Answers for Practice Test IB, Math 291 Spring 2010

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1: Let $\mathbf{a} = (1,1,1)$

(a) Find a vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (-7,2,5) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

(b) There is no vector $\mathbf{x}$ such that

$$\mathbf{x} \times \mathbf{a} = (1,0,0) \qquad \text{and} \qquad \mathbf{x} \cdot \mathbf{a} = 0 \ .$$

Show that no such vector exists.

SOLUTION: The vector $\mathbf{x} = (x,y,z)$ is orthogonal to $\mathbf{a} = (1,1,1)$ if and only if $x + y + z = 0$, which means $z = -x - y$. Computing the cross product $(x,y,-x-y) \times (1,1,1)$ we find

$$(x,y,-x-y) \times (1,1,1) = (2y + x, -2x - y, x - y)$$

Setting this equal to $(-7,2,5)$ we see

$$\begin{aligned} 2y + x &= -7 \\ 2x + y &= -2 \\ y - x &= -5 \end{aligned}$$

Adding the first and third equation, we see $3y = -12$, so $y = -4$. Then form the third equation we see $x = 1$. One now sees that all three equations are satisfied with $x = 1$ and $y = -4$, and ths is the unique solution. Thus

$$\mathbf{x} = (x,y,-x-y) = (1,-4,3) \ .$$

There are many **other approaches**. For example, one can use the triple cross product identity

$$\mathbf{a} \times (\mathbf{x} \times \mathbf{a}) = \|\mathbf{a}\|^2\mathbf{x} - (\mathbf{a} \cdot \mathbf{x})\mathbf{a}$$

together with the orthogonality of $\mathbf{x}$ and $\mathbf{a}$ to deduce

$$\mathbf{x} = \frac{\mathbf{1}}{\|\mathbf{a}\|^{\mathbf{2}}}\mathbf{a} \times (\mathbf{x} \times \mathbf{a}) \ .$$

Given that $\mathbf{x} \times \mathbf{a} = (-7, 2, 5)$, this becomes

$$\mathbf{x} = \frac{1}{3}(1, 1, 1) \times (-7, 2, 5) = (1, -4, 3) \ ,$$

as we found above. There are several other ways to find the vector asked for in part **(a)**. If your approach was different, but got the same answer, it was probably just fine.

As for part **(b)**, $\mathbf{x} \times \mathbf{a}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{a}$. But $(1, 0, 0)$ is not orthogonal to $\mathbf{a}$, and so $\mathbf{x} \times \mathbf{a} = (1, 0, 0)$ is impossible.

2: Let P_1 denote the plane through the three points $\mathbf{a_1} = (1, 2, 1)$ $\mathbf{a_2} = (-1, 2, -3)$ and $\mathbf{a_3} = (2, -3, -2)$. Let P_2 denote the plane through the three points $\mathbf{b_1} = (1, 1, 0)$ $\mathbf{b_2} = (1, 0, 1)$ and $\mathbf{b_3} = (0, 1, 1)$.

(a) Find equations for the planes P_1 and P_2.

(b) Parameterize the line given by $P_1 \cap P_2$, and find the distance between this line and the point $\mathbf{a_1}$.

(c) Consider the line through $\mathbf{b_1}$ and $\mathbf{b_2}$. Determine the point of intersection of this line with the plane P_1, and find a parametrization of the line given by reflecting this line off plane P_1.

SOLUTION: The plane contains $\mathbf{a}_1$, and its normal vector is $(\mathbf{a}_2 - \mathbf{a}_1) \times (\mathbf{a}_3 - \mathbf{a}_1) = (-20, -10, 10)$. Hence an equation for P_1 is

$$(-2, -1, 1) \cdot (x - 1, y - 2, z - 1) = -2x - y + z = -3 \ .$$

You can check this: Each of the three given points do satisfy this equation.

In the same way, or by inspection, using the symmetry of the points specifying P_2, we find that an equation for it is

$$x + y + z = 2 \ .$$

To parameterize the line $P_1 \cap P_2$, we need a base point. Let us look for one with $z = 0$, and then we must have

$$\begin{aligned} -2x - y &= -3 \\ x + y &= 2 \end{aligned}$$

. Hence, $-3x = -5$, so $x = 3/5$ and $y = 7/5$. Our base point is $\mathbf{x}_0 = (3/5, 7/5, 0)$.

The direction vector is

$$\mathbf{v} = (-2, -1, 1) \times (1, 1, 1) = (-2, 3, -1) \ .$$

Hence the line is

$$\mathbf{x}(t) = (3/5 - 2t, 7/5 + 3t, -t) \ .$$

The distance between this line and $\mathbf{a}_1$ is the length of the component of $\mathbf{a}_1 - \mathbf{x}_0$ that is orthogonal to $\mathbf{v}$; i.e.,

$$(\mathbf{a}_1 - \mathbf{x}_0)_\perp = (\mathbf{a}_1 - \mathbf{x}_0) - \frac{1}{\|\mathbf{v}\|^2}[(\mathbf{a}_1 - \mathbf{x}_0) \cdot \mathbf{v}]\mathbf{v} \ .$$

Since

$$\mathbf{a}_1 - \mathbf{x}_0 = (1, 2, 1) - (3/5, 7/5, 0) = (2/5, 3/5, 1) \ ,$$

which is orthogonal to $\mathbf{v}$, one has

$$(\mathbf{a}_1 - \mathbf{x}_0)_\perp = (2/5, 3/5, 1) \ ,$$

and so the distance is

$$\frac{1}{5}\sqrt{38} \ .$$

Finally, for part **(c)**, The line through $\mathbf{b_1}$ and $\mathbf{b_2}$ is parameterized by

$$\mathbf{x}(t) = \mathbf{b_1} + t(\mathbf{b_2} - \mathbf{b_1}) = (1, 1-t, t) \ .$$

Plugging this into our equation for P_1, we have

$$-2(1) - (1-t) + t = -3$$

which means $t = 0$. Indeed, you can see that $\mathbf{b}_1$ does satisfy the equation for P_1. Had you notices this at the outset, you could have used $\mathbf{b}_1$ as the base point for the line in part **(b)**.

