

Partial Differential Equations — Problem Sheet 1

1. True or false? Fully explain your answers.

The function	is a solution of
$u(x, y) = A(y)$	$u_y = 0$
$u(x, y) = A(y)$	$u_{xy} = 0$
$u(t, x) = A(x)B(t)$	$u_{xy} = 0$
$u(t, x) = A(x)B(t)$	$uu_{xt} = u_x u_t$
$u(t, x, y) = A(x, y)$	$u_t = 0$
$u(x, t) = A(x+ct) + B(x-ct)$	$u_{tt} + c^2 u_{xx} = 0$
$u(x, y) = e^{kx} \sin(ky)$	$u_{xx} + u_{yy} = 0$

where A and B are arbitrary functions and c and k are constants.

2. Find PDEs that are satisfied by each of the following functions:

- (a) $u(t, x) = e^t \cos x$
- (b) $u(x, y) = x^2 + y^2$
- (c) $u(t, x) = x^2 t$
- (d) $u(t, x) = x^2 t^2$
- (e) $u(x, y) = e^{-x^2}$
- (f) $u(x, y) = \ln(x^2 + y^2)$

In each case try to find more than one suitable PDE.

3. In each of the following cases, find a PDE which is satisfied by the the function given

- (a) $u(t, x) = A(x + ct) + B(x - ct)$ where c is a constant
- (b) $u(t, x) = A(x) + B(t)$
- (c) $u(t, x) = A(x)/B(t)$
- (d) $u(t, x) = A(xt)$
- (e) $u(t, x) = A(x^2 t)$
- (f) $u(t, x) = A(x^2/t)$

4. For each of the cases in question 3 can you determine the functions $A(\cdot)$ and/or $B(\cdot)$ using the initial condition

$$u(0, x) = f(x)$$

where $f(\cdot)$ is some given specified function? Explain each answer and give expressions for $A(\cdot)$ and/or $B(\cdot)$ wherever they can be determined.

5. Repeat question 4 for the condition $u(1, x) = g(x)$.

Partial Differential Equations — Answer Sheet 1

1.

Function	Comment	Conclusion
$u(x, y) = A(y)$	$u_y = A'(y)$	False
$u(x, y) = A(y)$	$u_{xy} = 0$	True
$u(t, x) = A(x)B(t)$	$u_{xy} = 0$ concerns different independent variables!	False
$u(t, x) = A(x)B(t)$	$uu_{xt} = ABA'B' = u_x u_t$	True
$u(t, x, y) = A(x, y)$	$u_t = \partial_t A(x, y) = 0$	True
$u(x, t) = A(x+ct) + B(x-ct)$	$u_{tt} + c^2 u_{xx} = 2c^2(A'' + B'')$	False
$u(x, y) = e^{kx} \sin(ky)$	$u_{xx} = k^2 e^{kx} \sin(ky), u_{yy} = -k^2 e^{kx} \sin(ky)$	True

2. These are not the only possible cases; you might find other PDEs:

- (a) $u(t, x) = e^t \cos x$: $u_{t^n} = e^t \cos x = u$ (any n), $u_x = -e^t \sin x$, $u_{xx} = -e^t \cos x$ so $u_{t^n} = u$ or $u_{t^n} + u_{xx} = 0$, etc.
- (b) $u(x, y) = x^2 + y^2$: $u_{xx} = 2$, $u_{yy} = 2$ so $u_{xx} - u_{yy} = 0$. Also $u_{xy} = 0$, etc.
- (c) $u(t, x) = x^2 t$: $u_x = 2xt$, $u_t = x^2$, $u_{tx} = 2x$ so $2tu_t - xu_x = 0$ or $u_x = tu_{tx}$, etc.
- (d) $u(t, x) = x^2 t^2$: $u_{tx} = 4xt$, $u_{ttxx} = 4$ so $u_{tx} = 16u^2$ or $u_{tttx} = 0$, etc.
- (e) $u(x, y) = e^{-x^2}$: $u_x = -2xe^{-x^2}$, $u_{xx} = (4x^2 - 2)e^{-x^2}$, $u_y = 0$ so $u_y = 0$ or $u_{xx} = (4x^2 - 2)u$, etc.
- (f) $u(x, y) = \ln(x^2 + y^2)$: $u_x = \frac{2x}{x^2+y^2}$, $u_y = \frac{2y}{x^2+y^2}$, $u_{xx} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$, $u_{yy} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$ so $yu_x - xu_y = 0$, $u_{xx} + u_{yy} = 0$, etc.

3. These are not the only possible cases; you might find other PDEs:

- (a) $u(t, x) = A(x+ct) + B(x-ct)$: $u_t = cA'(x+ct) - cB'(x-ct)$,
 $u_x = A'(x+ct) + B'(x-ct)$, $u_{tt} = c^2 A''(x+ct) + c^2 B''(x-ct)$,
 $u_{xx} = A''(x+ct) + B''(x-ct)$ so $u_{tt} - c^2 u_{xx} = 0$ (wave equation).
- (b) $u(t, x) = A(x) + B(t)$: $u_t = B'(t)$ so $u_{tx} = 0$.
- (c) $u(t, x) = A(x)/B(t)$: $\ln u = \ln A(x) - \ln B(t)$ so $(\ln u)_{tx} = 0$ or $uu_{tx} - u_t u_x = 0$.
- (d) $u(t, x) = A(xt)$: $u_t = xA'(xt)$, $u_x = tA'(xt)$, so $tu_t - xu_x = 0$.
- (e) $u(t, x) = A(x^2 t)$: $u_t = x^2 A'(x^2 t)$, $u_x = 2xtA'(x^2 t)$ so $2tu_t - xu_x = 0$.
- (f) $u(t, x) = A(x^2/t)$: $u_t = -\frac{x^2}{t^2} A'(x^2/t)$, $u_x = -\frac{2x}{t} A'(x^2/t)$ so $2tu_t + xu_x = 0$.

4. (a) $u(0, x) = A(x) + B(x) = f(x)$ so $A(x) + B(x) = f(x)$. There is not enough information to determine both $A(\cdot)$ and $B(\cdot)$.
- (b) $u(0, x) = A(x) + B(0) = f(x)$ so $A(x) = f(x) - B(0)$. We would only need to know one value of B , namely $B(0)$ to determine A . However, the initial conditions gives no information about B .
- (c) $u(0, x) = A(x)/B(0) = f(x)$ so $A(x) = B(0)f(x)$. We would only need to know one value of B , namely $B(0)$ to determine A , but we have no information about B .

- (d) $u(0, x) = A(0) = f(x)$ so $A(0) = f(x)$. This tries to set a constant $A(0)$ to something that is not constant $f(x)$, which is not possible!
 - (e) (exactly the same)
 - (f) $u(0, x) = A(\infty) = f(x)$ so $A(\infty) = f(x)$. This tries to set a constant $A(\infty)$ to something that is not constant $f(x)$, which is again not possible!
- 5.
- (a) $u(1, x) = A(x + c) + B(x - c) = g(x)$ so $A(x + 1) + B(x - 1) = g(x)$. Again more information is needed to find both $A(\cdot)$ and $B(\cdot)$.
 - (b) $u(1, x) = A(x) + B(1) = g(x)$ so $A(x) = g(x) - B(1)$. We only need to know $B(1)$ to determine A . We obtain no information about B .
 - (c) $u(1, x) = A(x)/B(1) = g(x)$ so $A(x) = B(1)g(x)$. We only need to know $B(1)$ to determine A . There is no information about B .
 - (d) $u(1, x) = A(x) = g(x)$ so $A(x) = g(x)$.
 - (e) $u(1, x) = A(x^2) = g(x)$ so $A(x^2) = g(x)$. This determines $A(\xi)$ for $\xi \geq 0$ provided $g(\xi) = g(-\xi)$.
 - (f) $u(1, x) = A(x^2) = g(x)$ so $A(x^2) = g(x)$. Again, this determines $A(\xi)$ for $\xi \geq 0$ provided $g(\xi) = g(-\xi)$.

Partial Differential Equations — Problem Sheet 2

1. Categorise the following PDEs by order, linearity or degree of nonlinearity, and (if linear) whether homogeneous or inhomogeneous:

(a) $u_t - (x^2 + u)u_{xx} = x - t$
(b) $u^2 u_{tt} - \frac{1}{2}u_x^2 + (uu_x)_x = e^u$
(c) $u_t - \nabla^2 u = u^3$
(d) $(u_{xy})^2 - u_{xx} + u_t = 0$
(e) $u_t + u_x - u_y = 10$

2. Categorise the following 2nd order PDEs as elliptic, parabolic or hyperbolic. Also state their degree of nonlinearity and (if linear) whether homogeneous or inhomogeneous:

(a) $u_t + u_{tx} - u_{xx} + u_x^2 = \sin u$
(b) $u_x + u_{xx} + u_y + u_{yy} = \sin(xy)$
(c) $u_x + u_{xx} - u_y - u_{yy} = \cos(xyu)$
(d) $u_{tt} + xu_{xx} + u_t = f(x, t)$
(e) $u_t + uu_{xx} + u^2 u_{tt} - u_{tx} = 0$

3. Laplace's equation for $u(x, y)$, which is

$$u_{xx} + u_{yy} = 0$$

with the boundary conditions

$$u(x, 0) = \gamma \cos(x/\gamma), \quad u_y(x, 0) = 0$$

has the unique solution

$$u(x, y) = \gamma \cosh(y/\gamma) \cos(x/\gamma).$$

[Conform that this is a solution satisfying the conditions].

If these solutions were to vary continuously with the boundary conditions we would have:

For any (x, y) and any $\delta > 0$, $\exists \epsilon > 0$ such that for all $|u(x, 0)| < \epsilon$ and $|u_y(x, 0)| < \epsilon$, we have $|u(x, y)| < \delta$.

Show that this is not the case for the solutions given above for Laplace's equation.

Partial Differential Equations — Answer Sheet 2

1. (a) $u_t - (x^2 + u)u_{xx} = x - t$ is 2nd order and quasilinear.
 (b) $u^2u_{tt} - \frac{1}{2}u_x^2 + (uu_x)_x = e^u$ is 2nd order and quasilinear.
 (c) $u_t - \nabla^2 u = u^3$ is 2nd order and semilinear.
 (d) $(u_{xy})^2 - u_{xx} + u_t = 0$ is 2nd order and fully nonlinear.
 (e) $u_t + u_x - u_y = 10$ is 1st order, linear and inhomogeneous.
2. We consider the operators to contain the terms $au_{tt} + bu_{tx} + cu_{xx}$ (or different subscripts for different independent variables):
 - (a) $u_t + u_{tx} - u_{xx} + u_x^2 = \sin u$. $b^2 - 4ac = 1 > 0$ so: hyperbolic and semilinear.
 - (b) $u_x + u_{xx} + u_y + u_{yy} = \sin(xy)$. $b^2 - 4ac = -4 < 0$ so: elliptic, linear and inhomogeneous.
 - (c) $u_x + u_{xx} - u_y - u_{yy} = \cos(xyu)$. $b^2 - 4ac = 4 > 0$ so: hyperbolic and semilinear.
 - (d) $u_{tt} + xu_{xx} + u_t = f(x, t)$. $b^2 - 4ac = -4x$ so: elliptic for $x > 0$, hyperbolic for $x < 0$, parabolic for $x = 0$, linear and inhomogeneous.
 - (e) $u_t + uu_{xx} + u^2u_{tt} - u_{tx} = 0$. $b^2 - 4ac = 1 - 4u^3$ so: elliptic for $u^3 > \frac{1}{4}$, hyperbolic for $u^3 < \frac{1}{4}$, parabolic for $u^3 = \frac{1}{4}$, and quasilinear.

3. Confirming that $u = \gamma \cosh(y/\gamma) \cos(x/\gamma)$ is a solution:

- At $y = 0$ we have $u(x, 0) = \gamma \cosh(0) \cos(x/\gamma) = \gamma \cos(x/\gamma)$ and $u_y(x, 0) = \sinh(0) \cos(x/\gamma) = 0$ so the boundary conditions are satisfied.
- Also, $u_{xx} = -\gamma^{-1} \cosh(y/\gamma) \cos(x/\gamma)$ and $u_{yy} = \gamma^{-1} \cosh(y/\gamma) \cos(x/\gamma)$ so that the PDE $u_{xx} + u_{yy} = 0$ is satisfied.

Hence $u = \gamma \cosh(y/\gamma) \cos(x/\gamma)$ is a solution satisfying the conditions.

Note that if $\gamma = 0$, then we have the solution $u \equiv 0$.

Let us examine any value of $y > 0$, say $y = \alpha$, and any value of x .

We can always find arbitrarily large values of γ such that $|\cos(x/\gamma)| > \frac{1}{2}$.

For any such value of γ we have $|u(x, y)| > \frac{\gamma}{2} \cosh(\alpha/\gamma) = \frac{\gamma}{4}(e^{\alpha/\gamma} + e^{-\alpha/\gamma}) > \frac{\alpha}{4} \frac{\gamma}{\alpha} e^{\alpha/\gamma}$.

Since $\nu e^{-\nu} \rightarrow 0$ as $\nu \rightarrow \infty$ we must have (setting $\nu = \alpha/\gamma$): $\frac{\alpha}{\gamma} e^{-\alpha/\gamma} \rightarrow 0$ as $\gamma \rightarrow 0$; and hence $|\frac{\gamma}{\alpha} e^{\alpha/\gamma}| \rightarrow \infty$.

It follows that $|u(x, y)| \rightarrow \infty$ as $\gamma \rightarrow 0$, for the chosen values of x and y .

Hence, given any $\delta > 0$ and $y > 0$ there can be no value of $\epsilon > 0$ such that $|\gamma| < \epsilon \implies |u(x, y)| < \delta$; if it did, then we would have $|u(x, y)| \rightarrow 0$ as $\gamma \rightarrow 0$ and this is not the case.

Hence the statement required for continuous dependence on boundary conditions is violated by the solution we have.

Note: In general, Laplace's equation is ill-posed if it is subjected to conditions on a boundary that does not *entirely surround* the domain in which the equation is to be satisfied.

Partial Differential Equations — Problem Sheet 3

1. Consider the “negative” heat equation for $u(t, x)$, corresponding to having the thermal diffusivity coefficient $\kappa = -1$:

$$u_t = -u_{xx}.$$

Confirm that, for constant values of A and T ,

$$u = \frac{AT^{1/2}}{(T-t)^{1/2}} \exp\left(-\frac{x^2}{4(T-t)}\right)$$

is a solution for any $t < T$. Use this to show that solutions can exist with, initially, $|u(0, x)| \leq \epsilon$ for any $\epsilon > 0$ but which become infinite in value after any given time later on.

Is the negative heat equation well-posed for $t > 0$ when subjected to initial conditions at $t = 0$?

Can you suggest conditions for which the equation might be well-posed?

2. Use the method of characteristics to find *general solutions* for the following PDEs for $u(t, x)$ both in terms of a characteristic variable and one of t or x , and in terms of t and x . In each case sketch the paths of the characteristics.

- (a) $u_t - u_x = 0$
- (b) $u_t + tu_x = u$
- (c) $tu_t - u_x = 1$
- (d) $u_t + xu_x = -u$
- (e) $xu_t - u_x = t$
- (f) $tu_t + xu_x = x$
- (g) $tu_t - xu_x = t$
- (h) $xu_t - tu_x = xt$
- (i) $xu_t + tu_x = -xu$

3. For the general solutions you have obtained from question 2, apply the following boundary conditions, (a) to (a), (b) to (b), etc., and try to obtain unique solutions. For what values of t and x is each solution valid?

- (a) $u(0, x) = \cos(x)$
- (b) $u(0, x) = \sin(x)$
- (c) $u(t, 0) = \exp(-t^2)$
- (d) $u(0, x) = x^2$
- (e) $u(t, 0) = \ln(1 + t^2)$
- (f) $u(1, x) = x^3$
- (g) $u(1, x) = 1/(1 + x^2)$
- (h) $u(0, x) = 1 + x$ for $x \geq 0$
- (i) $u(0, x) = 1 - x$ for $x \geq 0$

Partial Differential Equations — Answer Sheet 3

1. Confirming that $u = AT^{1/2}(T-t)^{-1/2}e^{-x^2/4(T-t)}$ is a solution of $u_t = -u_{xx}$:

$$u_t = \frac{1}{2}AT^{1/2}(T-t)^{-3/2}e^{-x^2/4(T-t)} - \frac{1}{4}AT^{1/2}x^2(T-t)^{-5/2}e^{-x^2/4(T-t)},$$

$$u_x = -\frac{1}{2}AT^{1/2}x(T-t)^{-3/2}e^{-x^2/4(T-t)},$$

$$u_{xx} = -\frac{1}{2}AT^{1/2}(T-t)^{-1/2}e^{-x^2/4(T-t)} + \frac{1}{4}AT^{1/2}x^2(T-t)^{-5/2}e^{-x^2/4(T-t)} = -u_t$$

so that $u_t = -u_{xx}$.

