Practice Final Exam, Math 291 Spring 2010

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Material from Part 1 of the Course:

1: **(a)** Let $\mathbf{a}$ and $\mathbf{b}$ be orthogonal vectors. Define a sequence of vectors $\{\mathbf{b}_n\}$ by

$$\mathbf{b}_{n+1} = \mathbf{a} \times \mathbf{b}_n \qquad \text{and} \qquad \mathbf{b}_0 = \mathbf{b}\ .$$

Show that for all positive integers m

$$\mathbf{b}_{2m+2} = (-1)^m \|\mathbf{a}\|^{2m}\mathbf{b}\ .$$

How do you have to adjust the formula if the hypothesis that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal is dropped?

(b) Let $\mathbf{a} = \dfrac{1}{3}(2,-1,2)$ and $\mathbf{b} = (1,1,1)$. With $\mathbf{b}_n$ defined as in part **(a)**, compute $\mathbf{b}_{99}$.

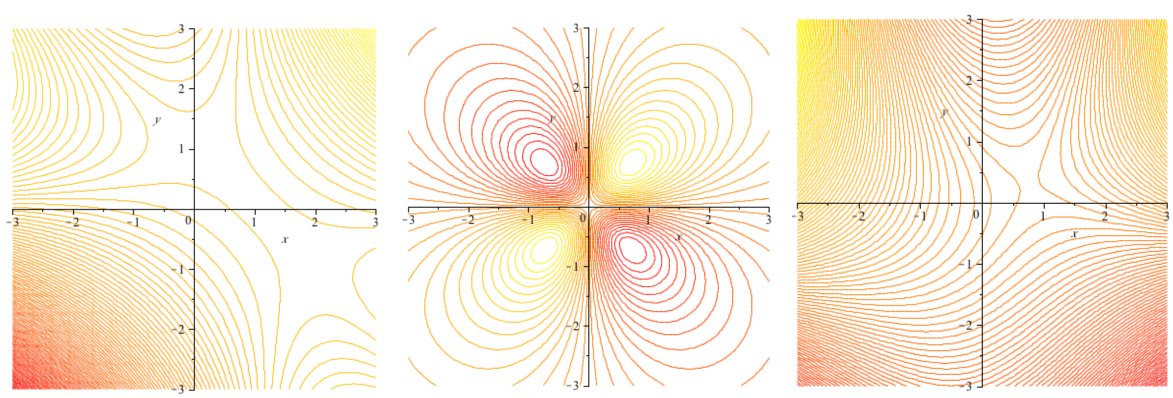
2: Let $f(x,y) = \dfrac{xy}{(1+x^2+y^2)^2}$.

(a) Find all of the critical points of f, and find the value of f at each of the critical points.

(b) Does f have a maximum value? Explain why or why not. If it does, find all points at which the value of f is maximal; i.e, find all maximizers.

(c) Does f have a minimum value? Explain why or why not. If it does, find all points at which the value of f is minimal; i.e, find all minimizers.

(d) One of the following is a contour plot for f. Which one is it? Explain your answer to receive credit.



3: Let $\mathbf{x}(t)$ be the curve given by $\mathbf{x}(t) = (\cos(t), \sin(t), t)$.

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$, and find an arc-length parameterization of this curve

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at time $t = 0$

4: Let $f(x, y) = x^2 y + (x - 1)(y - 1)^2$.

(a) Find the equations for the tangent planes to the graph of $z = f(x, y)$ at $\mathbf{x}_1 = (2, -1)$ and at $\mathbf{x}_2 = (-1, 1)$.

(b) Parameterize the line that is the intersection of the two planes found in **(a)**.

(c) Let $\mathbf{x}_0 = (-3, 0, 5)$. Compute the distance from $\mathbf{x}_0$ to the line found in **(b)**, and from $\mathbf{x}_0$ to the second tangent plane found in **(a)**. The first distance should be smaller than the second. Explain why.

Material from Part 2 of the Course:

5: Let $f(x, y) = 4x^2 + y^2 + 4xy - (x - 1)^4 - (y - 1)^4 - 4x - 4y$.

(a) The point $(0, 0)$ is a critical point of f. Compute the Hessian of f at this point, and determine whether it is a local minimum, a local maximum, a saddle point, or if it cannot be classified through a computation of the Hessian.

(b) Let $\mathbf{u} = (u, v)$ be a unit vector, and consider the directional second derivative

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(tu, tv)\right|_{t=0} .$$

Which choice of the unit vector (u, v) makes this as large as possible? What is the largest possible value? Also, which choice of the unit vector (u, v) makes this as small as possible, and what is the smallest possible value?

(c) Sketch a contour plot of f near $(0, 0)$.

6: Let $f(x, y) = \dfrac{xy}{(1 + x^2 + y^2)^2}$, as in problem **2.** Find the minimum and maximum values of f in the set where

$$|x| + |y| \leq 1 .$$

Also, find all of the minimizers and maximizers.

7: (a) Let Ω be the set in the positive quadrant of $\mathbb{R}^2$ that bounded by

$$\begin{aligned} y &= x \\ y &= \sqrt{3}x \\ y &= x^2 + y^2 \end{aligned}$$

Let $f(x, y) = \sqrt{1 + x^2 + y^2}$. Compute $\int_\Omega f(x, y)\mathrm{d}A$.

(b) Let Ω be the set in $\mathbb{R}^2$ that is given by

$$1 \leq \frac{y}{x^2} \leq 2 \qquad \text{and} \qquad 1 \leq \frac{x}{y^2} \leq 2 .$$

Let $f(x,y) = \dfrac{1}{x^2y^2}$. Compute $\int_\Omega f(x,y)\mathrm{d}A$.

8: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2+y^2+z^2=4$, and above the graph of $z = 1/\sqrt{x^2+y^2}$. Compute the volume of $\mathcal{V}$ **and** the total surface area of its boundary. (There are two pieces to the boundary.)

Material from Part 3 of the Course:

9: Consider the two vector fields

$$\mathbf{F} = (y+z^2, x+z^2, 2zx+2zy) \qquad \text{and} \qquad \mathbf{G} = (y+z^2, x+z^2, 2x+2y)\ .$$

(a) Compute the divergence and curl of **F** and **G**.

(b) Let $\mathcal{S}$ be the unit sphere, and **N** its outward normal. Compute **either**

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \qquad \textbf{or} \qquad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S\ .$$

The choice is yours. Do whichever one you find easier, and justify your answer to receive credit.

(c) One of the vector fields **F** and **G** is equal to $\nabla\varphi$ for some potential function φ. Which one is it? Find such a potential function.

(d) Let C be the curve that is given by

$$x^2+y^2+y^2=4 \qquad \text{and} \qquad x+y+z=1\ .$$

Orient C so that it is traversed in the counter-clockwise direction when viewed from above. Compute **either**

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s \qquad \textbf{or} \qquad \int_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s\ .$$

The choice is yours. Do whichever one you find easier, and justify your answer to receive credit.

10: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2+y^2+z^2=4$, and above the graph of $z = 1/\sqrt{x^2+y^2}$, as in problem **8**. Let $\mathbf{F} = (y+z^2, x+z^2, 2z(x+y))$ and let **N** be the outward normal to $\mathcal{S}$, the boundary of $\mathcal{V}$. Compute the total flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S\ .$$

11: Let C be the contour that runs from $(1,0,0)$ to $(0,1,0)$, and from there to $(0,0,1)$, and from there back to $(1,0,0)$. Let $\mathbf{G} = (y+z^2, x+z^2, 2x+2y)$. Compute the total circulation

$$\oint_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s\ .$$