requirement if and only if $\varrho = \varrho(s_0)$, $\mathbf{u}_1 = \mathbf{N}(s_0)$, $\mathbf{u}_2 = \mathbf{T}(s_0)$ and $\mathbf{c} = \mathbf{x}(s_0) + \varrho(s_0)\mathbf{N}(s_0)$, where $\varrho(s_0)$, $\mathbf{T}(s_0)$ and $\mathbf{N}(s_0)$ are the radius of curvature, the unit tangent vector, and the unit normal vector of $\mathbf{x}(s)$ at $s = s_0$. This is where the formulae in Definition 11 come from. □

We have derived all of our formulas for the osculating circle using the arc length parameterization. However, now that we have derived them, we can use them with the original t parameterization. First of all, we have seen that $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ as well as the curvature and torsion, are (essentially) independent of the parameterization: You can compute $\mathbf{T}$, $\mathbf{N}$ and κ usng any parameterization you find convenient, and then use the formulas from Definition 11 as they are.

To get the good "second order" fit from Theorem 9, one does however have to "match" the parameterizations. If $\mathbf{z}(s)$ is the parameterization of the osculating circle for the arc length parameterization, simply substitute

$$s(t) = s_0 + \int_{t_0}^{t} v(r)\mathrm{d}r$$

into $\mathbf{z}(s)$, getting the reparameterization $\mathbf{z}(s(t))$.

Since the original curve $\mathbf{x}(t)$ is assumed to be twice differentiable, the speed v is differentiable, and therefore continuous. Since we assume $v(t_0) > 0$, so that $\mathbf{T}(t_0)$ is well defined, we have

$$\lim_{t\to t_0} \frac{s(t) - s_0}{t - t_0} = \lim_{t\to t_0} \frac{1}{t - t_0} \int_{t_0}^{t} v(r)\mathrm{d}r = v(t_0) \ .$$

Then, since Theorem 9 says that $\lim_{s\to s_0} \dfrac{\|\mathbf{x}(s) - \mathbf{z}(s)\|}{|s - s_0|^2} = 0$, we have

$$\lim_{t\to s_0} \frac{\|\mathbf{x}(s(t)) - \mathbf{z}(s(t))\|}{|t - t_0|^2} = \left(\lim_{t\to t_0} \left(\frac{s(t) - s_0}{t - t_0}\right)^2\right) \left(\lim_{s\to s_0} \frac{\|\mathbf{x}(s) - \mathbf{z}(s)\|}{|s - s_0|^2}\right) = 0 \ .$$

Thus, substituting $s(t)$ in for s to parameterize both the curve and its osculating circle in the time parameter, we have the same second order fit here too:

$$\lim_{t\to t_0} \frac{|\mathbf{x}(t) - \mathbf{z}(t)|}{|t - t_0|^2} = 0 \ . \tag{2.1.54}$$

2.1.8 Torsion in torus knots

A *knot* is the graph of a function $\mathbf{x}$ from $\mathbb{R}$ to $\mathbb{R}^3$ with the property that that for some $T > 0$.

$$\mathbf{x}(t + T) = \mathbf{x}(t)$$

for all $t \in \mathbb{R}^3$, and such that if $|t - s| < T$, and $s \neq t$, then $\mathbf{x}(s) \neq \mathbf{x}(t)$.

That is, the curve $\mathbf{x}(t)$ comes back upon itself in time T, but not before. T is called the *period* of the parameterization.

Example 14 (Torus knots)**.** *For each pair of natural numbers* p, q, *define the curve* $\mathbf{x}_{p,q}(t)$ *in* $\mathbb{R}^3$ *by*

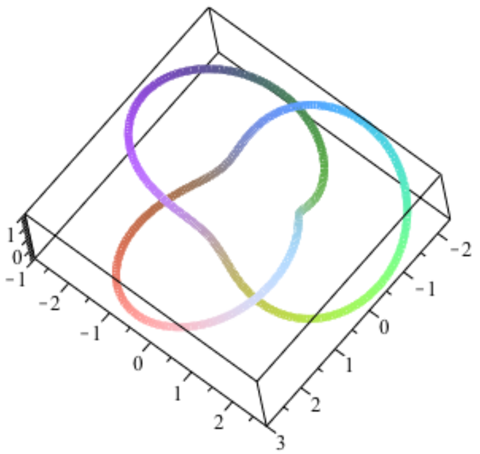
$$\begin{aligned}\mathbf{x}_{p,q}(t) &= \left[\left(2+\cos\left(\frac{q}{p}t\right)\right)\cos(t)\right]\mathbf{e}_1\\ &+ \left[\left(2+\cos\left(\frac{q}{p}t\right)\right)\sin(t)\right]\mathbf{e}_2\\ &+ \left[\sin\left(\frac{q}{p}t\right)\right]\mathbf{e}_3\ .\end{aligned}$$

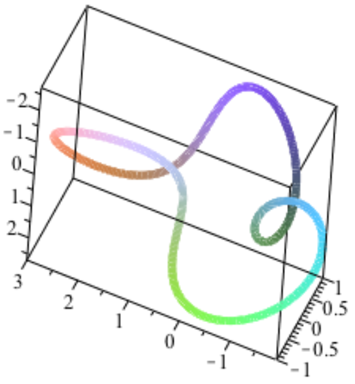
These are called torus knots because if (x, y, z) is a point on the graph of the knot, and we define $r = \sqrt{x^2+y^2}$, we have

$$(r-2)^2 + z^2 = 1\ ,$$

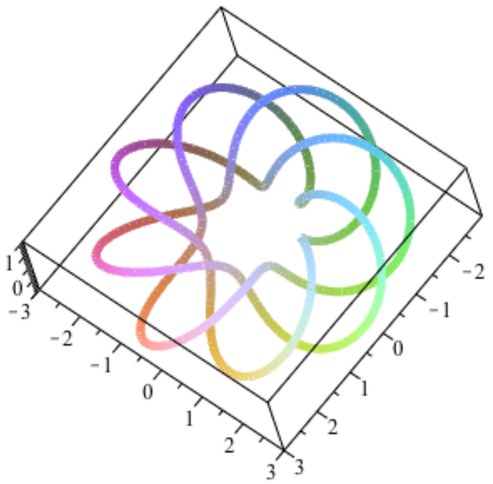
which is the equation of a circle of radius 1 centered at $(2, 0)$ in the r, z plane. Rotating this circle around the z axis yields a torus, and thus all of the points of the torus knots lies on this torus.