Note that with this solution $|u(0, x)| \leq A$ and as $t \rightarrow T$, $u(t, 0) \rightarrow \infty$.

Hence, given any $\epsilon > 0$ and any position (T, X) , the solution

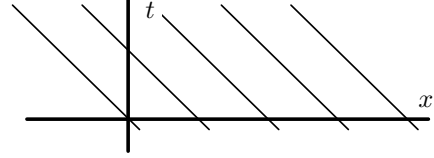
$$u = \epsilon T^{1/2}(T-t)^{-1/2}e^{-(x-X)^2/4(T-t)}, \text{ satisfying the initial condition}$$

$$u(0, x) = \epsilon e^{-(x-X)^2/4T}, \text{ for which } |u(0, x)| \leq \epsilon, \text{ becomes infinite as } (t, x) \rightarrow (T, X).$$

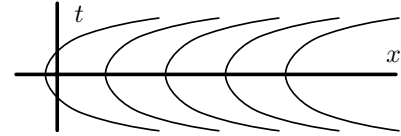
Because of this, the negative heat equation is ill-posed for $t > 0$ when subjected to initial conditions at $t = 0$; we can always find solutions that are arbitrarily small initially but that become infinite at any chosen time later on.

The negative heat equation is well posed when subjected to *final conditions* at some time (say) $t = t_f$ for times *before* the final time, $t < t_f$.

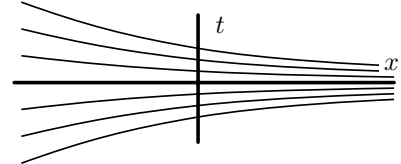
2. (a) $u_t - u_x = 0$: $\frac{dt}{1} = \frac{dx}{-1} = \frac{du}{0}$ so
 $\frac{dx}{dt} = -1$ and $\frac{du}{dt} = 0$ giving, in terms
of (t, k) : $x = k - t$ and $u = A(k)$.
In terms of (t, x) : $u = A(x + t)$.



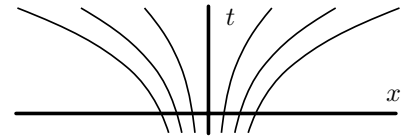
- (b) $u_t + tu_x = u$: $\frac{dt}{1} = \frac{dx}{t} = \frac{du}{u}$ so
 $\frac{dx}{dt} = t$ and $\frac{du}{dt} = u$ giving, in terms
of (t, k) : $x = k + \frac{1}{2}t^2$ and $u = A(k)e^t$.
In terms of (t, x) : $u = A(x - \frac{1}{2}t^2)e^t$.



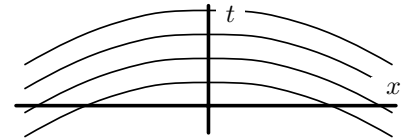
- (c) $tu_t - u_x = 1$: $\frac{dt}{t} = \frac{dx}{-1} = \frac{du}{1}$ so
 $\frac{dt}{dx} = -t$ and $\frac{du}{dx} = -1$, giving, in terms
of (x, k) : $t = ke^{-x}$ and $u = A(k) - x$.
In terms of (t, x) : $u = A(te^x) - x$.



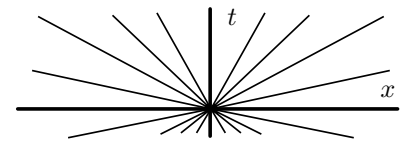
- (d) $u_t + xu_x = -u$: $\frac{dt}{1} = \frac{dx}{x} = \frac{du}{-u}$ so
 $\frac{dx}{dt} = x$ and $\frac{du}{dt} = -u$ giving, in terms
of (t, k) : $x = ke^t$ and $u = A(k)e^{-t}$.
In terms of (t, x) : $u = A(xe^{-t})e^{-t}$.



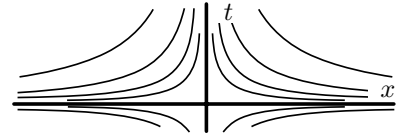
- (e) $xu_t - u_x = t$: $\frac{dt}{x} = \frac{dx}{-1} = \frac{du}{t}$ so $\frac{dt}{dx} = -x$
and $\frac{du}{dx} = -t$ giving, $t = k - \frac{1}{2}x^2$ and so
 $\frac{du}{dx} = -k + \frac{1}{2}x^2$. Hence in terms of (x, k) :
 $t = k - \frac{1}{2}x^2$ and $u = A(k) - kx + \frac{1}{6}x^3$.
In terms of (t, x) :
 $u = A(t + \frac{1}{2}x^2) - (t + \frac{1}{2}x^2)x + \frac{1}{6}x^3$.



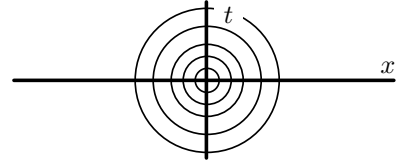
- (f) $tu_t + xu_x = x$: $\frac{dt}{t} = \frac{dx}{x} = \frac{du}{x}$ so
 $\int \frac{dt}{t} = \int \frac{dx}{x}$ and $\frac{du}{dx} = 1$ giving, in terms
of (x, k) : $t = kx$ and $u = A(k) + x$.
In terms of (t, x) : $u = A(t/x) + x$.



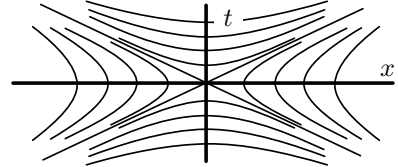
- (g) $tu_t - xu_x = t$: $\frac{dt}{t} = \frac{dx}{-x} = \frac{du}{t}$ so
 $\int \frac{dt}{t} = -\int \frac{dx}{x}$ and $\frac{du}{dt} = 1$ giving, in terms
of (t, k) : $x = k/t$ and $u = A(k) + t$.
In terms of (t, x) : $u = A(tx) + t$.



- (h) $xu_t - tu_x = xt$: $\frac{dt}{x} = \frac{dx}{-t} = \frac{du}{xt}$ so
 $\int t dt = -\int x dx$ and $\frac{du}{dt} = t$ giving, in terms
of (t, k) : $x^2 = k - t^2$ and $u = A(k) + \frac{1}{2}t^2$.
In terms of (t, x) : $u = A(x^2 + t^2) + \frac{1}{2}t^2$.



- (i) $xu_t + tu_x = -xu$: $\frac{dt}{x} = \frac{dx}{t} = \frac{du}{-xu}$ so
 $\int t dt = \int x dx$ and $\frac{du}{dt} = -u$ giving, in terms
of (t, k) : $x^2 = k + t^2$ and $u = A(k)e^{-t}$.
In terms of (t, x) : $u = A(x^2 - t^2)e^{-t}$.



3. Note that the solutions below are only valid for those values of (t, x) where characteristics passing through (t, x) also pass through the given boundary data without either going through infinity or crossing other characteristics. Check this against the sketches of the paths of the characteristics

- (a) $u = A(x + t)$ with $u(0, x) = \cos(x)$ gives $A(x) = \cos(x)$.
Hence $u = \cos(x + t)$, for all values of (t, x) .
- (b) $u = A(x - \frac{1}{2}t^2)e^{-t}$ with $u(0, x) = \sin(x)$ gives $A(x) = \sin(x)$.
Hence $u = \sin(x - \frac{1}{2}t^2)e^{-t}$, for all values of (t, x) .
- (c) $u = A(te^x) - x$ with $u(t, 0) = \exp(-t^2)$ gives $A(t) = \exp(-t^2)$.
Hence $u = \exp(-t^2e^{2x}) - x$, for all values of (t, x) .
- (d) $u = A(xe^{-t})e^{-t}$ with $u(0, x) = x^2$ gives $A(x) = x^2$.
Hence $u = x^2e^{-3t}$, for all values of (t, x) .
- (e) $u = A(t + \frac{1}{2}x^2) - (t + \frac{1}{2}x^2)x + \frac{1}{6}x^3$ with $u(t, 0) = \ln(1 + t^2)$ gives $A(t) = \ln(1 + t^2)$.
Hence $u = \ln[1 + (t + \frac{1}{2}x^2)^2] - (t + \frac{1}{2}x^2)x + \frac{1}{6}x^3$, for all values of (t, x) .
- (f) $u = A(t/x) + x$ with $u(1, x) = x^3$ gives $A(1/x) + x = x^3$ or $A(z) = z^{-3} - 1/z$.
Hence $u = (x/t)^3 - x/t + x$, for $t > 0$.
- (g) $u = A(tx) + t$ with $u(1, x) = 1/(1 + x^2)$ gives $A(x) + 1 = 1/(1 + x^2)$ or
 $A(x) = -x^2/(1 + x^2)$.
Hence $u = -x^2t^2/(1 + x^2t^2) + t$, for $t \geq 0$.
- (h) $u = A(x^2 + t^2) + \frac{1}{2}t^2$ with $u(0, x) = 1 + x$ for $x \geq 0$ gives $A(x^2) = 1 + x$ or
 $A(z) = 1 + \sqrt{z}$.
Hence $u = 1 + \sqrt{x^2 + t^2} + \frac{1}{2}t^2$, for all values of (t, x) .
- (i) $u = A(x^2 - t^2)e^{-t}$ with $u(0, x) = 1 - x$ for $x \geq 0$ gives $A(x^2) = 1 - x$ or
 $A(z) = 1 - \sqrt{z}$.
Hence $u = (1 - \sqrt{x^2 - t^2})e^{-t}$, for $x \geq |t|$.

Partial Differential Equations — Problem Sheet 4

1. Find general solutions for the following PDEs satisfied by $u(t, x)$ for $t \geq 0$ in terms of t and a characteristic variable.

- (a) $u_t + (u^2)_x = 0$
- (b) $u_t + uu_x = 2$
- (c) $u_t + \frac{1}{2}(u^2)_x = -u$
- (d) $u_t + (\ln u)_x = u$
- (e) $u_t + (e^u)_x = \frac{1}{1+t}$
- (f) $uu_t + u^3u_x = t$

2. For each of the PDEs given in question 1, how fast would a conserving shock propagate, given values of u immediately ahead of and behind the shock.
3. True or false? Explain your answers fully:

A conserving shock in the PDE	propagates at the speed
$u_t + (\ln u)_x = \ln x$	$[\ln u]/[u \ln u - u]$
$u_t + u^2u_x = u^3$	$\frac{1}{3}[u^3]/[u]$
$u_t + e^u u_x = 9$	$9[e^u]/[u]$
$u_t + \frac{u_x}{u} = \sin(xt)$	$-[\ln u]/[u]$
$u_t + nu^{n-1}u_x = xte^u, n \neq 1$	$[u^n]/[u]$

4. Solve the initial value problem for $u(t, x)$ at times $t \geq 0$ in terms of t and a characteristic variable:

$$u_t + \frac{u_x}{u} = 0 \quad \text{with} \quad u(0, x) = u_0(x) = \frac{1+x^2}{2+x^2}$$

Do characteristics cross for any $t \geq 0$ and if so where and when?

5. Solve the initial value problem for $u(t, x)$ at times $t \geq 0$ in terms of t and a characteristic variable:

$$u_t + uu_x = 0 \quad \text{with} \quad u(0, x) = u_0(x) = \frac{1+x^2}{2+x^2}$$

Do characteristics cross for any $t \geq 0$ and if so where and when?

6. Solve the initial value problem for $u(t, x)$ at times $t \geq 0$ in terms of t and a characteristic variable:

$$u_t + u^{1/2}u_x = 0 \quad \text{with} \quad u(0, x) = u_0(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ x^2 & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Do characteristics cross for any $t \geq 0$ and if so where and when?

7. Solve the initial value problem for $u(t, x)$ at times $t \geq 0$ in terms of t and a characteristic variable:

$$u_t + uu_x = 1 \quad \text{with} \quad u(0, x) = u_0(x) = 1 - \frac{1}{2} \tanh(x)$$

Do characteristics cross for any $t \geq 0$ and if so where and when?

Partial Differential Equations — Answer Sheet 4

1. (a) $u_t + (u^2)_x = 0$: $u_t + 2uu_x = 0$ so $\frac{dt}{1} = \frac{dx}{2u} = \frac{du}{0}$ and so $\frac{du}{dt} = 0 \implies u = A(k)$ and $\frac{dx}{dt} = 2u = 2A(k) \implies x = k + 2A(k)t$. Hence, in terms of t and the characteristic variable k : $u = A(k)$ with $x = k + 2A(k)t$.
 - (b) $u_t + uu_x = 2$: so $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{2}$ and so $\frac{du}{dt} = 2 \implies u = A(k) + 2t$ and $\frac{dx}{dt} = u = A(k) + 2t \implies x = k + A(k)t + t^2$. Hence, in terms of t and the characteristic variable k : $u = A(k) + 2t$ with $x = k + A(k)t + t^2$.
 - (c) $u_t + \frac{1}{2}(u^2)_x = -u$: $u_t + uu_x = -u$ so $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{-u}$ and so $\frac{du}{dt} = -u \implies u = A(k)e^{-t}$ and $\frac{dx}{dt} = u = A(k)e^{-t} \implies x = k - A(k)e^{-t}$. Hence, in terms of t and the characteristic variable k : $u = A(k)e^{-t}$ with $x = k - A(k)e^{-t}$.
 - (d) $u_t + (\ln u)_x = u$: $u_t + \frac{u_x}{u} = u$ so $\frac{dt}{1} = u dx = \frac{du}{u}$ and so $\frac{du}{dt} = u \implies u = A(k)e^t$ and $\frac{dx}{dt} = \frac{1}{u} = e^{-t}/A(k) \implies x = k - e^{-t}/A(k)$. Hence, in terms of t and the characteristic variable k : $u = A(k)e^t$ with $x = k - e^{-t}/A(k)$.
 - (e) $u_t + (e^u)_x = \frac{1}{1+t}$: $u_t + e^u u_x = \frac{1}{1+t}$ so $\frac{dt}{1} = \frac{dx}{e^u} = (1+t)du$ and so $\frac{du}{dt} = \frac{1}{1+t} \implies u = A(k) + \ln(1+t)$ and $\frac{dx}{dt} = e^u = e^{A(k)}(1+t) \implies x = k + e^{A(k)}(t + \frac{1}{2}t^2)$. Hence, in terms of t and the characteristic variable k : $u = A(k) + \ln(1+t)$ with $x = k + e^{A(k)}(t + \frac{1}{2}t^2)$.
 - (f) $uu_t + u^3 u_x = t$: so $\frac{dt}{u} = \frac{dx}{u^3} = \frac{du}{t}$ and so $\frac{du}{dt} = \frac{t}{u} \implies u^2 = A(k) + t^2$ and $\frac{dx}{dt} = u^2 = A(k) + t^2 \implies x = k + A(k)t + \frac{1}{3}t^3$. Hence, in terms of t and the characteristic variable k : $u = \pm \sqrt{A(k) + t^2}$ with $x = k + e^{A(k)}(t + \frac{1}{2}t^2)$.
2. (a) $u_t + (u^2)_x = 0$ is in conservation form. A conserving shock at $x = s(t)$ would travel at speed $\frac{ds}{dt} = [u^2]/[u]$ (which equals $u^+ + u^-$).
 - (b) $u_t + uu_x = 2$ becomes $u_t + \frac{1}{2}(u^2)_x = 2$ in conservation form. A conserving shock at $x = s(t)$ would travel at speed $\frac{ds}{dt} = [\frac{1}{2}u^2]/[u]$ (which equals $\frac{1}{2}(u^+ + u^-)$).
 - (c) $u_t + \frac{1}{2}(u^2)_x = -u$ is in conservation form. A conserving shock at $x = s(t)$ would travel at speed $\frac{ds}{dt} = [\frac{1}{2}u^2]/[u]$ (which equals $\frac{1}{2}(u^+ + u^-)$).
 - (d) $u_t + (\ln u)_x = u$ is in conservation form. A conserving shock at $x = s(t)$ would travel at speed $\frac{ds}{dt} = [\ln u]/[u]$.
 - (e) $u_t + (e^u)_x = \frac{1}{1+t}$ is in conservation form. A conserving shock at $x = s(t)$ would travel at speed $\frac{ds}{dt} = [e^u]/[u]$.
 - (f) $uu_t + u^3 u_x = t$ becomes $u_t + \frac{1}{3}(u^3)_x = \frac{t}{u}$ in conservation form. A conserving shock at $x = s(t)$ travels at speed $\frac{ds}{dt} = [\frac{1}{3}u^3]/[u]$, the same as $\frac{1}{3}(u^{+2} + u^+u^- + u^{-2})$.
3.
 - $u_t + (\ln u)_x = \ln x$ is in conservation form. A conserving shock would travel at speed $[\ln u]/[u] \neq [\ln u]/[u \ln u - u]$. (False)
 - $u_t + u^2 u_x = u^3$ becomes $u_t + (\frac{1}{3}u^3)_x = u^3$ in conservation form. A conserving shock would travel at speed $[\frac{1}{3}u^3]/[u] = \frac{1}{3}[u^3]/[u]$. (True)
 - $u_t + e^u u_x = 9$ becomes $u_t + (e^u)_x = 9$ in conservation form. A conserving shock would travel at speed $[e^u]/[u] \neq 9[e^u]/[u]$. (False)
 - $u_t + \frac{u_x}{u} = \sin(xt)$ becomes $u_t + (\ln |u|)_x = \sin(xt)$ in conservation form. A conserving shock would travel at speed $[\ln |u|]/[u] \neq -[\ln u]/[u]$. (False)
 - $u_t + nu^{n-1}u_x = xte^u$ becomes $u_t + (u^n)_x = xte^u$ in conservation form. A conserving shock would travel at speed $[u^n]/[u]$. (True)

Note: if $n = 1$ or $n = 0$ the problem is semilinear and shocks do not arise. Otherwise, n can have any real value.