The direction vector of the line though $\mathbf{b_1}$ and $\mathbf{b_2}$ is $(0, -1, 1)$. Reflecting this about the plane P_1, we get

$$\begin{aligned}
(0,-1,1) \quad - \quad & 2\frac{1}{\|(-2,-1,1)\|^2}[(0,-1,1)\cdot(-2,-1,1)](-2,-1,1) = \\
= \quad & (0,-1,1) - 2\frac{1}{6}2(-2,-1,1) \\
= \quad & (4/3, -1/3, 1/3) \ .
\end{aligned}$$

We take as our base point the point of intersection, found above to be $(1, 1, 0)$, and so the reflected line is

$$(1,1,0) + t(4/3, -1/3, 1/3) = (1 + (4/3)t, 1 - (1/3)t, t/3) \ .$$

3: Let $\mathbf{x}(t)$ be the curve given by

$$\mathbf{x}(t) = (e^t\cos(t), e^t\sin(t), e^t) \ .$$

(a) Compute the arc length $s(t)$ as a function of t, measured from the starting point $\mathbf{x}(0)$, and find an arc-length parameterization of this curve

(b) Compute curvature $\kappa(t)$ and torsion $\tau(t)$ as a function of t.

(c) Find an equation for the osculating plane at time $t = 0$

SOLUTION: We compute

$$\mathbf{v}(t) = (e^t(\cos(t) - \sin(t)), e^t(\sin(t) + \cos(t)), e^t) \ .$$

We now comoute

$$v(t) = \|\mathbf{v}(t)\| = \sqrt{3}e^t \ .$$

Integrating,

$$s(t) = \int_0^t \sqrt{3}e^r \mathrm{d}r = \sqrt{3}(d^t - 1) \ .$$

Next, for **(b)**, computing with the usual formulas we find

$$\kappa(t) = \frac{\sqrt{2}}{3}e^{-t} \ ,$$

and

$$\tau(t) = \frac{1}{3}e^{-t} \ ,$$

Next, for **(c)**, we need a base point, and a normal vector. For the base point, we take

$$\mathbf{x}(0) = (1, 0, 1) \ .$$

For the normal vector we take $\mathbf{n} = \mathbf{v}(0) \times \mathbf{a}(0)$. We first compute

$$\mathbf{a}(t) = (-2e^t \sin(t), 2e^t \cos(t), e^t) \ .$$

We then find

$$\mathbf{n} = (1, 1, 1) \times (0, 2, 1) = (-1, -1, 2) \ .$$

The equation is

$$(-1, -1, 2) \cdot (x - 1, y, z - 1) = 0$$

which reduces to

$$-x - y + 2z = 1 \ .$$

4: (a) Let $f(x, y)$ be given by

$$f(x, y) = \begin{cases} \dfrac{x}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) \neq (0, 0) \ . \end{cases}$$

Does

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

exist? If so, evaluate the limit. If not, explain why not.

(b) Let $g(x, y)$ be given by

$$g(x, y) = \begin{cases} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} & (x, y) \neq (0, 0) \\ 0 & (x, y) \neq (0, 0) \ . \end{cases}$$

Does

$$\lim_{(x,y)\to(0,0)} g(x, y)$$

exist? If so, evaluate the limit. If not, explain why not.

SOLUTION: Solved in class.

5: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

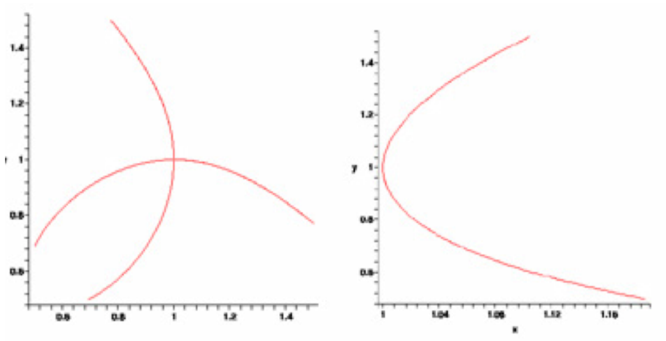
$$f(x, y) = x^2y + yx - xy^2 \ .$$

(a) Compute the gradient of f, and find all critical points of f..

(b) Find a parameterization of the tangent line to the level curve of f through the point $(1,1)$.

(c) Find the equation of the tangent plane to the graph f at the point $(1,1)$.

(d) Could the following be a contour plot of f? Explain your answer.



SOLUTION: Solved in class.

Extra Credit: Let $f(x,y) - 3xy - x^3 - y^3$. Find all points (x,y) at which the tangent plane to the graph of f is orthogonal to the line parameterized by $t(3,3,1)$.

SOLUTION: Solved in class.