Solutions for Practice Test IIA, Math 291 Spring 2010

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1: Let $f(x,y) = x^2 + y^2 - 2yx^2$.

(a) Find all of the critical points of f. Evaluate the Hessian matrix of f at each of these critical points, and determine where each is a local maximum, a local minimum, a saddle, or undecidable from the Hessian.

SOLUTION We compute $\nabla f(x,y) = 2(x(1-2y), y - x^2)$. Therefore, a critical point (x,y) must satisfy

$$\begin{aligned} x(1-2y) &= 0 \\ y - x^2 &= 0 \end{aligned}$$

From the first equation, either $x = 0$ or $y = 1/2$. From the second equation, if $x = 0$, then $y = 0$, and if $y = 1/2$, then $x = \pm 1/\sqrt{2}$. Hence there are three critical points:

$$(0,0) \qquad (1/\sqrt{2}, 1/2) \quad \text{and} \quad (-1/\sqrt{2}, 1/2)\ .$$

We next compute

$$\mathrm{Hess}_f(x,y) = \begin{bmatrix} 2-4y & -4x \\ -4x & 2 \end{bmatrix}\ .$$

Evaluating this at $(0,0)$, we find $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The two principle curvatures are both 2. Since both are positive, the surface curves upward from this critical point, which is a local minimum. Since the two principle curvatures are equal, the contour curves will look circular near this critical point.

Evaluating the Hessian at $(\pm 1/\sqrt{2}, 1/2)$, we find $\mp \begin{bmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & 2 \end{bmatrix}$. In either case, the principle curvatures are the roots of

$$t(t-2) - 8 = 0$$

which are 4 and -2. Hence theses two critical points are saddle points: The surface curves upwards in some directions, and downwards in others.

(b) Sketch a contour plot of f in the vicinity of each of the critical points. Show the computations that lead to the plots to get credit.

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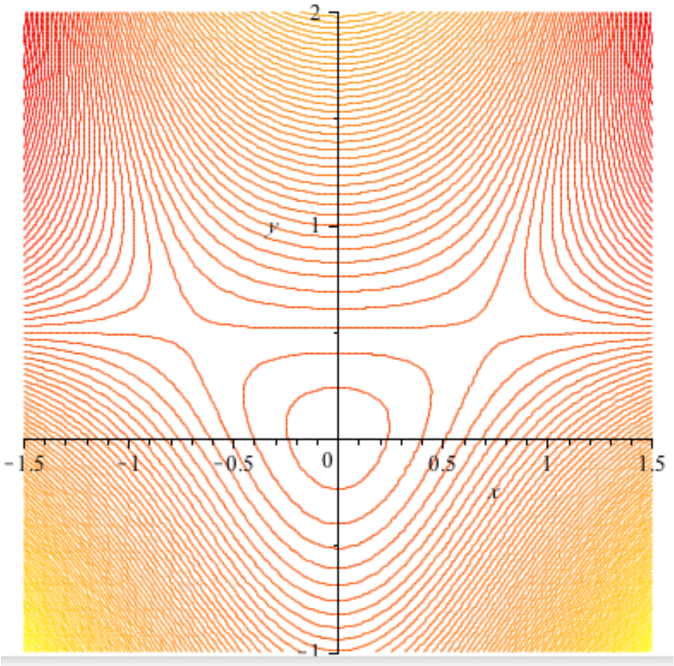
SOLUTION For the critical point $(0,0)$, the principle curvatures are 2 and 2, so in an orthogonal system of coordinates centered on $(0,0)$, the quadratic approximation takes the from $f(0,0)+2\tilde{x}^2+2\tilde{y}^2$. So the curves are of the quadratic approxmation are of the from $2\tilde{x}^2+2\tilde{y}^2 = \text{constant}$. These are circles and we do not need the axes to draw them.

For the critical point $(1/\sqrt{2}, 1/2)$, the principle curvatures are 4 and -2, and hence in an orthogonal system of coordinates centered on $(0,0)$, the quadratic approximation takes the from $f(0,0)+2\tilde{x}^2-2\tilde{y}^2$. To get the direction of the $\tilde{x}$ axis, we form

$$\mathrm{Hess}_f(1/\sqrt{2}, 1/2) - \begin{bmatrix} -2 & 0 \\ 0 & -22 \end{bmatrix} = \begin{bmatrix} 2 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{bmatrix}$$

and take $\mathbf{v}_1 = (2, -2\sqrt{2})$. That is, we subtract the *other* principle curvature form the diagonal of the Hessian, and that the tope row of what is left. This points along the $\tilde{x}$ axis, and the tilde y axis is orthogonal to that. The remaining critical point is handled the same way. Here is the resulting sketch:



2: Let $f(x,y) = (x+y)^4 + (x-y)^2$. Find the minimum and maximum values of f on the unit circle $x^2+y^2=1$, and all of the places on the circle at which f takes on these values.