4. $u_t + \frac{u_x}{u} = 0$ so $\frac{dt}{1} = u dx = \frac{du}{0}$ and so $\frac{du}{dt} = 0 \implies u = A(k)$
and $\frac{dx}{dt} = \frac{1}{u} = 1/A(k) \implies x = k + t/A(k)$.
Hence, in terms of t and k : $u = A(k)$ with $x = k + t/A(k)$.
Initially, at $t = 0$, $x = k$ and $u = A(x) = u_0(x) = \frac{1+x^2}{2+x^2}$
so $u = \frac{1+k^2}{2+k^2}$ with $x = k + t \frac{2+k^2}{1+k^2}$.
Characteristics cross if $0 = x_k = 1 + t \frac{(1+k^2)2k - (2+k^2)2k}{(1+k^2)^2} = 1 - t \frac{2k}{(1+k^2)^2}$.
That is, when $t = \frac{(1+k^2)^2}{2k}$, for $k > 0$, at $x = k + \frac{(1+k^2)^2}{2k} \frac{2+k^2}{1+k^2} = k + \frac{(1+k^2)(2+k^2)}{2k}$.
5. $u_t + uu_x = 0$ so $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$ and so $\frac{du}{dt} = 0 \implies u = A(k)$
and $\frac{dx}{dt} = u = A(k) \implies x = k + A(k)t$.
Hence, in terms of t and k : $u = A(k)$ with $x = k + A(k)t$.
Initially, at $t = 0$, $x = k$ and $u = A(x) = u_0(x) = \frac{1+x^2}{2+x^2}$ so $u = \frac{1+k^2}{2+k^2}$ with $x = k + t \frac{1+k^2}{2+k^2}$.
Characteristics cross if $0 = x_k = 1 + t \frac{(2+k^2)2k - (1+k^2)2k}{(2+k^2)^2} = 1 + t \frac{2k}{(2+k^2)^2}$.
That is, when $t = -\frac{(2+k^2)^2}{2k}$, for $k < 0$, at $x = k - \frac{(2+k^2)^2}{2k} \frac{1+k^2}{2+k^2} = k - \frac{(1+k^2)(2+k^2)}{2k}$.
6. $u_t + u^{1/2}u_x = 0$ so $\frac{dt}{1} = \frac{dx}{u^{1/2}} = \frac{du}{0}$ and so $\frac{du}{dt} = 0 \implies u = A(k)$
and $\frac{dx}{dt} = u^{1/2} = \sqrt{A(k)} \implies x = k + t\sqrt{A(k)}$.
Hence, in terms of t and k : $u = A(k)$ with $x = k + t\sqrt{A(k)}$.
Initially, at $t = 0$, $x = k$ and $u = A(x) = u_0(x) = \begin{cases} 1 & \text{for } |x| \geq 1 \\ x^2 & \text{for } |x| \leq 1 \end{cases}$
so $u = \begin{cases} 1 & \text{for } |k| \geq 1 \\ k^2 & \text{for } |k| \leq 1 \end{cases}$ with $x = k + t \times \begin{cases} 1 & \text{for } |k| \geq 1 \\ |k| & \text{for } |k| \leq 1. \end{cases}$
Characteristics cross if $0 = x_k = 1 + t \times \begin{cases} 0 & \text{for } |k| > 1 \\ 1 & \text{for } 0 < k < 1 \\ -1 & \text{for } -1 < k < 0. \end{cases}$
That is, when $t = 1$, for $-1 < k < 0$, at $x = k + t|k| = 0$.
7. $u_t + uu_x = 1$ so $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{1}$ and so $\frac{du}{dt} = 1 \implies u = A(k) + t$
and $\frac{dx}{dt} = u = A(k) + t \implies x = k + A(k)t + \frac{1}{2}t^2$.
Hence, in terms of t and k : $u = A(k) + t$ with $x = k + A(k)t + \frac{1}{2}t^2$.
Initially, at $t = 0$, $x = k$ and $u = A(x) = u_0(x) = 1 - \frac{1}{2} \tanh x$
so $u = 1 - \frac{1}{2} \tanh k + t$ with $x = k + t(1 - \frac{1}{2} \tanh k) + \frac{1}{2}t^2$.
Characteristics cross if $0 = x_k = 1 - \frac{1}{2}t \operatorname{sech}^2 k$.
That is, when $t = 2/\operatorname{sech}^2 k$, for any $k \in \mathbb{R}$, at $x = k + \frac{2 - \tanh k}{\operatorname{sech}^2 k} + 2/\operatorname{sech}^4 k$.

Partial Differential Equations — Problem Sheet 5

1. Completely solve the initial value problem describing $u(t, x)$ for $t \geq 0$

$$u_t + uu_x = 1 - u \quad \text{with} \quad u(0, x) = \pi + \tan^{-1} x$$

in terms of t and a characteristic variable.

(You must demonstrate that your solution is valid for all $t \geq 0$).

2. Solve the initial value problem for $u(t, x)$ at times $t \geq 0$ in terms of t and a characteristic variable:

$$u_t + u^{1/3}u_x = 0 \quad \text{with} \quad u(0, x) = u_0(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ -x^3 & \text{for } -1 \leq x \leq 0 \\ 0 & \text{for } x \geq 0 \end{cases}$$

Where and when do characteristics cross?

Assuming that a conserving shock is formed, find the full solution after the birth of the shock.

3. Consider the initial value problem describing $u(t, x)$ for $t \geq 0$

$$u_t + uu_x = 0 \quad \text{with} \quad u(0, x) = \begin{cases} 1 - \sqrt{1+x} & \text{if } x \geq 0 \\ \sqrt{1-x} - 1 & \text{if } x < 0 \end{cases}$$

- (a) Sketch the initial data. Is the data anti-symmetric about any value of x ?
 (b) Show that the solution, in terms of t and a characteristic variable k , can be written as

$$u = \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0, \end{cases} \quad x = k + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0. \end{cases}$$

- (c) i. Sketch the way in which $x(t, k)$ varies with k as t increases.
 ii. Sketch the way in which $u(t, x)$ varies with x as t increases.

State what symmetry properties are maintained by these solutions.

- (d) Show that characteristics first cross when $t = 2$, at $k = 0$, where $x = 0$.
 (e) Anticipating that a conserving shock forms and follows $x = s(t)$ for $t \geq 2$, and that $u^+ = -u^-$ across the shock, show from the conservation form of the PDE that we should have $\frac{ds}{dt} = 0$. Hence find the path that the shock would follow.
 (f) Show that for $t \geq 2$, the values of k for which $x = 0$ are:

$$k = 0 \quad \text{or} \quad k = k^+(t) = t(2-t) \quad \text{or} \quad k = k^-(t) = -t(2-t).$$

Hence show that $u^+ = -u^-$ for a shock at $x = s(t) = 0$ for $t \geq 2$.

Hence show that the full solution for $t \geq 2$ is given by

$$u = \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t), \end{cases} \quad x = k + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t). \end{cases}$$

4. Consider the initial value problem describing $u(t, x)$ for $t \geq 0$

$$u_t + uu_x = 1 \quad \text{with} \quad u(0, x) = \begin{cases} 1 - \sqrt{1+x} & \text{if } x \geq 0 \\ \sqrt{1-x} - 1 & \text{if } x < 0 \end{cases}$$

(a) Show that

$$u = t + \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0, \end{cases} \quad x = k + \frac{1}{2}t^2 + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0. \end{cases}$$

(b) By defining $u(t, x) = t + v(t, z)$ with $x = \frac{1}{2}t^2 + z$ show that

$$v_t + vv_z = 0 \quad \text{with} \quad v(0, z) = \begin{cases} 1 - \sqrt{1+z} & \text{if } z \geq 0 \\ \sqrt{1-z} - 1 & \text{if } z < 0 \end{cases}$$

(c) Hence deduce from your answer to Question 1 that a conserving shock follows the path $x = s(t) = \frac{1}{2}t^2$ for $t \geq 2$ and that the full solution for $t \geq 2$ is given by

$$u = t + \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t), \end{cases} \quad x = k + \frac{1}{2}t^2 + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t). \end{cases}$$

5. In the following 1st order initial value problems describing $u(t, x)$ for $t \geq 0$, the initial data takes the discontinuous (piecewise constant) initial form

$$u(0, x) = \begin{cases} u_1 & \text{if } x < x_0 \\ u_2 & \text{if } x \geq x_0 \end{cases}$$

(a) $u_t + uu_x = 0 \quad u_1 = 1 \quad u_2 = 0 \quad x_0 = -1$

(b) $u_t + \frac{1}{4}u^3u_x = 0 \quad u_1 = -1 \quad u_2 = 1 \quad x_0 = 2$

(c) $u_t + u^{2/3}u_x = 0 \quad u_1 = 27 \quad u_2 = 8 \quad x_0 = 0$

(d) $u_t + \frac{u_x}{u^2} = 0 \quad u_1 = 2 \quad u_2 = 3 \quad x_0 = 1$

(e) $u_t + u^3u_x = 0 \quad u_1 = 1 \quad u_2 = -1 \quad x_0 = 0$

(f) $u_t + 2uu_x = 0 \quad u_1 = 0 \quad u_2 = 1 \quad x_0 = 1$

(g) $u_t + \frac{u_x}{u^2} = 0 \quad u_1 = 3 \quad u_2 = 2 \quad x_0 = -1$

(h) $u_t + \frac{3}{5}u^{2/3}u_x = 0 \quad u_1 = 8 \quad u_2 = 27 \quad x_0 = 2$

In each case, find a solution, either in the form of a propagating shock or in the form of an expansion fan. Explain why each solution takes the form it has.

(Assume that shocks conserve the quantity u).

6. Solve the initial value problem

$$u_t + (u(2-u))_x = 0 \quad \text{with} \quad \begin{cases} 0 & \text{if } x < -2 \\ 2 & \text{if } -2 \leq x \leq 0 \\ 0 & \text{if } x > 0. \end{cases}$$

7. Some shocks are non-conservative. That is, they alter the quantity that is discontinuous across them as they propagate. For example a sonic-boom creates entropy.

Given the PDE $u_t + (f(u))_x = g$ with only one shock at $x = s(t)$ within the range $a < x < b$, and assuming that the shock “produces” u at the rate

$$\gamma[u] \frac{ds}{dt}$$

show that the shock-speed is given by

$$\frac{ds}{dt} = \frac{1}{1 + \gamma} \frac{[f(u)]}{[u]}.$$

Hint:- You may need to use the formula

$$\frac{d}{dt} \int_{x_1}^{x_2} u \, dx = \int_{x_1}^{x_2} u_t \, dx + \frac{dx_2}{dt} v(t, x_2) - \frac{dx_1}{dt} u(t, x_1)$$

Partial Differential Equations — Answer Sheet 5

1. $u_t + uu_x = 1 - u$ with $u(0, x) = \pi + \tan^{-1} x$: On a characteristic $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{1-u}$. Hence ... $u = 1 + A(k)e^{-t}$ and $\frac{dx}{dt} = u = 1 + A(k)e^{-t}$ so that $x = k + t - A(k)(e^{-t} - 1)$ (choosing the "constant of integration" to be $k + A(k)$). Hence, at $t = 0$: $x = k$ and $u(0, x) = 1 + A(x) = \pi + \tan^{-1} x \implies A(x) = \pi - 1 + \tan^{-1} x$. Thus the solution becomes

$$u = 1 + e^{-t}(\pi - 1 + \tan^{-1} k) \quad \text{with} \quad x = k + t + (1 - e^{-t})(\pi - 1 + \tan^{-1} k).$$

Characteristics cross if $x_k = 0$. We have ... $x_k = 1 + \frac{1-e^{-t}}{1+k^2}$ which is positive for all k and all $t \geq 0$. Thus no shocks form and the solution remains valid for all $t \geq 0$.

2. $u_t + u^{1/3}u_x = 0$: On a characteristic $\frac{dt}{1} = \frac{dx}{u^{1/3}} = \frac{du}{0}$ so $u = A(k)$ and $\frac{dx}{dt} = u^{1/3} = A^{1/3}(k)$ so that $x = k + tA^{1/3}(k)$. At $t = 0$: $x = k$ so $u(0, x) = A(x) = u_0(x)$. Thus, in terms of t and k :
- $$u = \begin{cases} 1 & \text{for } k \leq -1 \\ -k^3 & \text{for } -1 \leq k \leq 0 \\ 0 & \text{for } k \geq 0 \end{cases} \quad \text{with} \quad x = k + t \times \begin{cases} 1 & \text{for } k \leq -1 \\ -k & \text{for } -1 \leq k \leq 0 \\ 0 & \text{for } k \geq 0. \end{cases}$$

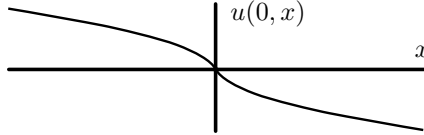
Characteristics cross when $x_k = 0$.

$$\text{Since} \quad x_k = 1 + t \times \begin{cases} 0 & \text{for } k \leq -1 \\ -1 & \text{for } -1 \leq k \leq 0 \\ 0 & \text{for } k \geq 0. \end{cases}$$

this only happens for $-1 \leq k \leq 0$ when $t = 1$ where ... $x = 0$. So, at $t = 1$ we have $u(1, x) = 1$ for $x < 0$ and $u(1, x) = 0$ for $x > 0$. The PDE is $u_t + (\frac{3}{4}u^{4/3})_x = 0$ in conservation form, so that a shock at $x = s(t)$ has speed $\frac{ds}{dt} = [\frac{3}{4}u^{4/3}]/[u]$, and with $u^+ = 0$, $u^- = 1$ this gives $\frac{ds}{dt} = \frac{3}{4}$. Since $s(1) = 0$ we have $s(t) = \frac{3}{4}(t - 1)$ for $t \geq 1$. Thus, after the birth of the shock (for $t \geq 1$):

$$u = \begin{cases} 1 & \text{for } x < \frac{3}{4}t \\ 0 & \text{for } x > \frac{3}{4}t. \end{cases}$$

3. (a) The initial data are antisymmetric about $x = 0$

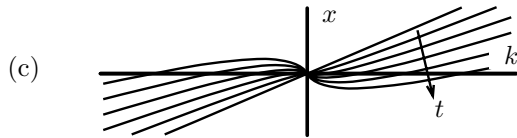


- (b) $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$:

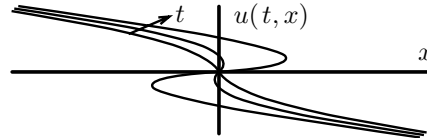
Hence $\frac{du}{dt} = 0 \implies u = A(k)$ and $\frac{dx}{dt} = u = A(k) \implies x = k + A(k)t$.

At $t = 0$: $x = k$ so that $u(0, x) = A(x) = \begin{cases} 1 - \sqrt{1+x} & \text{if } x \geq 0 \\ \sqrt{1-x} - 1 & \text{if } x < 0. \end{cases}$

Hence: $u = \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0, \end{cases} \quad x = k + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0. \end{cases}$



x is anti-symmetric about $k = 0$ for any $t \geq 0$.



u remains anti-symmetric about $x = 0$ for any $t \geq 0$

- (d) Characteristics cross (i.e. $x(t, k)$ becomes multivalued) when $x_k = 0$.

That is $0 = 1 + t \times \begin{cases} -\frac{1/2}{\sqrt{1+k}} & \text{if } k \geq 0 \\ -\frac{1/2}{\sqrt{1-k}} & \text{if } k < 0 \end{cases} \implies 0 = 1 - \frac{t/2}{\sqrt{1+|k|}}$ or $t = 2\sqrt{1+|k|}$. So

characteristics first cross when $t = 2$, at $k = 0$, where (*exercise*) also $x = 0$.