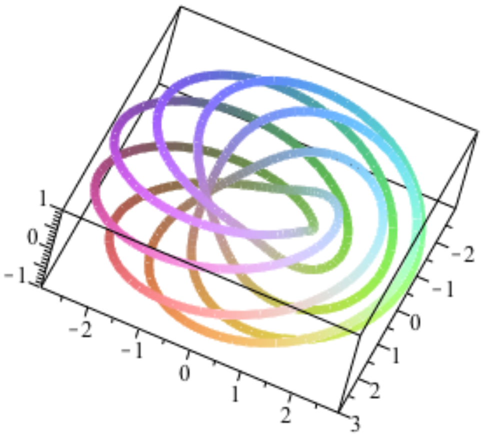
Here is a some plots of torus knots for various values of p and q: Below on the left is the $p = 2$, $q = 3$ torus knot, known as the *trefoil knot*. Below on the right is the $p = 1$, $q = 3$ torus knot, which you can see it is really an *unknot*: It can be continuously deformed into a planar circle without crossing itself. ("Unknot" is the actual technical term for this.)



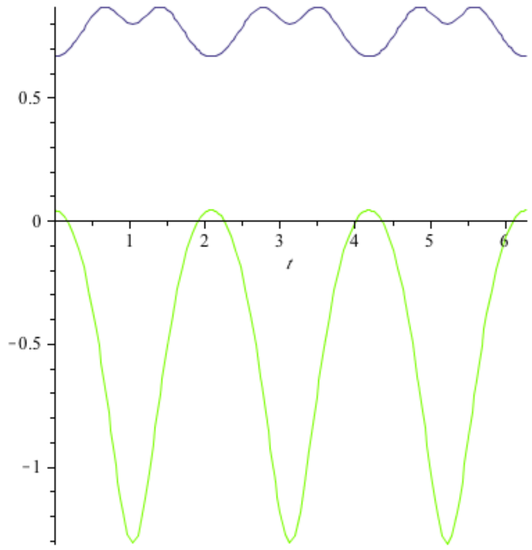


Here are some plots for larger values of p and q: Below on the left is the $p = 3$, $q = 8$ torus knot, and below on the right is the $p = 7$, $q = 6$ torus knot.

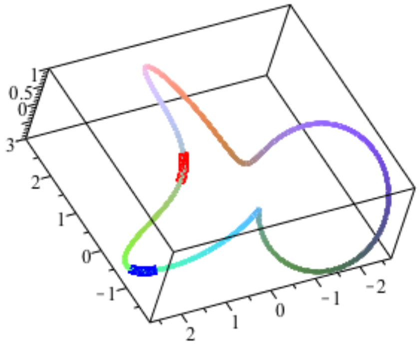




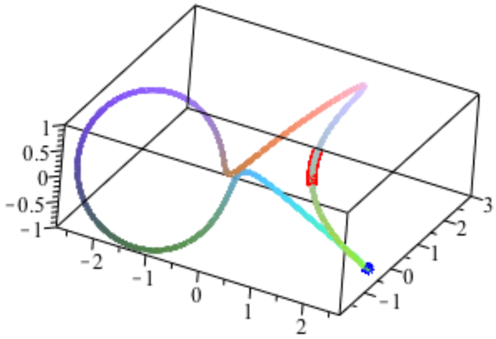
Our goal in this subsection is to use such 3 dimensional plots to develop a visual understanding of what torsion is. Let us go back to the $p = 1$, $q = 3$ torus knot (or unknot), and identify a short segment of the knot along with the torsion is high, and another along which the torsion is low. We use the formulas from Theorems 6 and 7 for this. The computations are possible, but a bit messy to do on paper, and in fact, we have used Maple to do this. The resulting formulas are complicated, and the results are best understood by plotting a graph of the curvature and torsion. Below is a plot of the curvature and torsion for the $p = 1$, $q = 3$ torus knot. The torsion plot is the very oscillatory plot in the lower part, and the curvature is the less oscillatory plot at the top.



As you can see, the torsion takes on its largest *absolute* value somewhere in a small interval about $t = 1$, and is zero somewhere in a small interval around $t = 1.8$. Let us look at those parts of the knot, and see if we can "see" the different values of the torsion there. We will "highlight" the potions of the curve for $t \in [0.9, 1.1]$ and $t \in [1.7, 1.9]$ by plotting them in thick segments. Here is the result:



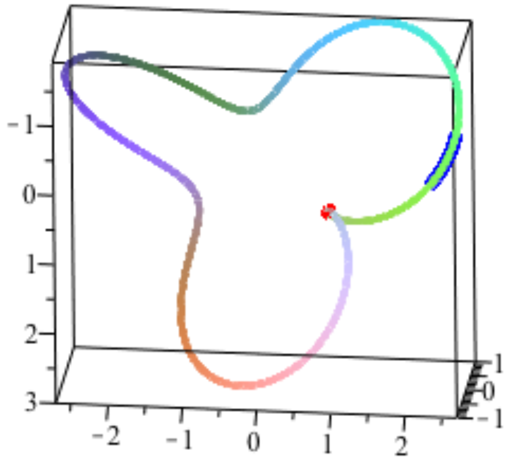
Which of the thick segments is the high torsion segment, and which is the low torsion segment? From this perspective, it is not easy to see the torsion. But if we change our perspective, and look straight along the segment, we can see how much torsion there is. Here is one such view:



The segment on which we are focusing shows up as a dark dot on the lower right portion of the curve. It shows up as a dot since we a looking straight along the segment itself.

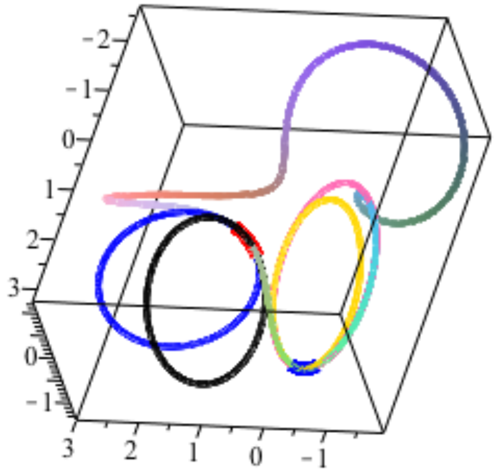
Now notice how the parts of the knot coming into and out of this line segment do so at essentially the same angle in this plot. The plane of motion; i.e., the osculating plane, does not twist very much at all along this short segment. This is low torsion.

Let us look at the other segment, the high torsion segment, from such a perspective:



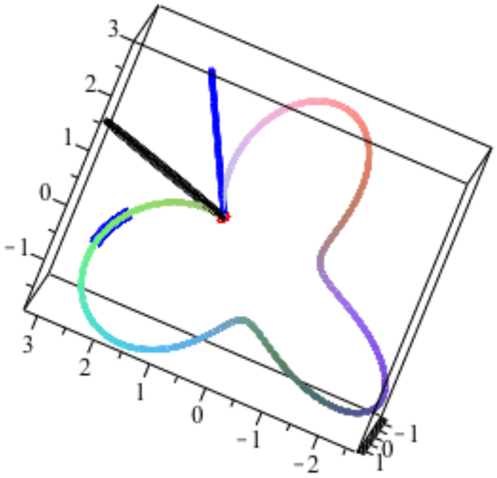
Now the segment in question shows up as a dot at the cusp near the center of the plot. Here you see that the plane of motion has twisted considerably along the short segment; the incoming and outgoing angles of approach are quite different.