SOLUTION We compute

$$\nabla f(x,y) = (4(x+y)^3 + 2(x-y), 4(x+y)^3 - 3(x-y)) \ .$$

With the constraint written as $g(x,y)=0$, we have $g(x,y) = x^2+y^2-1$, and we compute

$$\nabla g(x,y) = 2(x,y)$$

The Lagrange equation $\nabla f(x,y) = \lambda \nabla g(x,y)$ gives us

$$2(x+y)^3 y + (x-y)y = 2(x+y)^3 x - (x-y)x$$

or

$$(x+y)(x-y) = 2(x-y)(x+y)^3 \ .$$

This is solved if either $x = -y$ or $x = y$. We can now eliminate y in the equation $x^2 + y^2 = 1$ to find $2x^2 = 1$. Hence our list of solutions of the Lagrange equations is:

$$(1/\sqrt{2}, 1/\sqrt{2}) \qquad (-1/\sqrt{2}, 1/\sqrt{2}) \qquad (1/\sqrt{2}, -1/\sqrt{2}) \qquad (-1/\sqrt{2}, -1/\sqrt{2}) \ .$$

Call these points $\mathbf{x}_1$ through $\mathbf{x}_4$.

To find the rest, let us assume that neither $x = y$ nor $x = -y$. Then we can cancel $(x+y)(x-y)$ from both sides of our simplified Lagrange equation, yielding

$$1 = 2(x+y)^2 \ .$$

This means $y = -x/4$. Using this to eliminate y from the constraint equation we find

$$x^2 + \frac{1}{16x^2} = 1 \ .$$

Multiplying by x^2, we get an equation that is quadratic in x^2. Completing the square,

$$(x^2 - 1/2)^2 = 3/16$$

and so

$$x^2 = 1/2 \pm \sqrt{3}/4 \ .$$

Hence we get four additional values of x, $\pm\sqrt{1/2 \pm \sqrt{3}/4}$, and from $y = -x/4$, we get the corresponding y values. Call these points $\mathbf{x}_5$ through $\mathbf{x}_8$.

Plugging in, we find the maximum value, 4, occurs at $\pm(1/\sqrt{2}, 1/\sqrt{2})$, and the minimum value

$$\left(\frac{3\sqrt{6}}{16} - \frac{3\sqrt{2}}{16}\right)^4 + \left(\frac{5\sqrt{6}}{16} - \frac{5\sqrt{2}}{16}\right)^2$$

occurs at

$$\pm\left(\sqrt{1/2 - \sqrt{3}/4}, -\sqrt{1/2 - \sqrt{3}/4}/4\right) \ .$$

3: Let Ω be the region in $\mathbb{R}^3$ that is bounded above by the sphere $x^2+y^2+z^2 = 4$, and below by the cone $4z = 4 - \sqrt{x^2+y^2}$. Let $f(x,y,z) = 1/(x^2+y^2+z^2)^2$. Compute $\int_\Omega f(x,y,z)\mathrm{d}V$. To get full credit, you must carry the computations through to the point that only one single variable integral remains to be done. To arrive at this point you will have to choose coordinates appropriately. Making a good choice of coordinates is an important first step; say why you make the choice you make.

SOLUTION The key to solving this is to draw a good side view picture. In the x,z plane, $y = 0$, and the bounding equations become $x^2+z^2 = 4$ and $z = 1-|x|/4$. To find the point of intersection, equate z^2 from both equations:

$$4 - x^2 = (1 - |x|/4)^2 \ .$$

This quadratic equation has two roots, only one of which is positive. Hence at the intersection $|x|$ is the positive root, namely $|x| = (4 + 8\sqrt{13})/17$, which is a bit less than 2. The z value at the intersection is (using the equation of the cone):

$$z = \frac{16 - 2\sqrt{13}}{17} \ . \tag{$*$}$$

Let us use spherical coordinates:

$$(x, y, z) = (\rho \sin\phi, \cos\theta, \rho \sin\phi, \sin\theta, \rho\cos\phi) \ . \tag{$**$}$$

The limit on θ is very easy:

$$0 \leq \theta \leq 2\pi \ .$$

To get the limit on ϕ we write $(*)$ in spherical coordinates, using the fact that at the intersection $\rho = 2$. We get

$$2\cos\phi = \frac{16 - 2\sqrt{13}}{17} \ .$$

In other words,

$$0 \leq \phi \leq \arccos\left(\frac{16 - 2\sqrt{13}}{34}\right) =: \phi_0 \ . \tag{$***$}$$

We have define the upper limit of ϕ to be ϕ_0 to simplify notation in what follows.