- (e) Conservation form of PDE is $u_t + (\frac{1}{2}u^2)_x = 0$. Hence the speed of a shock at $x = s(t)$, for $t \geq 2$, is $\frac{ds}{dt} = [\frac{1}{2}u^2]/[u]$ and so (*exercise*) $\frac{ds}{dt} = \frac{1}{2}(u^+ + u^-)$.

Anticipating that $u^+ = -u^-$ we would then have

$\frac{ds}{dt} = \frac{1}{2}(u^+ + u^-) = 0$ with $s(2) = 0 \implies s(t) = 0 \quad \forall t \geq 2$.

(f) At $x = 0$, k satisfies $0 = k + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0 \end{cases}$ or $\frac{|k|}{t} = \sqrt{1+|k|} - 1$. Hence

$(\frac{|k|}{t} + 1)^2 = 1 + |k|$ and so (*exercise*) $k = 0$ or $k = k^\pm = \pm t(2-t)$ for $t \geq 2$. Hence

$u^+ = u(t, k^+) = 1 - \sqrt{1+k^+} = 1 - \sqrt{1+t(2-t)}$
and $u^- = u(t, k^-) = \sqrt{1-k^-} - 1 = \sqrt{1+t(2-t)} - 1$.

We thus see that, indeed we have $u^+ = -u^-$.

It follows that, for $t \geq 2$, the full solution is given by

$$u = \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t), \end{cases} \quad x = k + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t). \end{cases}$$

4. $u_t + uu_x = 1$ with $u(0, x) = \begin{cases} 1 - \sqrt{1+x} & \text{if } x \geq 0 \\ \sqrt{1-x} - 1 & \text{if } x < 0: \end{cases}$

(a) $\frac{dt}{1} = \frac{dx}{u} = \frac{du}{1}$:

Hence $\frac{du}{dt} = 1 \implies u = t + A(k)$ and $\frac{dx}{dt} = u = t + A(k) \implies x = k + \frac{1}{2}t^2 + A(k)t$.

At $t = 0$: $x = k$ so that $u(0, x) = A(x) = \begin{cases} 1 - \sqrt{1+x} & \text{if } x \geq 0 \\ \sqrt{1-x} - 1 & \text{if } x < 0. \end{cases}$

Hence: $u = t + \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0, \end{cases} \quad x = k + \frac{1}{2}t^2 + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq 0 \\ \sqrt{1-k} - 1 & \text{if } k < 0. \end{cases}$

(b) Writing $u(t, x) = t + v(t, z(t, x))$ with $z(t, x) = x - \frac{1}{2}t^2$, the chain rule

gives $u_t = 1 + v_t + v_z z_t$ and $u_x = v_z z_x$, having $z_t = -t$ and $z_x = 1$.

Hence $u_t + uu_x = 1$ becomes $1 + v_t - tv_z + (t + v)v_z = 1$ or $v_t + vv_z = 0$.

At $t = 0$: $u = v$ and $x = z$ so that $v(0, z) = u(0, z) = \begin{cases} 1 - \sqrt{1+z} & \text{if } z \geq 0 \\ \sqrt{1-z} - 1 & \text{if } z < 0. \end{cases}$

(c) Question 1, is identical to the problem for $v(t, z)$.

It follows that a shock appears, following $z = 0$ for $t \geq 2$,

annihilating characteristics in the range $-t(2-t) \leq k < t(2-t)$.

In terms of x and u the shock follows $x = s(t) = \frac{1}{2}t^2$ for $t \geq 0$,

and the full solution for $t \geq 2$ satisfies:

$$u = t + \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t), \end{cases} \quad x = k + \frac{1}{2}t^2 + t \times \begin{cases} 1 - \sqrt{1+k} & \text{if } k \geq t(2-t) \\ \sqrt{1-k} - 1 & \text{if } k < -t(2-t). \end{cases}$$

5. (a) The PDE $u_t + uu_x = 0$ has characteristic speed $f'(u) = u$. For a shock we require $f'(u_1) > f'(u_2)$ or $1 > 0$, which is true. In conservation form, $u_t + (\frac{1}{2}u^2)_x = 0$ so a shock at $x = s(t)$ has speed $\frac{ds}{dt} = [\frac{1}{2}u^2]/[u] = \frac{1}{2}$, with $s(0) = x_0 = -1$, and so $s = -1 + \frac{1}{2}t$.

The solution is $u = \begin{cases} 1 & \text{for } x + 1 < \frac{1}{2}t \\ 0 & \text{for } x + 1 > \frac{1}{2}t. \end{cases}$

(b) The PDE $u_t + \frac{1}{4}u^3 u_x = 0$ has characteristic speed $f'(u) = \frac{1}{4}u^3$. For a shock we require $f'(u_1) > f'(u_2)$ or $\frac{1}{4}(-1)^3 > \frac{1}{4}1^3$, which is false. An expansion fan with $x - 2 = kt$ has $f'(u) = \frac{1}{4}u^3 = k$ so $u = (4k)^{1/3}$. Front characteristic has $u = 1$, so $k = \frac{1}{4}1^3 = \frac{1}{4}$, and rear characteristic has $u = -1$, so $k = \frac{1}{4}(-1)^3 = -\frac{1}{4}$.

Thus the solution is $u = \begin{cases} -1 & \text{for } x - 2 \leq -\frac{1}{4}t \\ (4\frac{x-2}{t})^{1/3} & \text{for } -\frac{1}{4}t \leq x - 2 \leq \frac{1}{4}t \\ 1 & \text{for } x - 2 \geq \frac{1}{4}t. \end{cases}$

(c) The PDE $u_t + u^{2/3}u_x = 0$ has characteristic speed $f'(u) = u^{2/3}$. For a shock we require $f'(u_1) > f'(u_2)$ or $27^{2/3} > 8^{2/3}$, which is true. In conservation form, $u_t + (\frac{3}{5}u^{5/3})_x = 0$ so a shock at $x = s(t)$ has speed $\frac{ds}{dt} = [\frac{3}{5}u^{5/3}]/[u] = \frac{3}{5}(32 - 243)(8 - 27) = \frac{633}{95}$, with $s(0) = 0$, and so $s = \frac{633}{95}t$.

The solution is $u = \begin{cases} 27 & \text{for } x < \frac{633}{95}t \\ 8 & \text{for } x > \frac{633}{95}t. \end{cases}$

- (d) The PDE $u_t + \frac{u_x}{u^2} = 0$ has characteristic speed $f'(u) = u^{-2}$. For a shock we require $f'(u_1) > f'(u_2)$ or $2^{-2} > 3^{-2}$, which is true. In conservation form, $u_t + (-u^{-1})_x = 0$ so a shock at $x = s(t)$ has speed $\frac{ds}{dt} = [-u^{-1}]/[u] = -(3^{-1} - 2^{-1})/(3 - 2) = \frac{1}{6}$, with $s(0) = x_0 = 1$, and so $s = 1 + \frac{1}{6}t$.

$$\text{The solution is } u = \begin{cases} 2 & \text{for } x < 1 + \frac{1}{6}t \\ 3 & \text{for } x > 1 + \frac{1}{6}t. \end{cases}$$

- (e) The PDE $u_t + u^3 u_x = 0$ has characteristic speed $f'(u) = u^3$. For a shock we require $f'(u_1) > f'(u_2)$ or $1^3 > (-1)^3$, which is true. In conservation form, $u_t + (\frac{1}{4}u^4)_x = 0$ so a shock at $x = s(t)$ has speed $\frac{ds}{dt} = [\frac{1}{4}u^4]/[u] = \frac{1}{4}(1 - 1)/(-1 - 1) = 0$, with $s(0) = x_0 = 0$, and so $s = 0$ for all $t \geq 0$.

$$\text{The solution is } u = \begin{cases} 1 & \text{for } x < 0 \\ -1 & \text{for } x > 0. \end{cases}$$

- (f) The PDE $u_t + 2uu_x = 0$ has characteristic speed $f'(u) = 2u$. For a shock we require $f'(u_1) > f'(u_2)$ or $2 \times 0 > 2 \times 1$, which is false. An expansion fan with $x - 1 = kt$ has $f'(u) = 2u = k$ so $u = \frac{1}{2}k$. Front characteristic has $u = 1$, so $k = 2$, and rear characteristic has $u = 0$, so $k = 0$.

$$\text{Thus the solution is } u = \begin{cases} 0 & \text{for } x - 1 \leq 0 \\ \frac{x-1}{2t} & \text{for } 0 \leq x - 1 \leq 2t \\ 1 & \text{for } x - 1 \geq 2t. \end{cases}$$

- (g) The PDE $u_t + \frac{u_x}{u^2} = 0$ has characteristic speed $f'(u) = u^{-2}$. For a shock we require $f'(u_1) > f'(u_2)$ or $3^{-2} > 2^{-2}$, which is false. An expansion fan with $x + 1 = kt$ has $f'(u) = u^{-2} = k$ so $u = k^{-1/2}$. Front characteristic has $u = 2$, so $k = 2^{-2} = \frac{1}{4}$, and rear characteristic has $u = 3$, so $k = 3^{-2} = \frac{1}{9}$.

$$\text{Thus the solution is } u = \begin{cases} 3 & \text{for } x + 1 \leq \frac{1}{9}t \\ (\frac{t}{x+1})^{1/2} & \text{for } \frac{1}{9}t \leq x + 1 \leq \frac{1}{4}t \\ 2 & \text{for } x + 1 \geq \frac{1}{4}t. \end{cases}$$

- (h) The PDE $u_t + \frac{3}{5}u^{2/3}u_x = 0$ has characteristic speed $f'(u) = \frac{3}{5}u^{2/3}$. For a shock we require $f'(u_1) > f'(u_2)$ or $\frac{3}{5}8^{2/3} > \frac{3}{5}27^{2/3}$, which is false. An expansion fan with $x - 2 = kt$ has $f'(u) = \frac{3}{5}u^{2/3} = k$ so $u = (\frac{5}{3}k)^{3/2}$. Front characteristic has $u = 8$, so $k = \frac{3}{5}8^{2/3} = \frac{12}{5}$, and rear characteristic has $u = 27$, so $k = \frac{3}{5}27^{2/3} = \frac{27}{5}$.

$$\text{Thus the solution is } u = \begin{cases} 8 & \text{for } x - 2 \leq \frac{12}{5}t \\ (\frac{5}{3}\frac{x-2}{t})^{3/2} & \text{for } \frac{12}{5}t \leq x - 2 \leq \frac{27}{5}t \\ 27 & \text{for } x - 2 \geq \frac{27}{5}t. \end{cases}$$

6. The initial value problem $u_t + (u(2 - u))_x = 0$ with $\begin{cases} 0 & \text{if } x < -2 \\ 2 & \text{if } -2 \leq x \leq 0 \\ 0 & \text{if } x > 0. \end{cases}$

This PDE has flux $f(u) = u(2 - u)$ and characteristic speed $f'(u) = 2 - 2u$.

For a shock at $x = -2$ we require $f'(0) > f'(2)$ or $0 > -2$ which is true.

The speed of the shock, following $x = s(t)$ is $\frac{ds}{dt} = [u(2 - u)]/[u]$ with $u^+ = 2$ and $u^- = 0$.

Thus $\frac{ds}{dt} = \frac{2(2-2) - 0(2-0)}{2-0} = 0$, so the shock stays at $s = -2$ until u^+ changes.

For a shock at $x = 0$ we require $f'(2) > f'(0)$ or $-2 > 0$ which is false. An expansion fan with $x = kt$ has characteristic speed $f'(u) = 2 - 2u = k$ so $u = 1 - \frac{1}{2}k$. Front characteristic has $u = 0$, so $k = 2$, and rear characteristic has $u = 2$, so $k = -2$. Thus the solution is

$$u = \begin{cases} 0 & \text{for } x < -2 \\ 2 & \text{for } -2 < x \leq -2t \\ 1 - \frac{x}{2t} & \text{for } -2t \leq x \leq 2t \\ 0 & \text{for } x \geq 2t \end{cases} \text{ until } t = 1 \text{ when the characteristic } x = -2t \text{ reaches the rear}$$

shock.

After $t = 1$, the shock speed is $\frac{ds}{dt} = [u(2 - u)]/[u]$ with $u^+ = 1 - \frac{s}{2t}$ and $u^- = 0$.

That is, $\frac{ds}{dt} = \frac{u^+(2-u^+) - 0(2-0)}{u^+ - 0} = 2 - u^+ = 2 - (1 - \frac{s}{2t}) = 1 + \frac{s}{2t}$. That is, $\frac{ds}{dt} = 1 + \frac{s}{2t}$.

Setting $v = s/t$ gives $\frac{ds}{dt} = t\frac{dv}{dt} + v = 1 + v/2$ so that $2t\frac{dv}{dt} = 2 - v$. Hence $\int \frac{dv}{2-v} = \int \frac{dt}{2t}$ or $\ln(2 - v) = -\frac{1}{2}\ln t + C_1$ so $2 - v = Ct^{-1/2}$ or $v = s/t = 2 - Ct^{-1/2}$ and hence $s = 2t - Ct^{1/2}$.

At $t = 1$, $s = -2$ so $C = 4$ giving $s = 2t - 4t^{1/2}$.

Hence, after $t = 1$ the solution is $u = \begin{cases} 0 & \text{for } x < 2t - 4t^{1/2} \\ 1 - \frac{x}{2t} & \text{for } 2t - 4t^{1/2} \leq x \leq 2t \\ 0 & \text{for } x \geq 2t. \end{cases}$

7. Integrating $u_t + (f(u))_x = g$ from a to b , in the absence of any shocks we would have

$$\int_a^b (u_t + f_x) dx = \int_a^b g dx \text{ giving } \frac{d}{dt} \int_a^b u dx = f|_{x=a} - f|_{x=b} + \int_a^b g dx$$

since a and b are fixed. If there is a shock at $x = s(t) \in (a, b)$ then

$$\begin{aligned} \frac{d}{dt} \int_a^b u dx &= \frac{d}{dt} \left(\int_a^s u dx + \int_s^b u dx \right) = (u^- - u^+) \frac{ds}{dt} + \int_a^s u_t dx + \int_s^b u_t dx \\ &= (u^- - u^+) \frac{ds}{dt} + \int_a^s (g - f_x) dx + \int_s^b (g - f_x) dx \\ &= (u^- - u^+) \frac{ds}{dt} - (f^- - f|_{x=a}) - (f|_{x=b} - f^+) + \int_a^b g dx. \end{aligned}$$

For the shock to be producing u at the rate G , this must exceed the formula for $\frac{d}{dt} \int_a^b u dx$ without the shock by G . That is, subtracting the right sides of the two formulae:

$$(u^- - u^+) \frac{ds}{dt} - (f^- - f^+) = G \text{ or } (u^+ - u^-) \frac{ds}{dt} = (f^+ - f^-) - G \text{ or } [u] \frac{ds}{dt} = [f] - G.$$

Given that $G = \gamma [u] \frac{ds}{dt}$ this gives $[u] \frac{ds}{dt} = [f] - \gamma [u] \frac{ds}{dt}$, so that $\frac{ds}{dt} = \frac{1}{1+\gamma} [f]/[u]$.

Partial Differential Equations — Problem Sheet 6

1. Find characteristics for the PDE describing $u(t, x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}$

$$u_{tt} + u_{tx} - 2u_{xx} = t$$

and hence show that the general solution is

$$u = A(x + t) + B(x - 2t) - \frac{1}{18}t(x + t)(x - 2t).$$

Find the solution that satisfies the initial conditions

$$u(0, x) \equiv 0 \quad \text{and} \quad u_t(0, x) \equiv 0.$$

2. Find characteristics for the PDE describing $u(t, x)$ for $t \in \mathbb{R}$ and $x \in [0, \pi]$

$$u_{tt} - u_{xx} = 0$$

and hence show that the general solution takes the form

$$u = A(t + x) + B(t - x).$$

If we are given that $u(t, 0) = 0$ for all t show that the solution takes the form

$$u = A(t + x) - A(t - x).$$

- (a) If we are also given that $u(t, \pi) = 0$ for all t show that the function $A(\cdot)$ must be periodic with period 2π .
- (b) If instead, we were given that $u_x(t, \pi) = 0$ show that the function $A(\cdot)$ must satisfy $A'(z + 2\pi) = -A'(z)$ for any $z \in \mathbb{R}$. Can you sketch such a function?
Show that $A(z) = \text{Const.}$, $A(z) = \sin\left(\frac{2n+1}{2}z\right)$ or $A(z) = \cos\left(\frac{2n+1}{2}z\right)$ are possible functions for $n = 0, 1, 2$, etc.