We can bring this all out more clearly by plotting the osculating circles at the beginning and ends of these segments. The point of this is the osculating circle lies in the osculating plane, and hence plotting the osculating circle helps us to visualize the osculating plane. Here is a general view showing the knot and the four osculating circles at the beginning and ends of the segments:

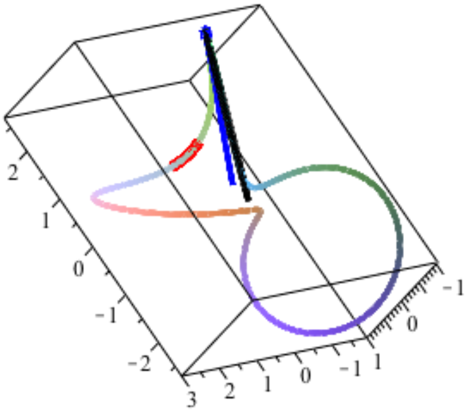


This plot is a bit too busy to see everything clearly, but you can see that the two osculating circles at ends of the segment on the left definitely lie in two quite different planes. Again, this is the result of the high torsion along this segment which has twisted the osculating plane. On the other hand, the other two osculating circles, the ones at the ends of the low torsion segment, are not so readily distinguishable in the plot. They are almost on top of each other due to low torsion, though one does have a noticeably larger radius than the other.

To see this more clearly, here is an "end view" plot of the high torsion segment. The two osculating circles show up in profile, and hence as line segments. But from this view we can clearly see how much of an angle the torsion has opened up between the two osculating planes.



Here is the same thing for the low torsion segment:



So far, our discussion has been qualitative and not quantitative. Let us relate the angles visible in the last two graphs to the magnitude of the torsion along the corresponding segments. Let $\mathbf{B}(t)$ denote the binormal vector at time t. Then for small $h > 0$,

$$\mathbf{B}(t_0 + h) \approx \mathbf{B}(t_0) + h\mathbf{B}'({}_0) = \mathbf{B}(t_0) - hv(t_0)\tau(t_0)\mathbf{N}(t_0)\ . \tag{2.1.55}$$

Let Θ denote the angle between $\mathbf{B}(t_0+h)$ and $\mathbf{B}(t_0)$. Then

$$\sin(\Theta) = \|\mathbf{B}(t_0+h)\times\mathbf{B}(t_0)\| \ .$$

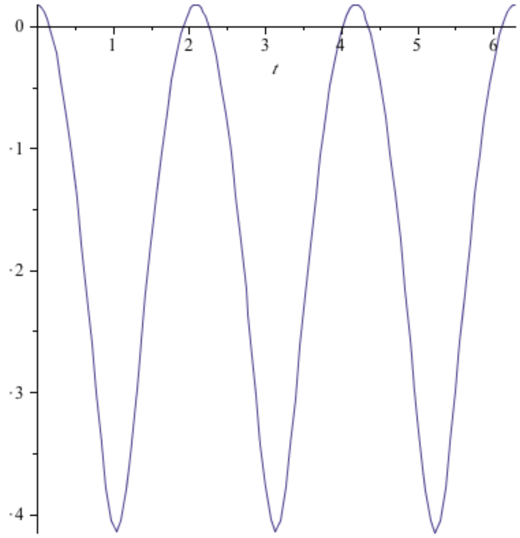
Using the first order approximation (2.1.55), we deduce

$$\sin(\Theta) \approx hv(t_0)|\tau(t_0)| \ , \tag{2.1.56}$$

and thus

$$\Theta \approx \arcsin(hv(t_0)|\tau(t_0)|) \ . \tag{2.1.57}$$

Here is a plot of $v(t)\tau(t)$ over the period of our knot:



Let us take $t_0 = 0.9$, the beginning of our high torsion segment. We see from the graph that the value of $|v(t_0)\tau(t_0)|$ is close to 4.0. We then use $h = 0.2$, time interval corresponding to the length of our segment, and compute

$$\Theta \approx \arcsin(0.2\times 4.0) = \arcsin(0.8) \approx 0.927 \text{ radians} \ .$$

This is about 53 degrees, which a good bit larger than the angle between the two osculating circles in the end view are plotted above; there the angle look to be closer to about 30 degrees. Of course, we are using a large relatively value of h, and should not expect a first order approximation to very accurate with such an h. We leave the similar calculation for the low torsion segment to the reader.

2.2 The prediction of motion

2.2.1 Newton's Second Law

Newton's Second Law states that if $\mathbf{x}(t)$ denote the position at time t of a particle of mass m, and at each time t a force $\mathbf{F}(t)$ is acting on the particle, the acceleration of the particle, $\mathbf{a}(t)$, satisfies

$$\mathbf{a}(t) = \frac{1}{m}\mathbf{F}(t) \ . \tag{2.2.1}$$

For example, a particle of mass m in $\mathbb{R}^3$ that is subject to a constant gravitational field (pointing downwards by convention) is acted upon by the force

$$\mathbf{F} = -mg\big(0, 0, 1\big) \ ,$$

where g is the gravitational constant, and is about 9.8 meters/second^2 on the Earth.

Since $\mathbf{a}(t)$ is the derivative of the velocity $\mathbf{v}(t) = \big(v_1(t), v_2(t), v_3(t)\big)$, this means that

$$\big(v_1'(t), v_2'(t), v_3'(t)\big) = -g\big(0, 0, 1\big) \ .$$

In other words,

$$\begin{aligned} v_1'(t) &= 0 \\ v_2'(t) &= 0 \\ v_3'(t) &= -g \end{aligned}$$

Then, by the Fundamental Theorem of Calculus applied to each of these equations, we deduce that

$$\begin{aligned} v_1(t) &= v_1(0) \\ v_2(t) &= v_2(0) \\ v_3(t) &= v_3(0) - gt \end{aligned}$$

Since $\mathbf{v}(t) = \mathbf{x}'(t)$, we can rewrite this as

$$\begin{aligned} x_1'(t) &= v_1(0) \\ x_2'(t) &= v_2(0) \\ x_3'(t) &= v_3(0) - gt \end{aligned}$$

Once more, by the Fundamental Theorem of Calculus applied to each of these equations, we deduce that

$$\begin{aligned} x_1(t) &= x_1(0) + tv_1(0) \\ x_2(t) &= x_2(0) + tv_2(0) \\ x_3(t) &= x_3(0) + tv_3(0) - \frac{1}{2}gt^2 \ . \end{aligned}$$

Going back to vector notation, we have deduces that the particle's trajectory is given by

$$\mathbf{x}(t) = \mathbf{x}(0) + t\mathbf{v}(0) - \frac{1}{2}gt^2\mathbf{e}_3 \ . \tag{2.2.2}$$

The term *ballistic motion* is used to describe the motion along such a trajectory where some "launching process" imparts the initial velocity, and after that the only force acting is gravity. Once you know that initial position and velocity, you know the whole trajectory, and can determine (or "predict") any aspect of the motion. For example, suppose that the initial position is at the origin; i.e., $\mathbf{x}(0) = \mathbf{0}$, and that $\mathbf{v}(0) = (a, 0, b)$ with $a, b > 0$. This corresponds to "launching" the particle with this initial velocity, upwards and outwards along the $\mathbf{e}_1$ axis. What is the maximum height of the particle along the trajectory? What is the distance from the origin when it "hits the ground"?