To get the limits on ρ, note rom the picture you drew that the upper limit comes from the sphere, $\rho = 4$. The lower limit comes from the cone. Using $(*)$ to translate the equation of the cone into spherical coordinates, we find

$$4\rho\cos\phi = 4 - \rho\sin\phi$$

and so

$$\rho = \frac{1}{4\cos\phi + \sin\phi} \ .$$

Thus,

$$\frac{1}{4\cos\phi + \sin\phi} \leq \rho \leq 2 \ .$$

Finally, we translate our integrand:

$$f(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^2} = \rho^{-4} \ .$$

Now remembering that the volume element in spherical coordinates is $\mathrm{d}V = \rho^2 \sin\phi \rho \mathrm{d}\phi \mathrm{d}\theta$, we get

$$\int_0^{\phi_0} \left(\int_{1/(4\cos\phi + \sin\phi)}^{2} \left(\int_0^{2\pi} \rho^{-2} \sin\phi \mathrm{d}\theta \right) \mathrm{d}\rho \right) \mathrm{d}\phi$$

The integrals over θ and ρ are easy, and this leaves

$$2\pi \int_0^{\phi_0} (4\cos\phi + \sin\phi) - 2) \sin\phi \mathrm{d}\phi \ ,$$

where ϕ_0 is given in $(***)$. The remaining integral is also easy, but we are done at this point.

4: Let Ω be the region in $\mathbb{R}^2$ that is bounded by the lines

$$y - x = 0 \qquad y - x = 3 \qquad x + 2y = 2 \quad \text{and} \quad x + 2y = 4 \ .$$

Compute

$$\int_\Omega \frac{x-y}{x^2+4xy+4y^2} \mathrm{d}A \ .$$

SOLUTION Let us take $u = y - x$ and $v = x + 2y$. Then we have $u + v = 3y$, so

$$y = \frac{u+v}{3} \qquad \text{and} x = \frac{v-2u}{3} \ .$$

The Jacobian determinant gives us

$$\mathrm{d}x\mathrm{d}y = \frac{1}{3}\mathrm{d}u\mathrm{d}v \ .$$

We next compute

$$\frac{x-y}{x^2+4xy+4y^2} = \frac{u}{v^2}$$

and so

$$\int_\Omega \frac{x-y}{x^2+4xy+4y^2} \mathrm{d}A = \frac{1}{3}\int_2^4 \left(\int_0^3 \frac{u}{v^2}\mathrm{d}u \right) \mathrm{d}v = \frac{1}{3}\frac{1}{4}\frac{9}{2} = \frac{3}{8} \ .$$

5: Let $\mathcal{S}$ be the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane $x + z = 1$.
(a) Compute $\int_\mathcal{S} f(x,y,z)\mathrm{d}S$ where $f(x,y,z) = y/\sqrt{x^2+y^2}$. To get full credit, carry the computations through to the point that only an integral over a single variable remains to be evaluated.
(b) Let $\mathbf{F}$ be the vector field $\mathbf{F}(x,y,z) = (xy, yz, zx)$. Compute the flux integral

$$\int_\mathcal{S} \mathbf{F} \cdot \mathbf{N} \mathrm{d}S$$

where $\mathbf{N}$ is the downward unit normal to the surface. That is, compute the flux across the surface from top to bottom.
SOLUTION The key to solving both parts is coming up with a good parameterization. To find the intersection of the plane and paraboloid, we equate their z values and find

$$1 - x^2 - y^2 = 1 - x$$

which is the same as

$$x^2 + y^2 = x \qquad \text{or} \qquad (x - 1/2)^2 + y^2 = 1/4 \ .$$

This is the circle bounding the disk in the x, y plane centered on $(1/2, 0)$ with radius $1/2$. This is what we would see in a top view diagram. Our surface $\mathcal{S}$ is the part of the paraboloid that lies above this disk. Let us use cylindrical coordinates:

$$(x, y, z) = (r\cos\theta, r\sin\theta, z) \ .$$

The equation for the paraboloid is $x = 1 - r^2$, and so we have our parameterization

$$\Phi(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r^2) \ .$$

The equation $x^2 + y^2 = x$ translates to $r^2 = r\cos\theta$, so the limits on our parameters are

$$0 \le r \le \cos\theta \qquad \text{and} \qquad -\pi/2 \le \theta \le \pi/2 \ .$$

We next compute

$$\frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} = (2r^2\cos\theta, 2r^2\sin\theta, r) \ . \tag{$*$}$$

