1. Let $(X,d)$ be a metric space. Suppose that whenever $f$ is a continuous real valued function on $X$, there exists an $x_0 \in X$ such that $f(x_0) \geq f(x)$ for all $x \in X$. Prove that $X$ is compact.

2. Let $X$ be a compact metric space and let $\mathcal{F}$ be any set of real valued functions on $X$ that is uniformly bounded and equicontinuous. Prove that the function

$$ g(x) = \sup\{ f(x) : f \in \mathcal{F} \} $$

is continuous.

3. Let $f$ and $g$ be two continuous real valued functions on $\mathbb{R}^n$, and suppose that $f(x) = g(x)$ for all $x$ in the complement of a set of Lebesgue measure zero. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}^n$.

4. Let $(\Omega, \mathcal{M}, \mu)$ be a finite measure space. Suppose that $\{f_n\}$ is a sequence of real valued measurable functions on $\Omega$, and that $f(x) = \lim_{n \to \infty} f_n(x)$ for almost every $x$. Suppose that $\|f\|_1 > 0$. Show that there for some $\epsilon > 0$, there is a strictly positive number $b$ so that for all $n$ sufficiently large there is a set $E_n$ with $\mu(E_n) > \epsilon$ and $|f_n(x)| > b$ for all $x \in E_n$.

5. Let $\mathcal{B}$ be the Borel sigma algebra on $[0,1]$, and $\mu$ be Lebesgue measure on $[0,1]$. Let

$$ g_n(t) = \sin(2\pi nt) \cdot $$

Show that for all $f \in L^1([0,1], \mathcal{B}, \mu)$,

$$ \lim_{n \to \infty} \int_{[0,1]} g_n f \, d\mu = 0 \cdot $$

6. Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Consider the set $\mathcal{F}$ of real-valued functions on $\Omega$ given by

$$ \mathcal{F} = \{ f \in L^1(\Omega, \mathcal{M}, \mu) : 0 \leq f(x) \leq 1 \text{ for almost every } x \} \cdot $$

Is $\mathcal{F}$ a closed subset of $L^1(\Omega, \mathcal{M}, \mu)$?

7. (a) Show that for any two complex numbers $a$ and $b$,

$$ ||a + b| - |a| - |b|| \leq 2|b| \cdot $$
(b) Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of complex functions in $L^1(\Omega, \mathcal{M}, \mu)$. Suppose that $\lim_{n \to \infty} f_n(x) = f(x)$ for almost every $x$. Show that

$$\lim_{n \to \infty} \int_{\Omega} ||f_n| - |f_n - f| - |f|| d\mu = 0 .$$

(c) Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of complex functions in $L^1(\Omega, \mathcal{M}, \mu)$. Suppose that $\lim_{n \to \infty} f_n(x) = f(x)$ for almost every $x$. Show that in this case,

$$\lim_{n \to \infty} \int_{\Omega} |f_n| d\mu = \int_{\Omega} |f| d\mu$$

if and only if

$$\lim_{n \to \infty} \int_{\Omega} |f_n - f| d\mu = 0 .$$