Example 15 (Making predictions for ballistic motion)**.** *Let us answer the questions raised just above concerning ballistic motion. To do this, plug our initial position* $\mathbf{0}$ *and initial velocity* $(a, 0, b)$ *into (2.2.2). We find that the height of the particle at time* t, $x_3(t)$, *is given by*

$$x_3(t) = tb - \frac{1}{2}gt^2 \ .$$

Completing the square,

$$x_3(t) = -\frac{g}{2}\left(\frac{b}{g} - t\right)^2 + \frac{1}{2}\frac{b^2}{g} \ . \tag{2.2.3}$$

The maximum height is achieved when the first term is zero, i.e., at $t = b/g$, *at which time the height is* $b^2/(2g)$.

The particle "hits the ground" when $x_3(t) = 0$ *for the second time. From (2.2.3), we see that* $x_3(t) = 0$ *if and only if*

$$\left(\frac{b}{g} - t\right)^2 = \frac{b^2}{g^2} \ .$$

The two solutions are $t = t_0 = 0$ *and* $t = t_1 = 2b/g$*. Since* $t_0 = 0$ *is the launch time,* $t_1 = 2b/g$ *is the time when the particle hits the ground. Evaluating* $\mathbf{x}(t_1)$ *at this time* t*, we find*

$$\mathbf{x}(t_1) = \left(\frac{2ab}{g}, 0, 0\right) \ .$$

Thus, the distance from the origin to where the particle hit the ground is $2ab/g$*.*

Let us go a little further with this, suppose that however you are launching the particle, the maximum launch speed you can achieve is 100 *meters/second. Then, to have the particle hit the ground as far away along the* $\mathbf{e}_1$ *axis as possible, you should use an initial velocity vector* $(a, 0, b)$ *with* $a^2 + b^2 = 100$*, but which one?*

Since $a^2 + b^2 - 2ab = (a - b)^2$*,* $2ab \leq a^2 + b^2$ *with equality if and only if* $a = b$*. Thus to maximize* $2ab$ *under the constraint* $a^2 + b^2 = 100$*, you should use* $a = b = \sqrt{50}$ *meters/second. This corresponds to the particle at an initial angle of* $\pi/4$ *with the horizontal.*

2.2.2 Motion in a constant magnetic field: helical motion

Ballistic motion is particularly simple, but is fundamentally important. Another fundamentally important example concerns the motion of a charged particle in a constant magnetic field $\mathbf{H}$. Then if the particle has charge q and mass m, the Lorentz force law says that force $\mathbf{F}(t)$ due to the magnetic field that acts on the particle at time t is

$$\mathbf{F}(t) = q\mathbf{v}(t) \times \mathbf{H} \ ,$$

and then Newton's Second Law yields

$$\mathbf{v}'(t) = \frac{q}{m}\mathbf{v}(t) \times \mathbf{H} \ . \tag{2.2.4}$$

Notice that the magnetic force is zero if the particle is not moving, or is moving parallel to $\mathbf{H}$. The magnetic force does not change the speed of the particle:

$$\frac{\mathrm{d}}{\mathrm{d}t}v^2(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(t) \cdot \mathbf{v}(t) = 2\mathbf{v}(t) \cdot \mathbf{v}'(t) = 2\frac{q}{m}\mathbf{v}(t) \cdot (\mathbf{v}(t) \times \mathbf{H}) = 0 \ .$$

We shall now solve (2.2.4) to find the particle trajectory. First, let us simplify our notation. Define a vector $\mathbf{b} \in \mathbb{R}^3$ by $\mathbf{b} = -\frac{q}{m}\mathbf{H}$ so that (2.2.4) becomes

$$\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t) \ , \tag{2.2.5}$$

and we assume $\mathbf{b} \neq \mathbf{0}$ to avoid trivialities.

We shall now find all solutions of (2.2.5) that satisfy the initial condition

$$\mathbf{v}(0) = \mathbf{v}_0 \tag{2.2.6}$$

where $\mathbf{v}_0$ is a given initial value of the velocity vector. Together, (2.2.5) and (2.2.6) specify an *initial value problem.*

It is easy to see that there is at most one solution to our initial value problem. Indeed if $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are any two solutions of the initial value problem, define $\mathbf{z}(t) = \mathbf{v}(t) - \mathbf{w}(t)$, and note that

$$\mathbf{z}'(t) = \mathbf{b} \times \mathbf{z}(t) \qquad \text{and} \qquad \mathbf{z}(0) = \mathbf{0}\ .$$

But then we compute, a bit redundantly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{z}(t)\|^2 = 2\mathbf{z}(t) \cdot \mathbf{z}'(t) = 2\mathbf{z}(t) \cdot (\mathbf{b} \times \mathbf{z}(t)) = 0\ , \tag{2.2.7}$$

Therefore, $\|\mathbf{z}(t)\| = \|\mathbf{z}(0)\| = 0$ for all t, and consequently, $\mathbf{v}(t) = \mathbf{w}(t)$ for all t: The two solutions are in fact the same. Hence, if we can find one solution of our initial value problem, it is the only solution there is, and the problem is completely solved.

To find a solution of our initial value problem, we first introduce a right-handed orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ that is adapted to the problem at hand. We build this out of two orthogonal vectors that come along with the problem, namely $\mathbf{b}$ itself, and $(\mathbf{v}_0)_\perp$, the component of $\mathbf{v}_0$ that is orthogonal to $\mathbf{b}$, which is

$$(\mathbf{v}_0)_\perp = \mathbf{v}_0 - \frac{1}{\|\mathbf{b}\|^2}(\mathbf{b} \cdot \mathbf{v}_0)\mathbf{b}\ .$$

Notice that if $(\mathbf{v}_0)_\perp = 0$, then $\mathbf{b} \times \mathbf{v}_0 = 0$, and the initial value problem has the simple solution $\mathbf{v}(t) = \mathbf{v}_0$ for all t. By what we have said above, this is the only solution in this case.