3. Find suitable characteristic variables for the following PDEs describing $u(t, x)$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}$. In each case write the PDE in terms of the characteristic variables and partial derivatives with respect to the characteristics.

- (a) $u_{tt} + (t - 1)u_{tx} - tu_{xx} = 0$
- (b) $u_{tt} + (1 + x)u_{tx} + xu_{xx} = 0$
- (c) $tu_{tt} + (x - t)u_{tx} - xu_{xx} = 0$
- (d) $xu_{tt} + (x - t)u_{tx} - tu_{xx} = 0$
- (e) $xu_{tt} + (1 + xt)u_{tx} + tu_{xx} = 0$
- (f) $txu_{tt} + (x^2 - t^2)u_{tx} - txu_{xx} = 0$

4. Find where each of the PDEs in question 3 fails to be hyperbolic. How does this relate to the characteristics you have obtained?

Partial Differential Equations — Answer Sheet 6

1. $u_{tt} + u_{tx} - 2u_{xx} = t$: Characteristic condition is $\phi_t^2 + \phi_t\phi_x - 2\phi_x^2 = 0$.
 Factorising gives $(\phi_t - \phi_x)(\phi_t + 2\phi_x) = 0$ so either:
 $\phi_t - \phi_x = 0$ giving $\frac{dt}{1} = \frac{dx}{-1} = \frac{d\phi}{0}$ so that $x = k_1 - t$ and $\phi = P(x + t)$; or
 $\phi_t + 2\phi_x = 0$ giving $\frac{dt}{1} = \frac{dx}{2} = \frac{d\phi}{0}$ so that $x = k_2 + 2t$ and $\phi = Q(x - 2t)$.
 We can choose the characteristics: $\xi = x + t$ and $\eta = x - 2t$
 Transforming $u(t, x) = U(\xi, \eta)$ then leads to $\beta U_{\xi\eta} = F$ where
 $\beta = (2\xi_t\eta_t) + (\xi_t\eta_x + \xi_x\eta_t) - 2(2\xi_x\eta_x) = -4 + (1 - 2) - 4 = -9$ and
 $F = t - (\xi_{tt} + \xi_{tx} - 2\xi_{xx})U_\xi - (\eta_{tt} + \eta_{tx} - 2\eta_{xx})U_\eta = t = \frac{1}{3}(\xi - \eta)$.
 Hence $-9U_{\xi\eta} = \frac{1}{3}(\xi - \eta)$ or $U_{\xi\eta} = \frac{1}{27}(\eta - \xi)$.
 Integrating w.r.t. ξ gives $U_\eta = B'(\eta) + \frac{1}{27}(\eta\xi - \frac{1}{2}\xi^2)$, and w.r.t. η gives
 $U = A(\xi) + B(\eta) + \frac{1}{27}(\frac{1}{2}\eta^2\xi - \frac{1}{2}\eta\xi^2) = A(\xi) + B(\eta) + \frac{1}{54}\eta\xi(\eta - \xi)$ so that
 $u = A(x + t) + B(x - 2t) - \frac{1}{18}(x - 2t)(x + t)t$.
 At $t = 0$ we have $u(0, x) = A(x) + B(x) = 0$ so that $B(x) = -A(x)$
 giving $u = A(x + t) - A(x - 2t) - \frac{1}{18}(x - 2t)(x + t)t$
 and $u_t = A'(x + t) + 2A'(x - 2t) - \frac{1}{18}[(x - 2t)(x + t) + (x - 2t)t - 2t(x + t)t]$.
 Hence at $t = 0$ we have $u_t(0, x) = A'(x) + 2A'(x) - \frac{1}{18}x^2 = 0$ so $A'(x) = \frac{1}{54}x^2$
 and hence $A(x) = \frac{1}{162}x^3 + c$ where c is any constant.
 The solution thus becomes $u = \frac{1}{162}(x + t)^3 - \frac{1}{162}(x - 2t)^3 - \frac{1}{18}(x - 2t)(x + t)t$.
2. $u_{tt} - u_{xx} = 0$: Characteristic condition is $\phi_t^2 - \phi_x^2 = (\phi_t + \phi_x)(\phi_t - \phi_x) = 0$ so either
 $\phi_t + \phi_x = 0$ giving $\frac{dt}{1} = \frac{dx}{1} = \frac{d\phi}{0}$ so that $x = k_1 + t$ and $\phi = P(x - t)$; or
 $\phi_t - \phi_x = 0$ giving $\frac{dt}{1} = \frac{dx}{-1} = \frac{d\phi}{0}$ so that $x = k_1 - t$ and $\phi = Q(x + t)$.
 We can choose the characteristics: $\xi = t - x$ and $\eta = t + x$
 Transforming $u(t, x) = U(\xi, \eta)$ then leads to $\beta U_{\xi\eta} = F$ where
 $\beta = (2\xi_t\eta_t) - (2\xi_x\eta_x) = 2 + 2 = 4$ and $F = -(\xi_{tt} + \xi_{tx} - 2\xi_{xx})U_\xi - (\eta_{tt} + \eta_{tx} - 2\eta_{xx})U_\eta = 0$.
 Hence $U_{\xi\eta} = 0$. Integrating w.r.t. ξ gives $U_\eta = A'(\eta)$ and w.r.t. η gives $U = A(\eta) + B(\xi)$
 so that $u = A(t + x) + B(t - x)$.
 Given that $u(t, 0) = 0$ we have $A(t) + B(t) = 0$ so that $B(z) = -A(z)$ and hence
 $u = A(t + x) - A(t - x)$.
- (a) If also, $u(t, \pi) = 0$ then $A(t + \pi) - A(t - \pi) = 0$ or $A(z + 2\pi) = A(z)$ for any $z \in \mathbb{R}$. That is $A(\cdot)$ is periodic with period 2π .
- (b) If instead, $u_x(t, \pi) = 0$ then $u_x = A'(t + x) + A'(t - x)$ gives $A'(t + \pi) + A'(t - \pi) = 0$ or $A'(z + 2\pi) = -A'(z)$ for any $z \in \mathbb{R}$.
 If $A(z) = \text{Const.}$ then $A'(z) = 0$, satisfying the condition. If $A(z) = \sin(\frac{2n+1}{2}z)$ or
 $A(z) = \cos(\frac{2n+1}{2}z)$ then $A'(z) = \frac{2n+1}{2}\cos(\frac{2n+1}{2}z)$ or $A'(z) = -\frac{2n+1}{2}\sin(\frac{2n+1}{2}z)$
 satisfying $A'(z + 2\pi) = \frac{2n+1}{2}\cos(\frac{2n+1}{2}(z + 2\pi)) = \frac{2n+1}{2}\cos(\frac{2n+1}{2}z + (2n+1)\pi)$
 $= -\frac{2n+1}{2}\cos(\frac{2n+1}{2}z) = -A'(z)$. (similarly for sin)
3. (a) $u_{tt} + (t - 1)u_{tx} - tu_{xx} = 0$: $\phi_t^2 + (t - 1)\phi_t\phi_x - t\phi_x^2 = (\phi_t - \phi_x)(\phi_t + t\phi_x) = 0$
 $\phi_t - \phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{-1} = \frac{d\phi}{0}$ so $x = k_1 - t$ and $\phi = P(x + t)$
 $\phi_t + t\phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{t} = \frac{d\phi}{0}$ so $x = k_2 + \frac{1}{2}t^2$ and $\phi = Q(x - \frac{1}{2}t^2)$
 Choosing $\xi = x + t$ and $\eta = x - \frac{1}{2}t^2$ gives $\beta u_{\xi\eta} = F$ with
 $\beta = (2\xi_t\eta_t) + (t - 1)(\xi_t\eta_x + \xi_x\eta_t) - t(2\xi_x\eta_x) = -2t + (t - 1)(1 - t) - 2t = -(1 - t)^2 - 4t = -(1 + t)^2$
 and $F = -(\xi_{tt} + (t - 1)\xi_{tx} - t\xi_{xx})u_\xi - (\eta_{tt} + (t - 1)\eta_{tx} - t\eta_{xx})u_\eta = -(-1)u_\eta = u_\eta$.
 So $-(1 + t)^2 u_{\xi\eta} = u_\eta$. Also $\xi - \eta = t + \frac{1}{2}t^2$ so $(1 + t)^2 = 1 + 2(\xi - \eta)$
 giving $(1 + 2\xi - 2\eta)u_{\xi\eta} + u_\eta = 0$.
- (b) $u_{tt} + (1 + x)u_{tx} + xu_{xx} = 0$: $\phi_t^2 + (1 + x)\phi_t\phi_x + x\phi_x^2 = (\phi_t + \phi_x)(\phi_t + x\phi_x) = 0$
 $\phi_t + \phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{1} = \frac{d\phi}{0}$ so $x = k_1 + t$ and $\phi = P(x - t)$
 $\phi_t + x\phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{x} = \frac{d\phi}{0}$ so $x = k_2 e^t$ and $\phi = Q(xe^{-t})$
 Choosing $\xi = x - t$ and $\eta = xe^{-t}$ gives $\beta u_{\xi\eta} = F$ with
 $\beta = (2\xi_t\eta_t) + (1 + x)(\xi_t\eta_x + \xi_x\eta_t) + x(2\xi_x\eta_x) = (2xe^{-t}) + (1 + x)(-e^{-t} - xe^{-t}) + x(2e^{-t})$
 $= (2x - (1 + x)^2 + 2x)e^{-t} = -(1 - x)^2 e^{-t}$ and
 $F = -(\xi_{tt} + (1 + x)\xi_{tx} + x\xi_{xx})u_\xi - (\eta_{tt} + (1 + x)\eta_{tx} + x\eta_{xx})u_\eta$
 $= -(xe^{-t} - (1 + x)e^{-t})u_\eta = e^{-t}u_\eta$.
 So $-(1 - x)^2 u_{\xi\eta} = e^{-t}u_\eta$.
 Eliminating t and x requires solving the transcendental equations
 $\xi = \eta e^t - t$ and $xe^{-x} = \eta e^{-\xi}$. This cannot be done algebraically.

- (c) $tu_{tt} + (x-t)u_{tx} - xu_{xx} = 0$: $t\phi_t^2 + (x-t)\phi_t\phi_x - x\phi_x^2 = (\phi_t - \phi_x)(t\phi_t + x\phi_x) = 0$
 $\phi_t - \phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{-1} = \frac{d\phi}{0}$ so $x = k_1 - t$ and $\phi = P(x+t)$
 $t\phi_t + x\phi_x = 0$ gives $\frac{dt}{t} = \frac{dx}{x} = \frac{d\phi}{0}$ so $x = k_2t$ and $\phi = Q(x/t)$
Choosing $\xi = x+t$ and $\eta = x/t$ gives $\beta u_{\xi\eta} = F$ with
 $\beta = t(2\xi_t\eta_t) + (x-t)(\xi_t\eta_x + \xi_x\eta_t) - x(2\xi_x\eta_x) = t(-2x/t^2) + (x-t)(1/t - x/t^2) - x(2/t)$
 $= -2x/t + (x-t)(t-x)/t^2 - 2x/t = -(x-t)^2/t^2 - 4x/t = -(x+t)^2$ and
 $F = -(t\xi_{tt} + (x-t)\xi_{tx} - x\xi_{xx})u_\xi - (t\eta_{tt} + (x-t)\eta_{tx} - x\eta_{xx})u_\eta$
 $= -(t(2x/t^3) + (x-t)(-1/t^2))u_\eta = -(2x/t^2 - x/t^2 + t/t^2)u_\eta = -((x+t)/t^2)u_\eta$.
So $-(x+t)^2u_{\xi\eta} = -((x+t)/t^2)u_\eta$ or $(x+t)u_{\xi\eta} = u_\eta/t^2$, provided $x \neq -t$.
Using $x = \eta t$, so that $\xi = (1+\eta)t$ and so $t = \xi/(1+\eta)$
gives $\xi u_{\xi\eta} = u_\eta(1+\eta)^2/\xi^2$ or $\xi^3u_{\xi\eta} = (1+\eta)^2u_\eta$, provided $\xi = x+t \neq 0$.
- (d) $xu_{tt} + (x-t)u_{tx} - tu_{xx} = 0$: $x\phi_t^2 + (x-t)\phi_t\phi_x - t\phi_x^2 = (\phi_t + \phi_x)(x\phi_t - t\phi_x) = 0$
 $\phi_t + \phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{1} = \frac{d\phi}{0}$ so $x = k_1 + t$ and $\phi = P(x-t)$
 $x\phi_t - t\phi_x = 0$ gives $\frac{dt}{x} = \frac{dx}{-t} = \frac{d\phi}{0}$ so $x^2 = k_2 - t^2$ and $\phi = Q(x^2 + t^2)$
Choosing $\xi = x-t$ and $\eta = x^2 + t^2$ gives $\beta u_{\xi\eta} = F$ with
 $\beta = x(2\xi_t\eta_t) + (x-t)(\xi_t\eta_x + \xi_x\eta_t) - t(2\xi_x\eta_x) = x(-4t) + (x-t)(-2x+2t) - t(4x)$
 $= -2(x-t)^2 - 8xt = -2(x+t)^2$ and
 $F = -(x\xi_{tt} + (x-t)\xi_{tx} - t\xi_{xx})u_\xi - (x\eta_{tt} + (x-t)\eta_{tx} - t\eta_{xx})u_\eta = -(2x-2t)u_\eta = -2(x-t)u_\eta$.
So $-2(x+t)^2u_{\xi\eta} = -2(x-t)u_\eta$ or, using $(x+t)^2 = \eta + 2xt$ and $\xi^2 = \eta - 2xt$,
 $(2\eta - \xi^2)u_{\xi\eta} = \xi u_\eta$.
- (e) $xu_{tt} + (1+xt)u_{tx} + tu_{xx} = 0$: $x\phi_t^2 + (1+xt)\phi_t\phi_x + t\phi_x^2 = (\phi_t + t\phi_x)(x\phi_t + \phi_x) = 0$
 $\phi_t + t\phi_x = 0$ gives $\frac{dt}{1} = \frac{dx}{t} = \frac{d\phi}{0}$ so $x = k_1 + \frac{1}{2}t^2$ and $\phi = P(x - \frac{1}{2}t^2)$
 $x\phi_t + \phi_x = 0$ gives $\frac{dt}{x} = \frac{dx}{1} = \frac{d\phi}{0}$ so $t = k_2 + \frac{1}{2}x^2$ and $\phi = Q(t - \frac{1}{2}x^2)$
Choosing $\xi = x - \frac{1}{2}t^2$ and $\eta = t - \frac{1}{2}x^2$ gives $\beta u_{\xi\eta} = F$ with
 $\beta = x(2\xi_t\eta_t) + (1+xt)(\xi_t\eta_x + \xi_x\eta_t) + t(2\xi_x\eta_x) = x(-2t) + (1+xt)(tx+1) + t(-2x)$
 $= (1+xt)^2 - 4tx = (1-xt)^2$ and
 $F = -(x\xi_{tt} + (1+xt)\xi_{tx} + t\xi_{xx})u_\xi - (x\eta_{tt} + (1+xt)\eta_{tx} + t\eta_{xx})u_\eta = -(-x)u_\xi - (-t)u_\eta$
 $= xu_\xi + tu_\eta$. So $(1-xt)^2u_{\xi\eta} = xu_\xi + tu_\eta$. Eliminating t and x requires solving cubic and quartic equations, which is difficult to do algebraically.
- (f) $txu_{tt} + (x^2 - t^2)u_{tx} - txu_{xx} = 0$: $tx\phi_t^2 + (x^2 - t^2)\phi_t\phi_x - tx\phi_x^2 = (t\phi_t + x\phi_x)(x\phi_t - t\phi_x) = 0$
 $t\phi_t + x\phi_x = 0$ gives $\frac{dt}{t} = \frac{dx}{x} = \frac{d\phi}{0}$ so $x = k_1t$ and $\phi = P(x/t)$
 $x\phi_t - t\phi_x = 0$ gives $\frac{dt}{x} = \frac{dx}{-t} = \frac{d\phi}{0}$ so $x^2 = k_2 - t^2$ and $\phi = Q(x^2 + t^2)$
Choosing $\xi = x/t$ and $\eta = x^2 + t^2$ gives $\beta u_{\xi\eta} = F$ with
 $\beta = tx(2\xi_t\eta_t) + (x^2 - t^2)(\xi_t\eta_x + \xi_x\eta_t) - tx(2\xi_x\eta_x) =$
 $tx(-4x/t) + (x^2 - t^2)(-2x^2/t^2 + 2) - tx(4x/t) = -2(x^2 - t^2)^2/t^2 - 8x^2 = -2(x^2 + t^2)^2/t^2$
and $F = -(tx\xi_{tt} + (x^2 - t^2)\xi_{tx} - tx\xi_{xx})u_\xi - (tx\eta_{tt} + (x^2 - t^2)\eta_{tx} - tx\eta_{xx})u_\eta$
 $= -(x^2/t^2 + 1)u_\xi$. So $2(x^2 + t^2)^2u_{\xi\eta} = (x^2 + t^2)u_\xi$ or $\eta u_{\xi\eta} = \frac{1}{2}u_\xi$ provided $\eta \neq 0$.
4. (a) $u_{tt} + (t-1)u_{tx} - tu_{xx} = 0$: $b^2 - 4ac = (t-1)^2 + 4t = (t+1)^2$ failing to be hyperbolic at $t = -1$. Characteristic $\xi = x+t$ has speed $\frac{dx}{dt} = -1$; characteristic $\eta = x - \frac{1}{2}t^2$ has speed $\frac{dx}{dt} = t$. So both have speed -1 at $t = -1$.
- (b) $u_{tt} + (1+x)u_{tx} + xu_{xx} = 0$: $b^2 - 4ac = (1+x)^2 - 4x = (1-x)^2$ failing to be hyperbolic at $x = 1$. Characteristic $\xi = x-t$ has speed $\frac{dx}{dt} = 1$; characteristic $\eta = xe^{-t}$ has speed $\frac{dx}{dt} = \eta e^t = x$. So both have speed 1 at $x = 1$.
- (c) $tu_{tt} + (x-t)u_{tx} - xu_{xx} = 0$: $b^2 - 4ac = (x-t)^2 + 4tx = (x+t)^2$ failing to be hyperbolic at $x = -t$. Characteristic $\xi = x+t$ has speed $\frac{dx}{dt} = -1$; characteristic $\eta = x/t$ has speed $\frac{dx}{dt} = \eta = x/t$. So both have speed -1 at $x = -t$.
- (d) $xu_{tt} + (x-t)u_{tx} - tu_{xx} = 0$: $b^2 - 4ac = (x-t)^2 + 4tx = (x+t)^2$ failing to be hyperbolic at $x = -t$. Characteristic $\xi = x-t$ has speed $\frac{dx}{dt} = 1$; characteristic $\eta = x^2 + t^2$ has speed $\frac{dx}{dt} = -t/x$. So both have speed 1 at $x = -t$.
- (e) $xu_{tt} + (1+xt)u_{tx} + tu_{xx} = 0$: $b^2 - 4ac = (1+xt)^2 - 4tx = (1-xt)^2$ failing to be hyperbolic at $x = 1/t$. Characteristic $\xi = x - \frac{1}{2}t^2$ has speed $\frac{dx}{dt} = t$; characteristic $\eta = t - \frac{1}{2}x^2$ has speed $\frac{dx}{dt} = 1/x$. So both have speed t at $x = 1/t$.
- (f) $txu_{tt} + (x^2 - t^2)u_{tx} - txu_{xx} = 0$: $b^2 - 4ac = (x^2 - t^2)^2 + 4t^2x^2 = (x^2 + t^2)^2$ failing to be hyperbolic at $x = t = 0$. Characteristic $\xi = x/t$ has speed $\frac{dx}{dt} = \xi = x/t$; characteristic $\eta = x^2 + t^2$ has speed $\frac{dx}{dt} = -t/x$. So both have undefined speed at $x = t = 0$.
- (In general, at points of parabolic behaviour characteristics are parallel.)

