Practice Final Exam Answers , Math 291 Spring 2010

Eric A. Carlen[1]
Rutgers University

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These are answers, with few steps indicated, for the problems that were not solved in class during the in-class review.

8: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2 + y^2 + z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2 + y^2}$. Compute the volume of $\mathcal{V}$ **and** the total surface area of its boundary. (There are two pieces to the boundary.)

SOLUTION The boundary equations are simple in cylindrical coordinates. From $r^2 + z^2 = 4$ and $z = 1/r$, we deduce

$$(r^2)^2 - 4(r^2) = -1 \ .$$

This quadratic equation for r^2 has the roots $r^2 = 2 \pm \sqrt{3}$. Hence the two bounding surfaces intersect at

$$r = \sqrt{2 - \sqrt{3}} \qquad \text{and} \qquad r = \sqrt{2 + \sqrt{3}} \ .$$

The limits of integration are thus

$$\begin{aligned} \sqrt{2 - \sqrt{3}} \ &\leq \ r \ \leq \ \sqrt{2 + \sqrt{3}} \\ 0 \ &\leq \ \theta \ \leq \ 2\pi \\ 1/r \ &\leq \ z \ \leq \ \sqrt{4 - r^2} \end{aligned}$$

Hence, the integral for the volume is

$$\begin{aligned} \int_{\sqrt{2-\sqrt{3}}}^{\sqrt{2+\sqrt{3}}} \left(\int_{1/r}^{\sqrt{4-r^2}} \left(\int_0^{2\pi} 1 \mathrm{d}\theta \right) \mathrm{d}z \right) r\mathrm{d}r \ &= \ 2\pi \int_{\sqrt{2-\sqrt{3}}}^{\sqrt{2+\sqrt{3}}} \left(\sqrt{4 - r^2} - 1/r \right) r\mathrm{d}r \\ &= \ \frac{4}{3}\pi\sqrt{2} \ . \end{aligned}$$

Material from Part 3 of the Course:

9: Consider the two vector fields

$$\mathbf{F} = (y + z^2, x + z^2, 2zx + 2zy) \qquad \text{and} \qquad \mathbf{G} = (y + z^2, x + z^2, 2x + 2y) \ .$$

(a) Compute the divergence and curl of $\mathbf{F}$ and $\mathbf{G}$.

SOLUTION We compute

$$\mathrm{curl}(\mathbf{F}) = \mathbf{0} \qquad \text{and} \qquad \mathrm{div}(\mathbf{F}) = 2(x+y)$$

and

$$\mathrm{curl}(\mathbf{G}) = (2-2z, 2z-2, 0) \qquad \text{and} \qquad \mathrm{div}(\mathbf{G}) = 0 \ .$$

(b) Let $\mathcal{S}$ be the unit sphere, and $\mathbf{N}$ its outward normal. Compute **either**

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \qquad \textbf{or} \qquad \int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S \ .$$

The choice is yours. Do whichever one you find easier, and justify your answer to receive credit.

SOLUTION This is a flux integral. Since the divergence of $\mathbf{G}$ is zero, the Divergence Theorem tells us that with $\mathcal{V}$ being the unit ball,

$$\int_{\mathcal{S}} \mathbf{G}\cdot\mathbf{N}\mathrm{d}S = \int_{\mathcal{V}} \mathrm{div}(\mathbf{G})\mathrm{d}V = 0 \ .$$

(c) One of the vector fields $\mathbf{F}$ and $\mathbf{G}$ is equal to $\nabla\varphi$ for some potential function φ. Which one is it? Find such a potential function.

SOLUTION A vector field on $\mathbb{R}^3$ is a gradient if and only if its curl is zero at every point in $\mathbb{R}^3$. Hence $\mathbf{F}$ is a gradient. Applying the usual procedure, we find

$$\mathbf{F}(x,y,z) = \nabla\varphi(x,y,z)$$

where

$$\varphi(x,y,z) = xy + z^2(x+y) \ .$$

(d) Let C be the curve that is given by

$$x^2+y^2+y^2 = 4 \qquad \text{and} \qquad x+y+z = 1 \ .$$

Orient C so that it is traversed in the counter-clockwise direction when viewed from above. Compute **either**

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s \qquad \textbf{or} \qquad \int_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s \ .$$

The choice is yours. Do whichever one you find easier, and justify your answer to receive credit.

SOLUTION The curve C is a closed curve, and $\mathbf{F}$ is a gradient vector field. Hence

$$\int_C \mathbf{F}\cdot\mathbf{T}\mathrm{d}s = 0 \ .$$

10: Let $\mathcal{V}$ be the region in $\mathbb{R}^3$ that lies inside the sphere $x^2+y^2+z^2 = 4$, and above the graph of $z = 1/\sqrt{x^2+y^2}$, as in problem **8**. Let $\mathbf{F} = (y+z^2, x+z^2, 2z(x+y))$ and let $\mathbf{N}$ be the outward normal to $\mathcal{S}$, the boundary of $\mathcal{V}$. Compute the total flux

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S \ .$$

SOLUTION As we have computed above, $\mathrm{div}(\mathbf{F}) = 2(x+y)$. By the Divergence Theorem, and by the computations of limits in problem **8**,

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \int_{\sqrt{2-\sqrt{3}}}^{\sqrt{2+\sqrt{3}}} \left(\int_{1/r}^{\sqrt{4-r^2}} \left(\int_0^{2\pi} (r\cos(\theta) + r\sin(\theta))\mathrm{d}\theta \right) \mathrm{d}z \right) r\mathrm{d}r \ .$$

Since

$$\int_0^{2\pi} \cos(\theta)\mathrm{d}\theta = \int_0^{2\pi} \sin(\theta)\mathrm{d}\theta = 0 \ ,$$

$$\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{N}\mathrm{d}S = 0 \ .$$

11: Let C be the contour that runs from $(1,0,0)$ to $(0,1,0)$, and from there to $(0,0,1)$, and from there back to $(1,0,0)$. Let $\mathbf{G} = (y+z^2, x+z^2, 2x+2y)$. Compute the total circulation

$$\oint_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s \ .$$

SOLUTION When asked to compute a work integral, unless the answer is totally obvious, the first step is to compute the curl of the vector field. Above we have found

$$\mathrm{curl}(\mathbf{G}) = (2-2z, 2z-2, 0) \ .$$

This is pretty simple, so it will be good to use Stoke's Theorem, which says,

$$\oint_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s = \int_{\mathcal{S}} \mathrm{curl}(\mathbf{G})\cdot\mathbf{N}\mathrm{d}S \ ,$$

where $\mathcal{S}$ is the triangle with the specified vertices.

The triangle $\mathcal{S}$ lies in the plane given by $x+y+z=1$, and for this plane the unit normal is

$$\mathbf{N} = \pm\frac{1}{\sqrt{3}}(1,1,1) \ .$$

Therefore, $\mathrm{curl}(\mathbf{G})\cdot\mathbf{N} = 0$, and so

$$\oint_C \mathbf{G}\cdot\mathbf{T}\mathrm{d}s = 0 \ .$$

Warning: You should not expect quite so many answers from this part on the actual exam to be zero. These problems have short solutions, but some sort of cleverness is required to see the short solution for some of them. There will not be quite so many problems whose efficient solution requires clever insights on the actual exam, though there will be some. These should give you good practice for dealing with those that do.