Therefore, without loss of generality, we assume $(\mathbf{v}_0)_\perp \neq 0$, and define

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{b}\|}\mathbf{b} \quad \text{and} \qquad \mathbf{u}_1 = \frac{1}{\|(\mathbf{v}_0)_\perp\|}(\mathbf{v}_0)_\perp\ .$$

(Here we are following a standard convention to take the direction of $\mathbf{b}$ as the third coordinate axis, but this is not essential.) Once $\mathbf{u}_3$ and $\mathbf{u}_1$ are defined, we have only one choice for $\mathbf{u}_2$ that makes $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ a right-handed orthonormal basis, namely

$$\mathbf{u}_2 = \mathbf{u}_3 \times \mathbf{u}_1\ .$$

Notice that with these definitions,

$$\mathbf{v}_0 \cdot \mathbf{u}_2 = 0 \qquad \text{and} \qquad \mathbf{v}_0 \cdot \mathbf{u}_1 \geq 0 \ . \tag{2.2.8}$$

Let us write $\mathbf{v}(t)$ in this basis, introducing real valued functions $y_1(t)$, $y_2(t)$ and $y_3(t)$ so that

$$\mathbf{v}(t) = y_1(t)\mathbf{u}_1 + y_2(t)\mathbf{u}_2 + y_3(t)\mathbf{u}_3 \ . \tag{2.2.9}$$

We then seek functions $y_1(t)$, $y_2(t)$ and $y_3(t)$ so that $\mathbf{v}(t)$, given by (2.2.9) satisfies (2.2.5) and such that $y_j(0) = \mathbf{v}_0 \cdot \mathbf{u}_j$, $j = 1, 2, 3$, for the given initial velocity $\mathbf{v}_0$. Plugging (2.2.9) into (2.2.5) and computing both sides, we find:

$$\begin{aligned}
y_1'(t)\mathbf{u}_1 + y_2'(t)\mathbf{u}_2 + y_3'(t)\mathbf{u}_3 &= y_1(t)\mathbf{b} \times \mathbf{u}_1 + y_2(t)\mathbf{b} \times \mathbf{u}_2 + y_3(t)\mathbf{b} \times \mathbf{u}_3 \\
&= \|\mathbf{b}\| \left(y_1(t)\mathbf{u}_3 \times \mathbf{u}_1 + y_2(t)\mathbf{u}_3 \times \mathbf{u}_2 + y_3(t)\mathbf{u}_3 \times \mathbf{u}_3 \right) \\
&= \|\mathbf{b}\| \left(y_1(t)\mathbf{u}_2 - y_2(t)\mathbf{u}_1 \right)
\end{aligned}$$

Thus we conclude

$$\begin{aligned}
y_1'(t) &= -\|\mathbf{b}\| y_2(t) \\
y_2'(t) &= \quad \|\mathbf{b}\| y_1(t) \\
y_3'(t) &= 0 \ .
\end{aligned} \tag{2.2.10}$$

Theorem 10. *There is one and only one solution to the system of equations (2.2.10) satisfying the initial condition*

$$y_1(0) = \mathbf{v}_0 \cdot \mathbf{u}_1 \qquad y_2(0) = \mathbf{v}_0 \cdot \mathbf{u}_2 = 0 \qquad \text{and} \qquad y_3(0) = \mathbf{v}_0 \cdot \mathbf{u}_3 \ .$$

It is given in explicit form by

$$\begin{aligned}
y_1(t) &= [\mathbf{v}_0 \cdot \mathbf{u}_1] \cos(\|\mathbf{b}\| t) \\
y_2(t) &= [\mathbf{v}_0 \cdot \mathbf{u}_1] \sin(\|\mathbf{b}\| t) \\
y_3(t) &= [\mathbf{v}_0 \cdot \mathbf{u}_3] \ .
\end{aligned} \tag{2.2.11}$$

The theorem says that the curve in $\mathbb{R}^2$ given by $\big(y_1(t), y_2(t)\big)$ satisfies

$$\big(y_1(t), y_2(t)\big) = [\mathbf{v}_0 \cdot \mathbf{u}_1]\big(\cos(\|\mathbf{b}\|t), -\sin(\|\mathbf{b}\|t)\big)\ ,$$

and this is the rotation through the angle $-\|\mathbf{b}\|t$ of the vector $[\mathbf{v}_0 \cdot \mathbf{u}_1]\big(1, 0\big)$. Thus, the motion is very simple when viewed in these coordinates: v_3 does not change, and $\big(v_1, v_2\big)$ simply moves around the circle of radius $|\mathbf{v}_0 \cdot \mathbf{u}_1|$ at constant speed.

Proof of Theorem 10: The last equation in (2.2.10) is easy: It must be the case that $y_3(t)$ is independent of t, and therefore always equal to $\mathbf{v}_0 \cdot \mathbf{u}_3$.

We have already observed that $v^2(t) = (y_1^2(t) + y_2^2(t) + y_3^2(t))$ is constant. Since we have just proved that $y_3^2(t)$ is constant, we know that $y_1^2(t) + y_2^2(t)$ is constant. That is,

$$y_1^2(t) + y_2^2(t) = y_1^2(0) + y_2^2(0) = (\mathbf{v}_0 \cdot \mathbf{u}_1)^2\ .$$

Therefore, at each time t, the vector $\big(y_1(t), y_2(t)\big) \in \mathbb{R}^2$ lies on the circle of radius $|\mathbf{v}_0 \cdot \mathbf{u}_1| = [\mathbf{v}_0 \cdot \mathbf{u}_1]$, and hence for some function $\theta(t)$,

$$\big(y_1(t), y_2(t)\big) = [\mathbf{v}_0 \cdot \mathbf{u}_1]\big(\cos\theta(t), \sin\theta(t)\big)\ .$$

Differentiating, we find $\big(y_1'(t), y_2'(t)\big) = |\mathbf{v}_0 \cdot \mathbf{u}_1|\theta'(t)\big(-\sin\theta(t), \cos\theta(t)\big)$ so that

$$\begin{aligned} y_1'(t) &= -\theta'(t)y_2(t) \\ y_2'(t) &= \quad \theta'(t)y_1(t) \end{aligned} \tag{2.2.12}$$

Comparing with (2.2.10), we see that $\theta'(t) = \|\mathbf{b}\|$, and hence

$$\theta(t) = \|\mathbf{b}\|t + \theta_0\ .$$

From the initial condition, we see that θ_0 must be an integer multiple of 2π, and since each integer multiple of 2π yields the same curve $(y_1(t), y_2(t))$, we may as well take $\theta_0 = 0$. Thus

$$(y_1(t), y_2(t)) = [\mathbf{v}_0 \cdot \mathbf{u}_1](\cos(\|\mathbf{b}\|t), \sin(\|\mathbf{b}\|t))\ .$$

Thus, $y_1(t)$ and $y_2(t)$ respectively must satisfy the first two equations in (2.2.10). $\square$

We remark that we have given two separate proofs of the uniqueness of the solution in the case $(\mathbf{v}_0)_\perp \neq 0$, since we proves the uniqueness of the functions $y_1(t)$, $y_2(t)$ and $y_3(t)$ in Theorem 10 without making use of the argument concerning (2.2.7).

What we have proved about the solution of the equation $\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t)$ suggests calling this equation *the rotation equation.* As we have said, the equation (2.2.5) arises in many contexts beside motion in a magnetic field. Therefore, let us pause to record our conclusions concerning it, before moving on to the position $\mathbf{x}(t)$ of our particle.