Partial Differential Equations — Problem Sheet 7

1. Suppose that the functions $p(x)$, $p_1(x)$, $p_2(x)$ and $q(x)$ are defined and continuous on $\alpha \leq x \leq \beta$ and that an inner product is defined as

$$(p, q) = \int_{\alpha}^{\beta} p(x) q(x) dx$$

show that:

- (a) $(p, q) = (q, p)$
 - (b) $(p, p) \geq 0$
 - (c) if $(p, p) = 0$ then $p(x) = 0$ for all $x \in [\alpha, \beta]$
 - (d) $(c_1 p_1 + c_2 p_2, q) = c_1 (p_1, q) + c_2 (p_2, q)$ for any constants c_1 and c_2
 - (e) if p_1 and p_2 are orthogonal with respect to the inner product (p_1, p_2) then p_1 and p_2 are linearly independent.
2. If the function $w(x)$ is continuous and positive on $\alpha \leq x \leq \beta$ show that the form of the inner product that is “weighted” by the function $w(x)$, as given by

$$(p, wq) = \int_{\alpha}^{\beta} p(x) w(x) q(x) dx$$

satisfies

- (a) $(p, wq) = (q, wp)$
- (b) $(p, wp) \geq 0$
- (c) if $(p, wp) = 0$ then $p(x) = 0$ for all $x \in [\alpha, \beta]$
- (d) $(c_1 p_1 + c_2 p_2, wq) = c_1 (p_1, wq) + c_2 (p_2, wq)$ for any constants c_1 and c_2 .
- (e) if p_1 and p_2 are orthogonal with respect to the weighted inner product (p_1, wp_2) then p_1 and p_2 are linearly independent.

These properties are also satisfied (under certain restrictions) if we relax the conditions that $p(x)$, $q(x)$, $p_1(x)$, $p_2(x)$ and $w(x)$ are continuous.

If we were to relax only the condition that $w(x)$ is positive, give at least two examples of possible forms of the function $w(x)$ for which at least one of the properties (a) to (d) would not be satisfied.

3. Using the method of separation of variables find at least three linearly independent solutions of the partial differential equation

$$(1 + t)u_t = D^2 u_{xx}$$

defined on $-\alpha \leq x \leq \alpha$ for $t \geq 0$, subject to the homogeneous boundary conditions

$$u(t, -\alpha) = 0 \quad \text{and} \quad u(t, \alpha) = 0.$$

If $u(t, x)$ also satisfies the initial condition

$$u(0, x) = A \cos(\pi x / 2\alpha)$$

find the exact solution for $u(t, x)$ throughout the “strip” $x \in [-\alpha, \alpha]$, $t \in [0, \infty)$.

4. Using the method of separation of variables find an infinite set of linearly independent solutions of the partial differential equation

$$u_t = u_{xx}$$

defined on $0 \leq x \leq a$ for $t \geq 0$, subject to the homogeneous boundary conditions

$$u(t, 0) = 0 \quad \text{and} \quad u_x(t, a) = 0.$$

If $u(t, x)$ also satisfies the initial condition

$$u(0, x) = a \quad \text{for all} \quad 0 < x < a$$

find the exact solution for $u(t, x)$ throughout the “strip” $x \in [0, a]$, $t \in [0, \infty)$, in the form of a Fourier expansion.

Partial Differential Equations — Answer Sheet 7

1. (a) $(p, q) = \int_{\alpha}^{\beta} p(x) q(x) dx = \int_{\alpha}^{\beta} q(x) p(x) dx = (q, p).$
 - (b) $(p, p) = \int_{\alpha}^{\beta} (p(x))^2 dx$ which is positive or zero because $(p(x))^2$ is positive or zero throughout the range of integration.
 - (c) Since $p(x)$ is continuous for all $x \in [\alpha, \beta]$ it follows that $(p(x))^2$ is continuous as well as non-negative. If $(p(a))^2 = b > 0$ at some point $\alpha \leq a \leq \beta$ then, because $(p(x))^2$ is continuous, for any $\delta > 0 \exists \epsilon > 0$ such that $|(p(x))^2 - b| < \delta$ for all $x \in (a - \epsilon, a + \epsilon) \cap [\alpha, \beta]$. Let us choose $\delta = b/2$ then $|(p(x))^2 - b| < b/2$ for all $x \in (a - \epsilon, a + \epsilon) \cap [\alpha, \beta]$, an interval of length at least ϵ , over which $(p(x))^2 > b/2$. This would contribute at least $\epsilon b/2 > 0$ to (p, p) , making it non-zero. Hence if $(p, p) = 0$ then $(p(x))^2$, and therefore $p(x)$, must be zero at all points $x \in [\alpha, \beta]$.
 - (d) $(c_1 p_1 + c_2 p_2, q) = c_1 \int_{\alpha}^{\beta} p_1(x) q(x) dx + c_2 \int_{\alpha}^{\beta} p_2(x) q(x) dx = c_1 (p_1, q) + c_2 (p_2, q).$
 - (e) For the functions p_1 and p_2 to be orthogonal they must be non-trivial, continuous functions on $[\alpha, \beta]$ that satisfy $(p_1, p_2) = 0$. If they were also linearly independent then there would be a non-zero constant c such that $p_2 = c p_1$, so that $(p_1, p_2) = (p_1, c p_1) = c (p_1, p_1) = 0$. Since $c \neq 0$ it follows that $(p_1, p_1) = 0$ so that p_1 would be trivial. Because, p_1 is not trivial it follows that the the functions must be linearly independent.
2. (a) $(p, wq) = \int_{\alpha}^{\beta} p(x) w(x) q(x) dx = \int_{\alpha}^{\beta} q(x) w(x) p(x) dx = (q, wp).$
 - (b) $(p, wp) = \int_{\alpha}^{\beta} w(x) (p(x))^2 dx$ which is positive or zero because both $w(x)$ and $(p(x))^2$ are positive or zero, and so $w(x)(p(x))^2$ is positive or zero, throughout the range of integration.
 - (c) If $p(a) = b \neq 0$ at some point $\alpha \leq a \leq \beta$ then, because $p(x)$ is continuous, $\exists \epsilon > 0$ such that $|p(x) - b| < |b|/2$ for all $x \in (a - \epsilon, a + \epsilon) \cap [\alpha, \beta]$, an interval of length at least ϵ , over which $(p(x))^2 > b^2/4$. If c is the minimum value of w in this interval, then $c > 0$ since $w(x)$ is positive throughout $[\alpha, \beta]$. It follows that integration over the interval would contribute at least $\epsilon c b^2/4 > 0$ to (p, wp) , making it non-zero. Hence if $(p, wp) = 0$ then $p(x)$ must be zero at all points $x \in [\alpha, \beta]$.
 - (d) $(c_1 p_1 + c_2 p_2, wq) = c_1 \int_{\alpha}^{\beta} p_1(x) w(x) q(x) dx + c_2 \int_{\alpha}^{\beta} p_2(x) w(x) q(x) dx = c_1 (p_1, wq) + c_2 (p_2, wq).$
 - (e) For the functions p_1 and p_2 to be orthogonal they must be non-trivial, continuous functions on $[\alpha, \beta]$ that satisfy $(p_1, wp_2) = 0$. If they were also linearly independent then there would be a non-zero constant c such that $p_2 = c p_1$, so that $(p_1, wp_2) = (p_1, w c p_1) = c (p_1, w p_1) = 0$. Since $c \neq 0$ it follows that $(p_1, w p_1) = 0$ so that p_1 would be trivial. Because, p_1 is not trivial it follows that the the functions must be linearly independent.

When $w(x)$ is not required to be positive for all $\alpha \leq x \leq \beta$:

- if we take $w(x) \equiv -1$ then $(p, wp) \leq 0$. Otherwise all of the properties remain unchanged.

- if we take $w(x) \equiv 0$ then $(p, wq) = 0$ in all cases. In fact only the properties (c) and (e) fail to hold; the rest hold trivially.
- if we take $w(x)$ to change sign somewhere in the interval then properties (b), (c) and (e) do not generally hold.

3. We have: $(1+t)u_t = D^2u_{xx}$ with the B.C.s: $u(t, -\alpha) = 0$ and $u(t, \alpha) = 0$.

Assuming $u = T(t)X(x)$ gives $(1+t)T'X = D^2TX''$ or $\frac{X''}{X} = \frac{1+t}{D^2} \frac{T'}{T} = \mu$ with the B.C.s giving $T(t)X(-\alpha) = 0$ and $T(t)X(\alpha) = 0$.

So $X'' - \mu X = 0$ and $(1+t)T' - \mu D^2T = 0$

with B.C.s: $X(-\alpha) = 0$ and $X(\alpha) = 0$, since we want to have $T(t) \neq 0$.

If $\mu = 0$ then $X = c_1x + c_2$ with $c_2 - c_1\alpha = 0$, $c_2 + c_1\alpha = 0$ so that $c_1 = c_2 = 0$.

If $\mu = \omega^2 > 0$ then $X = c_1e^{\omega x} + c_2e^{-\omega x}$ with $c_1e^{-\omega\alpha} + c_2e^{\omega\alpha} = 0$, $c_1e^{\omega\alpha} + c_2e^{-\omega\alpha} = 0$ so that $c_1 = -c_2e^{2\omega\alpha}$ and hence $-c_2e^{3\omega\alpha} + c_2e^{-\omega\alpha} = 0$, leading to $c_1 = c_2 = 0$.

If $\mu = -\omega^2 < 0$ then $X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$ and the B.C.s give:

$$\left. \begin{array}{l} c_1 \cos(\omega\alpha) - c_2 \sin(\omega\alpha) = 0 \\ c_1 \cos(\omega\alpha) + c_2 \sin(\omega\alpha) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 \cos(\omega\alpha) = 0 \\ c_2 \sin(\omega\alpha) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = 0 \quad \text{or} \quad \omega = (n + \frac{1}{2})\pi/\alpha \\ c_2 = 0 \quad \text{or} \quad \omega = n\pi/\alpha \end{array} \right.$$

Thus non-trivial solutions for $X(x)$ are found only for the eigenvalues of the separation constant: $\mu = -(\frac{1}{2}m\pi/\alpha)^2$, $m = 1, 2, 3, \dots$, taking the corresponding eigenfunctions:

$X = X_m(x) = \sin(\frac{1}{2}m\pi x/\alpha)$ for m even and $X = X_m(x) = \cos(\frac{1}{2}m\pi x/\alpha)$ for m odd.

Also, correspondingly, we have $\frac{T'}{T} = \frac{\mu D^2}{1+t}$

so that $\ln T = \mu D^2 \ln(1+t) + \text{const.}$ or $T = \text{const.} \times (1+t)^{\mu D^2}$.

Ignoring constant multiples, the eigensolutions are therefore

$$u = u_m(t, x) = (1+t)^{-\frac{1}{4}m^2\pi^2/\alpha^2 D^2} \times \left\{ \begin{array}{l} \sin(\frac{1}{2}m\pi x/\alpha) \quad : \quad m \text{ even} \\ \cos(\frac{1}{2}m\pi x/\alpha) \quad : \quad m \text{ odd} \end{array} \right.$$

These are all linearly independent because they are all non-trivial and because the eigenfunctions X_m are all orthogonal (from Sturm-Liouville theory) — any three different values of m will do.

With initial condition $u(0, x) = A \cos(\pi x/2\alpha)$ only the mode for $m = 1$ is non trivial, giving the unique solution

$$u(t, x) = A(1+t)^{-\frac{1}{4}\pi^2/\alpha^2 D^2} \cos(\pi x/2\alpha)$$

throughout the “strip” $x \in [-\alpha, \alpha]$, $t \in [0, \infty)$.

4. We have: $u_t = u_{xx}$ with the B.C.s $u(t, 0) = 0$ and $u_x(t, a) = 0$.

Assuming $u = T(t)X(x)$ and that $T(t) \neq 0$ gives $T'X = TX''$ or $\frac{X''}{X} = \frac{T'}{T} = \mu$ with the B.C.s giving $T(t)X(0) = 0$ and $T(t)X'(a) = 0$.

So $T' - \mu T = 0$ and $X'' - \mu X = 0$ with $X(0) = X'(a) = 0$.

If $\mu = 0$ then $X = c_1x + c_2$ with $c_2 = 0$, $c_1 = 0$ so that $c_1 = c_2 = 0$.

If $\mu = \omega^2 > 0$ then $X = c_1e^{\omega x} + c_2e^{-\omega x}$ with $c_1 + c_2 = 0$, $c_1\omega e^{\omega a} - c_2\omega e^{-\omega a} = 0$

so that $c_1 = c_2 = 0$.

If $\mu = -\omega^2 < 0$ then $X = c_1 \cos(\omega x) + c_2 \sin(\omega x)$ and the B.C.s give:

$c_1 = 0$ and $c_2 \omega \cos(\omega a) - c_1 \omega \sin(\omega a) = 0$ so that $c_1 = 0$ and $c_2 \cos(\omega a) = 0$, allowing $c_2 \neq 0$ only if $\cos(\omega a) = 0$ or $\omega a = (\frac{1}{2} + n)\pi$ for $n = 0, 1, 2, \dots$.

Thus non-trivial eigenfunctions $X = X_n(x) = \sin((\frac{1}{2} + n)\pi x/a)$ are found for the eigenvalues of the separation constant: $\mu = -((\frac{1}{2} + n)\pi/a)^2$, $n = 0, 1, 2, \dots$,

Also, correspondingly, we have $T' - \mu T = 0$ so that $T = \text{const.} \times e^{\mu t}$. The eigensolutions are therefore

$$u = u_n(t, x) = e^{-((\frac{1}{2}+n)\pi/a)^2 t} \sin((\frac{1}{2} + n)\pi x/a).$$

These are all linearly independent so that we have an infinite number of suitable solutions.