Now we are ready to do part (**a**). Translating $f(x,y,z)$ into cylindrical coordinates, we find $f(x,y,z) = \sin\theta$. From $(*)$, we find

$$\mathrm{d}S = r\sqrt{4r^2+1}\mathrm{d}r\mathrm{d}\theta \ .$$

Hence

$$\int_{\mathcal{S}} f(x,y,z)\mathrm{d}S = \int_{-\pi/2}^{\pi/2}\left(\int_0^{\cos\theta} r\sqrt{4r^2+1}\mathrm{d}r\right)\sin\theta\mathrm{d}\theta \ .$$

The inner integral is easily done by substitution:

$$\int_0^{\cos\theta} r\sqrt{4r^2+1}\mathrm{d}r = \frac{1}{8}\frac{2}{3}u^{3/2}\Big|_1^{4\cos^2\theta+1} = \frac{1}{12}((4\cos^2\theta+1)^{3/2}-1) \ .$$

We finally have

$$\int_{\mathcal{S}} f(x,y,z)\mathrm{d}S = \frac{1}{12}\int_{-\pi/2}^{\pi/2}((4\cos^2\theta+1)^{3/2}-1)\sin\theta\mathrm{d}\theta \ ,$$

which is easily evaluated, but this is all we are asked for.

Now we are ready to do part **(b)**: Translating $\mathbf{F}$ into cylindrical coordinates, we find

$$\mathbf{F} = (r^2\cos\theta\sin\theta, (r-r^3)\sin\theta, (r-r^3)\cos\theta) \ .$$

Thus from $(*)$ we have

$$\mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \pm(r^2\cos\theta\sin\theta, (r-r^3)\sin\theta, (r-r^3)\cos\theta)\cdot(2r^2\cos\theta, 2r^2\sin\theta, r)\mathrm{d}r\mathrm{d}\theta \ .$$

Since we are asked for the downward flus, we want the z component in $(*)$ to be negative, so we take the minus sign. We have

$$\int_{\mathcal{S}}\mathbf{F}\cdot\mathbf{N}\mathrm{d}S = \int_{-\pi/2}^{\pi/2}\left(\int_0^{\cos\theta}[r^2\cos\theta + r^4(\cos\theta - 2\cos^2\theta\sin\theta) + 2(r^3-r^5)\sin^2\theta]\mathrm{d}r\right)\mathrm{d}\theta \ .$$

From here, the integrals are easy.

Extra Credit: $\mathcal{S}$ be upper hemisphere of the unit sphere in $\mathbb{R}^3$. Let $f(x,y,z) = xyz$. Find the minimum and maximum values of f on $\mathcal{S}$, and all of the points at which f takes on these values. Explain how you are taking into account both of the constraints $x^2+y^2+z^2 = 1$ and $z \ge 0$.
SOLUTION The functions whose maximum and minimum we seek is pretty symmetric, so let us find the maximum and minimum on the whole sphere, and see if they occur in the upper hemisphere.

To do this we write the constrain in the form $g(x,y,z) = 1$ with

$$g(x,y,z) = x^2+y^2+z^2 \ .$$

The Lagrange condition then gives us

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \ .$$

This means

$$\nabla f(x,y,z) \times \nabla g(x,y,z) = 0 \ .$$

Computing the gradients and cross product, we get the equations

$$\begin{aligned} x(y^2 - z^2) &= 0 \\ y(x^2 - z^2) &= 0 \\ z(y^2 - x^2) &= 0 \end{aligned}$$

We also have the equation

$$x^2 + y^2 + z^2 = 1 \ .$$

We see that if any two coordinates are zero, the third must be ±1. This give the points

$$(\pm 1, 0, 0) \qquad (0, \pm 1, 0) \qquad (0, 0, , \pm 1) \ .$$

Now suppose that only one coordinate, say x is zero. Since y and z are non zero, we can divide by them and conclude

$$\begin{aligned} x^2 - z^2 &= 0 \\ y^2 - x^2 &= 0 \end{aligned}$$

Thus we have $x^2 = y^2 = z^2$, and all coordinates would be zero ,which is impossible. So the only other choice is all coordinates non-zero. Then we still have $x^2 = y^2 = z^2$. From the constraint equation we have that the remaining candidates are

$$(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}) \ .$$

Since the sphere is compact, and f is continuous, there will be a minimum and a maximum. The maximum is $3^{3/2}$, and is attained at the points $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ with an even number of minus signs. There are such points, namely $(-1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ and $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ in the upper hemisphere. Since these are maximizers on the whole sphere, they are certainly maximizers on the hemisphere. The minimum $3^{-3/2}$, and is attained at the points $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ with an odd number of minus signs. There are such points, namely $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ in the upper hemisphere. Since these are minimizers on the whole sphere, they are certainly maximizers on the hemisphere.