Theorem 11 (The rotation equation)**.** *Let* $\mathbf{b}$ *and* $\mathbf{v}_0$ *be given non-zero vectors in* $\mathbb{R}^3$*. Then there is one and only one curve* $\mathbf{v}(t)$ *in* $\mathbb{R}^3$ *satisfying*

$$\mathbf{v}'(t) = \mathbf{b} \times \mathbf{v}(t) \qquad \text{and} \qquad \mathbf{v}(0) = \mathbf{v}_0 \ .$$

If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ *is an orthonormal basis of* $\mathbb{R}^3$ *satisfying (2.2.8), then* $\mathbf{v}(t)$ *is given by*

$$\mathbf{v}(t) = [\mathbf{v}_0 \cdot \mathbf{u}_1] \cos(\|\mathbf{b}\|t)\mathbf{u}_1 + [\mathbf{v}_0 \cdot \mathbf{u}_1] \sin(\|\mathbf{b}\|t)\mathbf{u}_2 + [\mathbf{v}_0 \cdot \mathbf{u}_3]\mathbf{u}_3 \ .$$

Now that we have found $\mathbf{v}(t)$ for motion in the constant magnetic field, we can integrate to find the position. The key is that since for each j,

$$x_j'(t) = v_j(t),$$

the Fundamental Theorem of Calculus applies to yield

$$x_j(t) = x_j(0) + \int_0^t v_j(s)\mathrm{d}s \ .$$

We collect the components, and write this in vector form as:

$$\mathbf{x}(t) = \mathbf{x}(0) + \int_0^t \mathbf{v}(t) \ .$$

Moreover, we can do the integration coordinate by coordinate for *any* orthonormal basis.

Therefore, if our initial position is $\mathbf{x}_0$, so that $\mathbf{x}(0) = \mathbf{x}_0$, using our nice basis, we find

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{x}_0 + \int_0^t [y_1(s)\mathbf{u}_1 + y_2(s)\mathbf{u}_2 + y_3(s)\mathbf{u}_3]\mathrm{d}s \\
&= \mathbf{x}_0 + \left(\int_0^t y_1(s)\mathrm{d}s\right)\mathbf{u}_1 + \left(\int_0^t y_2(s)\mathrm{d}s\right)\mathbf{u}_2 + \left(\int_0^t y_3(s)\mathrm{d}s\right)\mathbf{u}_3 \ .
\end{aligned}$$

Doing the integrals we find

$$\begin{aligned}\int_0^t y_1(s)\mathrm{d}s &= -\frac{\mathbf{v}_0\cdot\mathbf{u}_1}{\|\mathbf{b}\|}\sin(\|\mathbf{b}\|t)\\ \int_0^t y_2(s)\mathrm{d}s &= \frac{\mathbf{v}_0\cdot\mathbf{u}_1}{\|\mathbf{b}\|}(1-\cos(\|\mathbf{b}\|t))\\ \int_0^t y_3(s)\mathrm{d}s &= [\mathbf{v}_0\cdot\mathbf{u}_3]t\ .\end{aligned} \tag{2.2.13}$$

Putting it altogether, we have that

$$\begin{aligned}\mathbf{x}(t) &= (\mathbf{x}_0\cdot\mathbf{u}_1)\mathbf{u}_1 - \frac{\mathbf{v}_0\cdot\mathbf{u}_1}{\|\mathbf{b}\|}\sin(\|\mathbf{b}\|t)\mathbf{u}_1\\ &= (\mathbf{x}_0\cdot\mathbf{u}_2)\mathbf{u}_2 - \frac{\mathbf{v}_0\cdot\mathbf{u}_1}{\|\mathbf{b}\|}(1-\cos(\|\mathbf{b}\|t))\mathbf{u}_2\\ &= (\mathbf{x}_0\cdot\mathbf{u}_3)\mathbf{u}_3 + t(\mathbf{v}_0\cdot\mathbf{u}_3)\mathbf{u}_3\ .\end{aligned} \tag{2.2.14}$$

The this trajectory is a helix: The $\mathbf{u}_3$ component increases at constant speed, while the the other components rotate around the $\mathbf{u}_3$ axis at a steady rate.

We close this subsection with an application of Theorem 11 to the Frenet-Seret formulae written in terms of the Darboux vector $\boldsymbol{\omega}$. Recall from (2.1.46) that

$$\mathbf{T}'(t) = \boldsymbol{\omega}(t)\times\mathbf{T}(t)$$

and similarly for $\mathbf{N}(t)$ and $\mathbf{B}(t)$. This means that instantaneously at any time t_0. The rates of change of $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are the same as if they were rotation round the axis through $\boldsymbol{\omega}(t_0)$ with angular speed $\|\boldsymbol{\omega}(t_0)\|$. That is, the Darboux vector $\boldsymbol{\omega}(t_0)$ describes the infinitesimal rotation that carries

$$\{\mathbf{T}(t_0),\mathbf{N}(t_0),\mathbf{B}(t_0)\}$$

into

$$\{\mathbf{T}(t_0+\mathrm{d}t),\mathbf{N}(t_0+\mathrm{d}t),\mathbf{B}(t_0+\mathrm{d}t)\}\ .$$

2.2.3 Planetary motion

Consider a planet of mass m orbiting a star of mass M. According to Newton's Universal Theory of Gravitation, the force that attracts the planet and the star has magnitude

$$\frac{GMm}{r^2}$$

where G is the *gravitational constant* and r is the distance between the centers of the star and the planet.

The center of mass will stay fixed, and since the star is generally much, much more massive than the planet, it is an excellent approximation to regard the star as fixed. We take its position to be the origin of our coordinate system, and denote the position of the planet at time t by $\mathbf{x}(t)$. Since the gravitational force acting on the planet is directed towards the star, and hence towards $\mathbf{0}$, Newton's second law tells us that the motion of the planet satisfies

$$\mathbf{x}''(t) = -\frac{GM}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t)\ . \tag{2.2.15}$$

We are going to determine the orbit of the planet, which is to say, the path traced out by $\mathbf{x}(t)$. The key to this, as in our study of motion in a constant magnetic field, is to find *constants of the motion*. The crucial initial observation for us there was that in a constant magnetic field $\mathbf{H}$, $v(t)$ is independent of t, and we then found that $\mathbf{H}\cdot\mathbf{v}(t)$ was independent of time. This gave us the plane in which $\mathbf{v}(t)$ was then found to rotate.