An infinite series solution is $u = \sum_{n=0}^{\infty} b_n e^{-((\frac{1}{2}+n)\pi/a)^2 t} \sin((\frac{1}{2} + n)\pi x/a)$

which, at $t = 0$ when $u(0, x) = a$ for $0 < x < a$, gives $a = \sum_{n=0}^{\infty} b_n \sin((\frac{1}{2} + n)\pi x/a)$.

Sturm-Liouville theory ensures that all of the eigenfunctions X_n are orthogonal under the inner product $(p, q) = \int_0^a p(x)q(x) dx$

so that $\int_0^a a \sin((\frac{1}{2} + n)\pi x/a) dx = b_n \int_0^a \sin^2((\frac{1}{2} + n)\pi x/a) dx$.

Evaluating: $\int_0^a a \sin((\frac{1}{2} + n)\pi x/a) dx = a \left[-\frac{a}{(\frac{1}{2} + n)\pi} \cos((\frac{1}{2} + n)\pi x/a) \right]_0^a = \frac{a^2}{(\frac{1}{2} + n)\pi}$

and $\int_0^a \sin^2((\frac{1}{2} + n)\pi x/a) dx = \int_0^a \frac{1}{2} \{1 - \cos(2(\frac{1}{2} + n)\pi x/a)\} dx$
 $= \frac{1}{2} \left[x - \frac{a}{(1 + 2n)\pi} \sin((1 + 2n)\pi x/a) \right]_0^a = \frac{1}{2} a$

leads to $b_n = \frac{a}{(1 + 2n)\pi}$ so that the full series solution becomes

$$u(t, x) = \sum_{n=0}^{\infty} \frac{a}{(1 + 2n)\pi} e^{-((\frac{1}{2}+n)\pi/a)^2 t} \sin((\frac{1}{2} + n)\pi x/a).$$

Partial Differential Equations — Problem Sheet 8

1. Find all of the eigenvalues and eigenfunctions for each of the following problems
(Be sure that you consider all possible ranges of values for μ .)

(a) $X'' - \mu X = 0$ with $X(0) = X(l) = 0$.

(b) $Y'' - \mu Y = 0$ with $Y'(0) = Y'(l) = 0$.

(c) $Z'' - \mu Z = 0$ with $Z'(0) = Z(l) = 0$.

(d) $F'' - \mu F = 0$ with $F(0) = F'(l) = 0$.

In each case, sketch the first three eigenfunctions (in order of increasing $|\mu|$).

2. Given that the eigenfunctions are orthogonal on $x \in [0, l]$ in each of the following expansions, find all of the coefficients in each case.

(a) $1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

(b) $x = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l}$

(c) $\pi = \sum_{n=0}^{\infty} a_n \cos \frac{(n + \frac{1}{2})\pi x}{l}$

(d) $x = \sum_{n=0}^{\infty} b_n \sin \frac{(n + \frac{1}{2})\pi x}{l}$

3. Consider Laplace's equation governing the function $u(x, y)$ in the rectangular domain $x \in [0, \pi]$ with $y \in [0, 1]$

$$u_{xx} + u_{yy} = 0$$

along with the boundary conditions

$$\begin{aligned} u(0, y) = u(\pi, y) = 0 & \quad \text{for } y \in [0, 1] \\ u(x, 0) = 0 & \quad \text{and } u(x, 1) = \pi & \quad \text{for } x \in [0, \pi]. \end{aligned}$$

Use the method of separation of variables to show that the solution satisfying all of the boundary conditions can be written as

$$u = \sum_{k=0}^{\infty} \frac{4}{2k+1} \frac{\sinh((2k+1)y)}{\sinh(2k+1)} \sin((2k+1)x).$$

4. Still, considering Laplace's equation governing the function $u(x, y)$ in the rectangular domain $x \in [0, \pi]$ with $y \in [0, 1]$

$$u_{xx} + u_{yy} = 0$$

find infinite series solutions that satisfy the boundary conditions

(a) $u_x(0, y) = u(\pi, y) = 0$ for $y \in [0, 1]$ with $u(x, 0) = 0$ and $u(x, 1) = \frac{1}{2}\pi$ for $x \in [0, \pi]$

(b) $u(0, y) = u_x(\pi, y) = 0$ for $y \in [0, 1]$ with $u_y(x, 0) = 0$ and $u(x, 1) = \frac{1}{2}$ for $x \in [0, \pi]$

Partial Differential Equations — Answer Sheet 8

1. (a) $X'' - \mu X = 0$ with $X(0) = X(l) = 0$, taking $X = X(x)$:

$\mu = 0$ gives $X'' = 0$, so $X = a + bx$. BCs give $X(0) = a = 0$, so $a = 0$, and $X(l) = bl = 0$, so $b = 0$. Hence $X \equiv 0$.

$\mu > 0$, with $\mu = \omega^2 \neq 0$, gives $X'' - \omega^2 X = 0$, so $X = ae^{\omega x} + be^{-\omega x}$.

BCs give $X(0) = a + b = 0$ and $X(l) = ae^{\omega l} + be^{-\omega l} = 0$, so $b = -a$ and $a(e^{\omega l} - e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} - e^{-\omega l} \neq 0$). Hence $X \equiv 0$.

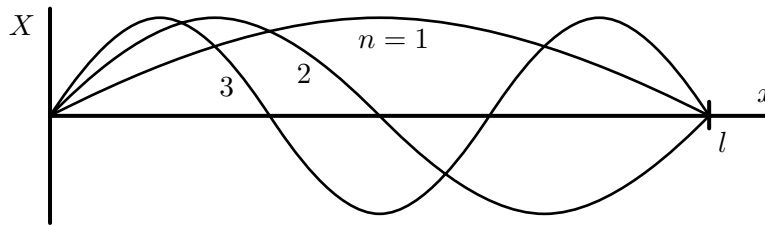
$\mu < 0$, with $\mu = -\omega^2 \neq 0$, gives $X'' + \omega^2 X = 0$, so $X = a \cos(\omega x) + b \sin(\omega x)$.

BCs give $X(0) = a \cos 0 + b \sin 0 = a = 0$, so $a = 0$, and $X(l) = b \sin(\omega l) = 0$, so it is possible to have $b \neq 0$ only if $\sin(\omega l) = 0$.

That is, if $\omega l = n\pi$ for $n = 1, 2, 3, \dots$ then $X = b \sin(n\pi x/l)$.

Hence we find the eigenvalues, $\mu = -(n\pi/l)^2$ for $n = 1, 2, 3, \dots$

and the corresponding eigenfunctions, $X = \sin(n\pi x/l)$.



- (b) $Y'' - \mu Y = 0$ with $Y'(0) = Y'(l) = 0$, taking $Y = Y(y)$:

$\mu = 0$ gives $Y'' = 0$, so $Y = a + by$. BCs give $Y'(0) = b = 0$, so $b = 0$, and $Y'(l) = b = 0$, so $b = 0$ (again). Thus any value of $a \neq 0$ is admissible and so $Y = a$ is a solution if $\mu = 0$.

$\mu > 0$, with $\mu = \omega^2 \neq 0$, gives $Y'' - \omega^2 Y = 0$, so $Y = ae^{\omega y} + be^{-\omega y}$.

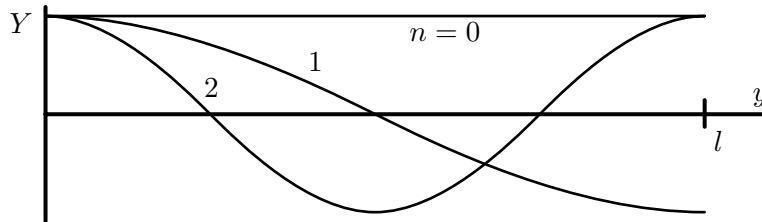
BCs give $Y'(0) = a\omega - b\omega = 0$ and $Y'(l) = a\omega e^{\omega l} - b\omega e^{-\omega l} = 0$, so $b = a$ and $a\omega(e^{\omega l} - e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} - e^{-\omega l} \neq 0$). Hence $Y \equiv 0$.

$\mu < 0$, with $\mu = -\omega^2 \neq 0$, gives $Y'' + \omega^2 Y = 0$, so $Y = a \cos(\omega y) + b \sin(\omega y)$.

BCs give $Y'(0) = -a\omega \sin 0 + b\omega \cos 0 = b\omega = 0$, so $b = 0$, and $Y'(l) = -a\omega \sin(\omega l) = 0$, so it is possible to have $a \neq 0$ only if $\sin(\omega l) = 0$.

That is, if $\omega l = n\pi$ for $n = 1, 2, 3, \dots$ then $Y = a \cos(n\pi y/l)$.

Hence we find the eigenvalues, $\mu = -(n\pi/l)^2$ for $n = 0, 1, 2, \dots$ and corresponding eigenfunctions, $Y = \cos(n\pi y/l)$. (Note: this is the constant 1 for $n = 0$).



- (c) $Z'' - \mu Z = 0$ with $Z'(0) = Z(l) = 0$, taking $Z = Z(z)$:

$\mu = 0$ gives $Z'' = 0$, so $Z = a + bz$. BCs give $Z'(0) = b = 0$, so $b = 0$, and $Z(l) = a = 0$, so $a = 0$. Hence $Z \equiv 0$.

$\mu > 0$, with $\mu = \omega^2 \neq 0$, gives $Z'' - \omega^2 Z = 0$, so $Z = ae^{\omega z} + be^{-\omega z}$.

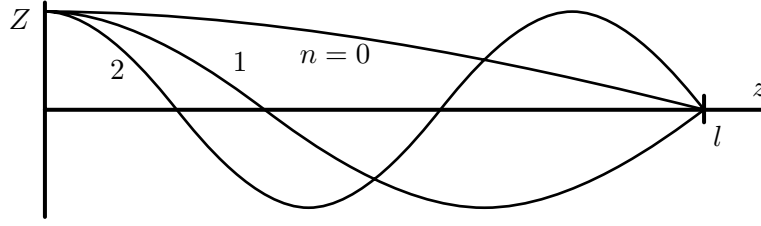
BCs give $Z'(0) = a\omega - b\omega = 0$ and $Z(l) = ae^{\omega l} + be^{-\omega l} = 0$, so $b = a$ and $a(e^{\omega l} + e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} + e^{-\omega l} \neq 0$). Hence $Z \equiv 0$.

$\mu < 0$, with $\mu = -\omega^2 \neq 0$, gives $Z'' + \omega^2 Z = 0$, so $Z = a \cos(\omega z) + b \sin(\omega z)$.

BCs give $Z'(0) = -a\omega \sin 0 + b\omega \cos 0 = b\omega = 0$, so $b = 0$, and $Z(l) = a \cos(\omega l) = 0$, so it is possible to have $a \neq 0$ only if $\cos(\omega l) = 0$.

That is, if $\omega l = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \dots$ then $Z = a \cos((n + \frac{1}{2})\pi z/l)$.

Hence we find the eigenvalues, $\mu = -((n + \frac{1}{2})\pi/l)^2$ for $n = 0, 1, 2, \dots$
and the corresponding eigenfunctions, $Z = \cos((n + \frac{1}{2})\pi z/l)$.



(d) $F'' - \mu F = 0$ with $F(0) = F'(l) = 0$, taking $F = F(f)$:

$\mu = 0$ gives $F'' = 0$, so $F = a + bf$. BCs give $F(0) = a = 0$, so $a = 0$, and $F'(l) = b = 0$, so $b = 0$. Hence $F \equiv 0$.

$\mu > 0$, with $\mu = \omega^2 \neq 0$, gives $F'' - \omega^2 F = 0$, so $F = ae^{\omega f} + be^{-\omega f}$.

BCs give $F(0) = a + b = 0$ and $F'(l) = a\omega e^{\omega l} - b\omega e^{-\omega l} = 0$, so $b = -a$
and $a\omega(e^{\omega l} + e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} + e^{-\omega l} \neq 0$). Hence $F \equiv 0$.

$\mu < 0$, with $\mu = -\omega^2 \neq 0$, gives $F'' + \omega^2 F = 0$, so $F = a \cos(\omega f) + b \sin(\omega f)$.

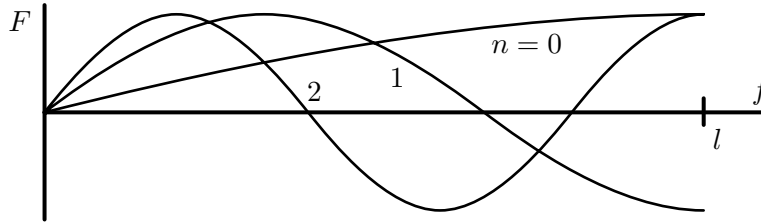
BCs give $F(0) = a \cos 0 + b \sin 0 = a = 0$, so $a = 0$, and

$F'(l) = b\omega \cos(\omega l) = 0$, so it is possible to have $b \neq 0$ only if $\cos(\omega l) = 0$.

That is, if $\omega l = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \dots$ then $F = b \sin((n + \frac{1}{2})\pi f/l)$.

Hence we find the eigenvalues, $\mu = -((n + \frac{1}{2})\pi/l)^2$ for $n = 0, 1, 2, \dots$

and the corresponding eigenfunctions $F = \sin((n + \frac{1}{2})\pi f/l)$.



2. (a) $1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ leads to

$$\int_0^l \sin \frac{m\pi x}{l} dx = \sum_{n=1}^{\infty} b_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx = b_m \int_0^l \sin^2 \frac{m\pi x}{l} dx$$

(through orthogonality). Evaluating:

$$\int_0^l \sin \frac{m\pi x}{l} dx = -\frac{l}{m\pi} \left[\cos \frac{m\pi x}{l} \right]_0^l = -\frac{l}{m\pi} ((-1)^m - 1) = \begin{cases} \frac{2l}{m\pi} & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even.} \end{cases}$$

$$\int_0^l \sin^2 \frac{m\pi x}{l} dx = \int_0^l \frac{1}{2} \left(1 - \cos \frac{2m\pi x}{l} \right) dx = \frac{1}{2}l - \frac{1}{2} \frac{l}{2m\pi} \left[\sin \frac{2m\pi x}{l} \right]_0^l = \frac{1}{2}l.$$

$$\text{Hence } \frac{2l}{(2k+1)\pi} = b_{2k+1} \times \frac{1}{2}l \text{ or } b_{2k+1} = \frac{4}{(2k+1)\pi} \text{ with } b_{2k} = 0$$

$$\text{and so } 1 = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin \frac{(2k+1)\pi x}{l}.$$

(b) $x = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l}$ leads to

$$\int_0^l x \cos \frac{m\pi x}{l} dx = \sum_{n=0}^{\infty} a_n \int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l} dx = a_m \int_0^l \cos^2 \frac{m\pi x}{l} dx$$

(through orthogonality). Evaluating:

$$\text{For } m = 0: \int_0^l x \cos \frac{m\pi x}{l} dx = \int_0^l x dx = \left[\frac{1}{2}x^2 \right]_0^l = \frac{1}{2}l^2$$

$$\text{and } \int_0^l \cos^2 \frac{m\pi x}{l} dx = \int_0^l dx = l \text{ so that } \frac{1}{2}l^2 = a_0 \times l \text{ giving } a_0 = \frac{1}{2}l.$$

Otherwise, for $m > 0$:

$$\begin{aligned}\int_0^l x \cos \frac{m\pi x}{l} dx &= \left[x \frac{l}{m\pi} \sin \frac{m\pi x}{l} \right]_0^l - \int_0^l \frac{l}{m\pi} \sin \frac{m\pi x}{l} dx = \left(\frac{l}{m\pi} \right)^2 \left[\cos \frac{m\pi x}{l} \right]_0^l \\ &= \left(\frac{l}{m\pi} \right)^2 ((-1)^m - 1) = \begin{cases} -2 \left(\frac{l}{m\pi} \right)^2 & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even.} \end{cases}\end{aligned}$$

$$\int_0^l \cos^2 \frac{m\pi x}{l} dx = \int_0^l \frac{1}{2} \left(1 + \cos \frac{2m\pi x}{l} \right) dx = \frac{1}{2}l + \frac{1}{2} \frac{l}{2m\pi} \left[\sin \frac{2m\pi x}{l} \right]_0^l = \frac{1}{2}l.$$

$$\text{Hence } -\frac{2l^2}{(2k+1)^2\pi^2} = a_{2k+1} \times \frac{1}{2}l \text{ or } a_{2k+1} = -\frac{4l}{(2k+1)^2\pi^2} \text{ with } a_{2k} = 0$$

$$\text{and so } x = \frac{1}{2}l - \sum_{k=0}^{\infty} \frac{4l}{(2k+1)^2\pi^2} \cos \frac{(2k+1)\pi x}{l}.$$