The key to easily solving (2.2.15) is to find constants of the motion. There are two *vector-valued* constants of the motion for (2.2.15):

Definition 12 (Momentum, angular momentum and the Runge-Lenz vector)**.** *Let* $\mathbf{x}(t)$ *be a twice differentiable curve in* $\mathbb{R}^3$ *representing the motion of a planet of mass* m *orbiting a star of mass* M*. Then the momentum* $\mathbf{p}(t)$*, the* angular momentum $\mathbf{L}(t)$ *and the* Runge-Lenz vector $\mathbf{A}(t)$ *of the planet are given by*

$$\mathbf{p}(t) = m\mathbf{x}'(t) = m\mathbf{v}(t), \tag{2.2.16}$$

$$\mathbf{L}(t) = \mathbf{x}(t) \times \mathbf{p}(t)\ . \tag{2.2.17}$$

and

$$\mathbf{A}(t) = \mathbf{p}(t) \times \mathbf{L}(t) - GMm^2\frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|}\ . \tag{2.2.18}$$

The momentum is not a constant of the motion for this system: computing the derivative, we find

$$\mathbf{p}'(t) = m\mathbf{x}''(t) = -\frac{GMm}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t)\ . \tag{2.2.19}$$

But notice that $\mathbf{p}'(t)$ is a multiple of $\mathbf{x}(t)$. Because of this and the fact that $\mathbf{p}(t)$ is a multiple of $\mathbf{x}'(t)$,

$$\mathbf{L}'(t) = \mathbf{x}'(t) \times \mathbf{p}(t) + \mathbf{x}(t) \times \mathbf{p}'(t) = \mathbf{0}\ .$$

This shows that $\mathbf{L}$ is a constant of the motion, and in fact, since it is vector valued, it provides us three scalar constants of the motion.

Next, let us compute $\mathbf{A}'(t)$ in two steps. First, since $\mathbf{L}'(t) = \mathbf{0}$,

$$\begin{aligned} (\mathbf{p}(t) \times \mathbf{L}(t))' &= \mathbf{p}'(t) \times \mathbf{L}(t) \\ &= -\frac{GMm}{\|\mathbf{x}(t)\|^3}\mathbf{x}(t) \times (\mathbf{x}(t) \times \mathbf{p}(t)) \ . \end{aligned} \tag{2.2.20}$$

Using the triple cross product identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{z})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{z}$, we find

$$\mathbf{x} \times (\mathbf{x} \times \mathbf{p}) = (\mathbf{x} \cdot \mathbf{p})\mathbf{x} - \|\mathbf{x}\|^2\mathbf{p} \ .$$

Therefore,

$$(\mathbf{p}(t) \times \mathbf{L}(t))' = -\frac{GMm}{\|\mathbf{x}(t)\|^3}(\mathbf{x}(t) \cdot \mathbf{p}(t))\mathbf{x}(t) + \frac{GMm}{\|\mathbf{x}(t)\|}\mathbf{p}(t) \ . \tag{2.2.21}$$

Second, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\|\mathbf{x}(t)\|}\mathbf{x}(t)\right) = -\frac{1}{\|\mathbf{x}(t)\|^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}(t)\|\right)\mathbf{x}(t) + \frac{1}{\|\mathbf{x}(t)\|}\mathbf{x}'(t)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{x}(t)\| = \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\mathbf{x}(t) \cdot \mathbf{x}(t)} = \frac{1}{\sqrt{\mathbf{x}(t) \cdot \mathbf{x}(t)}}\mathbf{x}(t) \cdot \mathbf{x}'(y) = \frac{1}{\|\mathbf{x}(t)\|}\mathbf{x}(t) \cdot \mathbf{x}'(y) \ .$$

Thus, altogether,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{GMm^2}{\|\mathbf{x}(t)\|}\mathbf{x}(t)\right) = -\frac{GMm}{\|\mathbf{x}(t)\|^3}(\mathbf{x}(t) \cdot \mathbf{p}(t))\mathbf{x}(t) + \frac{GMm}{\|\mathbf{x}(t)\|}\mathbf{p}(t) \ . \tag{2.2.22}$$

Combining (2.2.21), (2.2.22) and the definition of the Runge-Lenze vector $\mathbf{A}$, we see that $\mathbf{A}'(t) = \mathbf{0}$. Summarizing our conclusions, we have proved:

Theorem 12 (Constants of the motion for planetary orbits)**.** *Let* $\mathbf{x}(t)$ *be a solution of (2.2.15). Then the angular momentum vector* $\mathbf{L}(t)$ *and the Runge-Lenz vector* $\mathbf{A}(t)$ *are both constants of the motion:*

$$\mathbf{L}(t) = \mathbf{L}(0) = \mathbf{L} \qquad \text{and} \qquad \mathbf{A}(t) = \mathbf{A}(0) = \mathbf{A}$$

for all t*.*

We are now ready to solve for the orbits. Consider any orbit with given A and L. Notice that $L = 0$ if and only if the motion of the planet is straight towards or away from the star, and of course this does not describe an orbit. Therefore, to avoid trivialities, let us suppose that $L \neq 0$. Since by definition, the angular momentum is orthogonal to $\mathbf{x}(t)$, the orbit lies in the plane given by the equation

$$\mathbf{L} \cdot \mathbf{x} = 0 \ .$$

This plane is called the *orbital plane.* (It is, in fact, the osculating plane to the orbit at each time t.)

Next, suppose that $\mathbf{A} = 0$. Then $\mathbf{x}(t) \cdot \mathbf{A} = 0$ for all t, and by then the definition of $\mathbf{A}(t)$,

$$\mathbf{x}(t) \cdot \mathbf{A}(t) = \mathbf{x}(t) \cdot \mathbf{p}(t) \times \mathbf{L}(t) - GMm^2 \|\mathbf{x}(t)\| \ , \tag{2.2.23}$$

and so $\mathbf{x}(t) \cdot \mathbf{p}(t) \times \mathbf{L}(t) = GMm^2\|\mathbf{x}(t)\|$. Using the triple product identity

$$\mathbf{x}(t) \cdot \mathbf{p}(t) \times \mathbf{L}(t) = \mathbf{L}(t) \cdot (\mathbf{x}(t) \times \mathbf{p}(t)) = \mathbf{L}(t) \cdot \mathbf{L}(t) = \|\mathbf{L}\|^2 \ . \tag{2.2.24}$$

Thus, (2.2.23) reduces to

$$\|\mathbf{x}(t)\| = \frac{\|\mathbf{L}\|^2}{GMm^2} \ .$$

This means that $\mathbf{x}(t)$ traces out a circle around the star in the orbital plane, and with the radius

$$R = \frac{\|\mathbf{L}\|^2}{GMm^2} \ . \tag{2.2.25}$$

Since the orbit is circular, the velocity $\mathbf{x}'(t)$ is tangent to the circle and therefore orthogonal to $\mathbf{x}(t)$. It follows that, $\|\mathbf{L}\| = m\|\mathbf{x}(t)\|\|\mathbf{x}'(t)\| = mR\|\mathbf{x}'(t)\|$ and hence the speed $v = \|\mathbf{x}'(t)\|$ is constant and given by

$$v = \frac{\|\mathbf{L}\|}{mR} \ . \tag{2.2.26}$$

Thus, when $\mathbf{A} = 0$, the motion is constant speed circular motion in the orbital plane. We can get a relation between v and R by eliminating $\|\mathbf{L}\|$ between (2.2.25) and (2.2.26)

$$GMm^2R = \|\mathbf{L}\|^2 = v^2m^2R^2 \ ,$$

and so

$$v = \sqrt{\frac{GM}{R}} \ . \tag{2.2.27}$$

This is usually expressed in therms of the *period* of the orbit, since that is what one can most easily determine directly by astronomical observation. The period of the orbit, traditionally denoted T, is the time it takes the planet to complete one circular orbit. This is given by

$$T = \frac{2\pi R}{v} \ . \tag{2.2.28}$$

Then from (2.2.28) and (2.2.27) we obtain

$$T^2 = \left(\frac{4\pi}{GM}\right) R^3 \ , \tag{2.2.29}$$

which is the usual form of Kepler's Third Law in the case of circular orbits, with one very important contribution by Newton: Kepler's law stated that the square of the period was proportional to the cube of the radius, but did not provide a formula for the proportionality constant. Newton's derivation does, and moreover, the constant G, giving the strength of the gravitational force, can be measured on the Earth. Then, making astronomical observations from the Earth, one can determine the radii and the periods of the orbits of the other planets circling the Sun. Using this data in (2.2.29), we can calculate M, the mass of the Sun.

- *In this way, one can "weight" the Sun with a telescope, since the orbits of the planets are nearly circular.*

The estimates of the value of the Sun computed in this way would vary depending on which planet was used in part because the orbits are not exactly circular. To get a better treatment, let us now consider the case $\mathbf{A} \neq \mathbf{0}$.

From (2.2.23) and (2.2.24), we have

$$\mathbf{x}(t) \cdot \mathbf{A} = \|\mathbf{L}\|^2 - GMm^2\|\mathbf{x}(t)\| \ . \tag{2.2.30}$$

Identifying the orbital plane with $\mathbb{R}^2$ in such a way that $\mathbf{A}$ points in the $\mathbf{e}_1$ direction, and and writing $\mathbf{x}(t) = (x(t), y(t))$, we can rewrite (2.2.30) as

$$\sqrt{x^2(t) + y^2(t)} = \frac{\|\mathbf{L}\|^2 - \|\mathbf{A}\|x(t)}{GMm^2} \ ,$$

which, after some simple algebra, lead to the conclusion that $\big(x(t), y(t)\big)$ satisfies

$$(G^2M^2m^4 - \|\mathbf{A}\|^2)x^2 + G^2M^2m^4y^2 + 2\|\mathbf{L}\|^2\|\mathbf{A}\|x = \|\mathbf{L}\|^2$$

for all t. This is the equation of a conic section. It gives a closed orbit if and only if the coefficient of x^2 is strictly positive; i.e., if and only if $\|\mathbf{A}\|^2 < G^2M^2m^4$. In this case, the orbit will be an ellipse. Kepler's Second Law states that the obits of the planets are ellipses, and we have now explained how this may be derived from Newton's Universal Theory of Gravitation. Further results on planetary motion will be treated in the exercises.

2.3 Exercises

1 Let $\mathbf{x}(t) = (t+1\ ,\ t^2)$. This is a parameterization of the parabola $y = (x-1)^2$.

(a) Compute $\mathbf{v}(t) = \mathbf{x}'(t)$ and $\mathbf{a}(t) = \mathbf{x}''(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = 1$.

2 Let $\mathbf{x}(t) = \big(t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t\big)$ for $t > 0$.

(a) Compute $\mathbf{v}(t) = \mathbf{x}'(t)$ and $\mathbf{a}(t) = \mathbf{x}''(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = 1$.

3 Let $\mathbf{x}(t)$ and $\mathbf{y}(t)$ be two differentiable curves in $\mathbb{R}^3$. Show that

$$(\mathbf{x}(t) \times \mathbf{y}(t))' = \mathbf{x}'(t) \times \mathbf{y}(t) + \mathbf{x}(t) \times \mathbf{y}'(t)\ .$$

4 Let $\mathbf{x}(t) = \big(\cos(t)\ , \sin(t)\ , t/\pi\big)$ where $r > 0$. The curve $\mathbf{x}(t)$ is a helix in $\mathbb{R}^3$.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$, as well as $\mathbf{N}(t)$ and $\mathbf{B}(t)$.

(d) Compute the Darboux vector $\boldsymbol{\omega}(t)$.

(e) Find the tangent line to this curve at $t = \pi/4$, and the equation of the osculating plane to the curve at $t = \pi/2$. Find the intersection of this line and plane.

5 Consider the ellipse in $\mathbb{R}^2$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a, b > 0$.

(a) Show that the path traced out by the parameterized curve $\mathbf{x}(t) = \big(a\cos(t), b\sin(t)\big)$ is this ellipse. In other words, $\mathbf{x}(t) = \big(a\cos(t), b\sin(t)\big)$ is a parameterization of this ellipse.

(b) Compute the curvature $\kappa(t)$, and find the minimum and maximum values of curvature on the ellipse, and the places where the curvature takes on these values.

6 Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = \big(t\ ,\ \sqrt{2}\ln(t)\ ,\ 1/t\big)$ for $t > 0$.

(a) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(3)$.

(b) Find the arc length along the curve from $\mathbf{x}(1)$ to $\mathbf{x}(t)$ as a function of t.

(c) Find the arc length parameterization $\mathbf{x}(s)$ os this curve.

7 Find the arc length along the parabola $y = (x-1)^2$ from the point $(0,1)$ to the point $(1,0)$. (See Excercise 1.)

8 Find the arc length parameterization of the curve given by $\mathbf{x}(t) = (t^{-2}\ ,\ 4/\sqrt{t}\ ,\ t)$ for $t > 0$. (See Excercise 2.) What is the arc length along the segment of the curve joining $\mathbf{x}(1)$ and $\mathbf{x}(4)$?

9 Let $\mathbf{b} = (2,1,2)$. Let $\mathbf{x}(t)$ be the curve given satisfying the initial value problem

$$\mathbf{x}'(t) = \mathbf{b} \times \mathbf{x}(t) \qquad \text{and} \qquad \mathbf{x}(0) = (1,1,1)\ .$$

(a) Compute $\mathbf{x}(\pi)$ and ind the arc length along the curve from $\mathbf{x}(0)$ to $\mathbf{x}(\pi)$.

(b) Compute the curvature and torsion for this curve as a function of t.

10 Consider any thrice differentiable curve in $\mathbb{R}^3$ along which the curvature and torsion are both constant. In previous exercises we have seen that helices have this property. The point of this exercise is to show that *only* helices have this property.

(a) Show that when the curvature and torsion are both constant, so is the Darboux vector. Let $\boldsymbol{\omega}$ denote its constant value.

(b) Let $\mathbf{x}(s)$ denote the arc length parameterization of the curve so that $\mathbf{T}(s) = \mathbf{x}'(s)$. Show that

$$\mathbf{x}''(s) = \boldsymbol{\omega} \times \mathbf{x}'(s)\ .$$

(c) Show that $x(s)$ is a helix.