(c) $\pi = \sum_{n=0}^{\infty} a_n \cos \frac{(n+\frac{1}{2})\pi x}{l}$ leads to

$$\begin{aligned}\int_0^l \pi \cos \frac{(m+\frac{1}{2})\pi x}{l} dx &= \sum_{n=0}^{\infty} a_n \int_0^l \cos \frac{(n+\frac{1}{2})\pi x}{l} \cos \frac{(m+\frac{1}{2})\pi x}{l} dx \\ &= a_m \int_0^l \cos^2 \frac{(m+\frac{1}{2})\pi x}{l} dx\end{aligned}$$

(through orthogonality). Evaluating:

$$\int_0^l \pi \cos \frac{(m+\frac{1}{2})\pi x}{l} dx = \pi \frac{l}{(m+\frac{1}{2})\pi} \left[\sin \frac{(m+\frac{1}{2})\pi x}{l} \right]_0^l = \frac{(-1)^m l}{m+\frac{1}{2}}.$$

$$\begin{aligned}\int_0^l \cos^2 \frac{(m+\frac{1}{2})\pi x}{l} dx &= \int_0^l \frac{1}{2} \left(1 + \cos \frac{(2m+1)\pi x}{l} \right) dx \\ &= \frac{1}{2}l + \frac{1}{2} \frac{l}{(2m+1)\pi} \left[\sin \frac{(2m+1)\pi x}{l} \right]_0^l = \frac{1}{2}l.\end{aligned}$$

$$\text{Hence } \frac{(-1)^m l}{m+\frac{1}{2}} = a_m \times \frac{1}{2}l \text{ or } a_m = \frac{2(-1)^m}{m+\frac{1}{2}}$$

$$\text{and so } \pi = \sum_{n=0}^{\infty} \frac{2(-1)^n}{n+\frac{1}{2}} \cos \frac{(n+\frac{1}{2})\pi x}{l}.$$

(d) $x = \sum_{n=0}^{\infty} b_n \sin \frac{(n+\frac{1}{2})\pi x}{l}$ leads to

$$\int_0^l x \sin \frac{(m+\frac{1}{2})\pi x}{l} dx = \sum_{n=0}^{\infty} b_n \int_0^l \sin \frac{(n+\frac{1}{2})\pi x}{l} \sin \frac{(m+\frac{1}{2})\pi x}{l} dx = b_m \int_0^l \sin^2 \frac{(m+\frac{1}{2})\pi x}{l} dx$$

(through orthogonality). Evaluating:

$$\begin{aligned}\int_0^l x \sin \frac{(m+\frac{1}{2})\pi x}{l} dx &= \left[-x \frac{l}{(m+\frac{1}{2})\pi} \cos \frac{(m+\frac{1}{2})\pi x}{l} \right]_0^l + \int_0^l \frac{l}{(m+\frac{1}{2})\pi} \cos \frac{(m+\frac{1}{2})\pi x}{l} dx \\ &= \left[\left(\frac{l}{(m+\frac{1}{2})\pi} \right)^2 \sin \frac{(m+\frac{1}{2})\pi x}{l} \right]_0^l = \frac{(-1)^m l^2}{(m+\frac{1}{2})^2\pi^2}.\end{aligned}$$

$$\begin{aligned}\int_0^l \sin^2 \frac{(m+\frac{1}{2})\pi x}{l} dx &= \int_0^l \frac{1}{2} \left(1 - \cos \frac{(2m+1)\pi x}{l} \right) dx \\ &= \frac{1}{2}l - \frac{1}{2} \frac{l}{(2m+1)\pi} \left[\sin \frac{(2m+1)\pi x}{l} \right]_0^l = \frac{1}{2}l.\end{aligned}$$

$$\text{Hence } \frac{(-1)^m l^2}{(m+\frac{1}{2})^2\pi^2} = b_m \times \frac{1}{2}l \text{ or } b_m = \frac{2(-1)^m l}{(m+\frac{1}{2})^2\pi^2}$$

$$\text{and so } x = \sum_{n=0}^{\infty} \frac{2(-1)^n l}{(n+\frac{1}{2})^2\pi^2} \sin \frac{(n+\frac{1}{2})\pi x}{l}.$$

3. This answer includes full descriptions of the arguments involved.

$$u_{xx} + u_{yy} = 0 \text{ with } u(0, y) = u(\pi, y) = 0, u(x, 0) = 0 \text{ and } u(x, 1) = \pi:$$

- Assuming $u = X(x)Y(y)$ we can use the PDE with $u_{xx} = X''(x)Y(y)$ and $u_{yy} = X(x)Y''(y)$ to find $u_{xx} + u_{yy} = X''Y + XY'' = 0$.
Dividing by XY (assumed not equal to zero) gives $\frac{X''}{X} = -\frac{Y''}{Y} = \mu$.
Thus we have $X'' - \mu X = 0$ and $Y'' + \mu Y = 0$
with μ constant because $\frac{X''}{X}$ is independent of y and $\frac{Y''}{Y}$ is independent of x .
The homogeneous boundary conditions give
 $u(0, y) = X(0)Y(y) = 0$, $u(\pi, y) = X(\pi)Y(y) = 0$, $u(x, 0) = X(x)Y(0) = 0$.
Since we seek $XY \neq 0$ this gives $X(0) = X(\pi) = 0$ and $Y(0) = 0$.
The boundary condition $u(x, 1) = X(x)Y(1) = \pi$ cannot be used being non-homogeneous.

- Solving $X'' - \mu X = 0$ with $X(0) = X(\pi) = 0$:

$\mu = 0$: we have $X'' = 0$ giving $X = A + Bx$.

$X(0) = X(\pi) = 0$ give $A = 0$ and $B\pi = 0$ so that $A = B = 0$.

Only the trivial solution arises for $\mu = 0$.

$\mu = \omega^2 > 0$: we have $X'' - \omega^2 X = 0$ giving $X = Ae^{\omega x} + Be^{-\omega x}$. $X(0) = X(\pi) = 0$ gives
 $A + B = 0$ and $Ae^{\omega\pi} + Be^{-\omega\pi} = 0$,

i.e. (substituting) $A(e^{\omega\pi} - e^{-\omega\pi}) = 0$ so that $A = B = 0$ since $e^{\omega\pi} - e^{-\omega\pi} \neq 0$.

Only trivial solutions arise for $\mu > 0$.

$\mu = -\omega^2 < 0$: we have $X'' + \omega^2 X = 0$ giving $X = A \cos(\omega x) + B \sin(\omega x)$.

$X(0) = X(\pi) = 0$ give $A = 0$ and $B \sin(\omega\pi) = 0$.

Thus we can have $B \neq 0$ if and only if $\sin(\omega\pi) = 0$

or $\omega\pi = n\pi$ for any $n = 1, 2, 3, \dots$.

Hence the only eigenvalues are $\mu = \mu_n = -n^2$ for $n \in \mathbb{N}$

with corresponding eigenfunctions $X = X_n = \sin(nx)$.

- Solving $Y'' + \mu Y = 0$ with $Y(0) = 0$:

For any $\mu = -n^2$ we have $Y'' - n^2 Y = 0$. Thus $Y = Ae^{ny} + Be^{-ny}$.

The condition $Y(0) = 0$ gives $A + B = 0$ so, substituting,

$$Y = A(e^{ny} - e^{-ny}) = 2A \frac{e^{ny} - e^{-ny}}{2}$$

or $Y = Y_n(y) = \sinh(ny)$, multiplied by any constant.

- Since the solutions $u = X_n(x)Y_n(y)$ all satisfy a homogeneous PDE with homogeneous boundary conditions, the principle of superposition means that any linear combination of such solutions is also a solution. Thus a convergent sum

$$u = \sum_{n=1}^{\infty} A_n X_n Y_n = \sum_{n=1}^{\infty} A_n \sinh(ny) \sin(nx)$$

is also a solution, for constants A_n .

- At $y = 1$ we have $u(x, 1) = \pi$ so that

$$\sum_{n=1}^{\infty} A_n \sinh(n) \sin(nx) = \pi.$$

Since the eigenfunctions $X_n = \sin(nx)$ are orthogonal under the inner product

$$(f, g) = \int_0^{\pi} f(x)g(x) dx \text{ we have } A_n \sinh(n) \int_0^{\pi} \sin^2(nx) dx = \pi \int_0^{\pi} \sin(nx) dx.$$

Integrating $\int_0^{\pi} \sin^2(nx) dx = \int_0^{\pi} \frac{1}{2}(1 - \cos(2nx)) dx = \frac{\pi}{2} - [\frac{1}{2n} \sin(2nx)]_0^{\pi} = \frac{\pi}{2}$ and

$$\int_0^{\pi} \sin(nx) dx = [-\frac{1}{n} \cos(nx)]_0^{\pi} = \frac{2}{n} \text{ if } n \text{ is odd or } 0 \text{ if } n \text{ is even.}$$

Thus $A_n \sinh(n) \frac{\pi}{2} = \pi \frac{2}{n}$ or 0 so that $A_n = \frac{4}{n \sinh(n)}$ or 0.

Setting $n = 2k + 1$, the solution can therefore be written as

$$u = \sum_{k=0}^{\infty} \frac{4}{2k+1} \frac{\sinh((2k+1)y)}{\sinh(2k+1)} \sin((2k+1)x).$$

4. *These answers are more abbreviated.*

(a) $u_{xx} + u_{yy} = 0$ with $u_x(0, y) = u_x(\pi, y) = 0$, $u(x, 0) = 0$ and $u(x, 1) = \frac{1}{2}\pi$:

- Setting $u = X(x)Y(y)$ the PDE becomes $X''Y + XY'' = 0$ and so, for XY non-zero, $\frac{X''}{X} = -\frac{Y''}{Y} = \mu$ with $\mu = \frac{X''}{X}$ independent of y and $\mu = -\frac{Y''}{Y}$ independent of x so that μ is constant. Thus $X'' - \mu X = 0$ and $Y'' + \mu Y = 0$. Homogeneous BCs give $u_x(0, y) = X'(0)Y(y) = 0$, $u_x(\pi, y) = X(\pi)Y(y) = 0$, $u(x, 0) = X(x)Y(0) = 0$ so that, for $X(x)$ and $Y(y)$ non-zero, $X'(0) = X(\pi) = 0$ and $Y(0) = 0$.
- Solving $X'' - \mu X = 0$ with $X'(0) = X(\pi) = 0$
(as for question 1(c) — Exercise: repeat for this case)
gives eigenvalues $\mu = -(n + \frac{1}{2})^2$ for $n = 0, 1, 2, \dots$ and eigenfunctions $X = \cos((n + \frac{1}{2})x)$.
- Solving $Y'' - (n + \frac{1}{2})^2 Y = 0$ with $Y(0) = 0$ gives $Y = Ae^{(n + \frac{1}{2})y} + Be^{-(n + \frac{1}{2})y}$. BC gives $Y(0) = A + B = 0$ so that $B = -A$ and $Y = A(e^{(n + \frac{1}{2})y} - e^{-(n + \frac{1}{2})y}) = 2A \frac{e^{(n + \frac{1}{2})y} - e^{-(n + \frac{1}{2})y}}{2}$ or $Y = \sinh((n + \frac{1}{2})y)$ times any constant.
- Thus $u = \sinh((n + \frac{1}{2})y) \cos((n + \frac{1}{2})x)$ is a solution for any n
- Adding these solutions of the homogeneous problem gives the general solution $u = \sum_{n=0}^{\infty} A_n \sinh((n + \frac{1}{2})y) \cos((n + \frac{1}{2})x)$.
- At $y = 1$, using $u(x, 1) = \frac{1}{2}\pi$ gives $\frac{1}{2}\pi = \sum_{n=0}^{\infty} A_n \sinh(n + \frac{1}{2}) \cos((n + \frac{1}{2})x)$ so that, from orthogonality of the eigenfunctions, $\frac{1}{2}\pi \int_0^{\pi} \cos((n + \frac{1}{2})x) dx = A_n \sinh(n + \frac{1}{2}) \int_0^{\pi} \cos^2((n + \frac{1}{2})x) dx$. Integrating $\int_0^{\pi} \cos^2((n + \frac{1}{2})x) dx = \frac{\pi}{2}$ and $\int_0^{\pi} \cos((n + \frac{1}{2})x) dx = \frac{1}{n + \frac{1}{2}} [\sin((n + \frac{1}{2})x)]_0^{\pi} = \frac{(-1)^n}{n + \frac{1}{2}}$. Thus $\frac{1}{2}\pi \frac{(-1)^n}{n + \frac{1}{2}} = \frac{1}{2}\pi A_n \sinh(n + \frac{1}{2})$ so that $A_n = \frac{(-1)^n}{n + \frac{1}{2}} \frac{1}{\sinh(n + \frac{1}{2})}$ and so

$$u = \sum_{n=0}^{\infty} \frac{(-1)^n}{n + \frac{1}{2}} \frac{\sinh((n + \frac{1}{2})y)}{\sinh(n + \frac{1}{2})} \cos((n + \frac{1}{2})x).$$

(b) $u_{xx} + u_{yy} = 0$ with $u(0, y) = u_x(\pi, y) = 0$, $u_y(x, 0) = 0$ and $u(x, 1) = \frac{1}{2}$:

- Setting $u = X(x)Y(y)$ the PDE becomes $X''Y + XY'' = 0$ and so, for XY non-zero, $\frac{X''}{X} = -\frac{Y''}{Y} = \mu$ with $\mu = \frac{X''}{X}$ independent of y and $\mu = -\frac{Y''}{Y}$ independent of x so that μ is constant. Thus $X'' - \mu X = 0$ and $Y'' + \mu Y = 0$. Homogeneous BCs give $u(0, y) = X(0)Y(y) = 0$, $u_x(\pi, y) = X'(\pi)Y(y) = 0$, $u_y(x, 0) = X(x)Y'(0) = 0$ so that, for $X(x)$ and $Y(y)$ non-zero, $X(0) = X'(\pi) = 0$ and $Y'(0) = 0$.
- Solving $X'' - \mu X = 0$ with $X(0) = X'(\pi) = 0$
(as for question 1(d) — Exercise: repeat for this case)
gives eigenvalues $\mu = -(n + \frac{1}{2})^2$ for $n = 0, 1, 2, \dots$ and eigenfunctions $X = \sin((n + \frac{1}{2})x)$.
- Solving $Y'' - (n + \frac{1}{2})^2 Y = 0$ with $Y(0) = 0$ gives $Y = Ae^{(n + \frac{1}{2})y} + Be^{-(n + \frac{1}{2})y}$. BC gives $Y'(0) = (n + \frac{1}{2})A - (n + \frac{1}{2})B = 0$ so that $B = A$ and $Y = A(e^{(n + \frac{1}{2})y} + e^{-(n + \frac{1}{2})y}) = 2A \frac{e^{(n + \frac{1}{2})y} + e^{-(n + \frac{1}{2})y}}{2}$ or $Y = \cosh((n + \frac{1}{2})y)$ times any constant.
- Thus $u = \cosh((n + \frac{1}{2})y) \sin((n + \frac{1}{2})x)$ is a solution for any n
- Adding these solutions of the homogeneous problem gives the general solution $u = \sum_{n=0}^{\infty} A_n \cosh((n + \frac{1}{2})y) \sin((n + \frac{1}{2})x)$.
- At $y = 1$, using $u(x, 1) = \frac{1}{2}$ gives $\frac{1}{2} = \sum_{n=0}^{\infty} A_n \cosh(n + \frac{1}{2}) \sin((n + \frac{1}{2})x)$ so that, from orthogonality of the eigenfunctions, $\frac{1}{2} \int_0^{\pi} \sin((n + \frac{1}{2})x) dx = A_n \cosh(n + \frac{1}{2}) \int_0^{\pi} \sin^2((n + \frac{1}{2})x) dx$. Integrating $\int_0^{\pi} \sin^2((n + \frac{1}{2})x) dx = \frac{\pi}{2}$ and $\int_0^{\pi} \sin((n + \frac{1}{2})x) dx = -\frac{1}{n + \frac{1}{2}} [\cos((n + \frac{1}{2})x)]_0^{\pi} = \frac{1}{n + \frac{1}{2}}$. Thus $\frac{1}{2} \frac{1}{n + \frac{1}{2}} = \frac{1}{2}\pi A_n \cosh(n + \frac{1}{2})$ so that $A_n = \frac{1/\pi}{n + \frac{1}{2}} \frac{1}{\cosh(n + \frac{1}{2})}$ and so

$$u = \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}} \frac{\cosh((n + \frac{1}{2})y)}{\cosh(n + \frac{1}{2})} \sin((n + \frac{1}{2})x).$$