Assignment 1

The starred problems are for turn in.

1, 2*, 3, 5, 7*, 8*, 9*, 10, 11, 16*.

Here are some additional problems on Bounded variation (not for turn in).

1. Prove if \( f \) has a continuous first derivative \( f' \) on \([a, b]\), then

\[
V_a(f)(b) = \int_a^b |f'| \, dx,
\]

where \( V_a(f)(b) \) denotes the total variation function.

2. Prove for \( a < c < b \),

\[
V_a(f)(b) = V_a(f)(c) + V_c(f)(b).
\]

3. Given \( f \) is \( BV([a, b]) \). Prove that the total variation function \( V_a(f)(x) \) is continuous if and only if \( f \) is continuous on \([a, b]\).

**Hint:** In problem 10 of the book that is assigned but not for turn in, you are asked as a first step to prove Young's inequality. This states that for \( a, b \geq 0 \), and \( 1 < p < \infty \) and \( 1 < q < \infty \)

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Complete the proof outlined in the picture attached to arrive at Young's equality. The proof also shows that one has equality in Young's inequality, provided

\[
b^q = a^p.
\]
Hint for Problem 10

\[ y = x^{q-1}. \]

Fig. 1.

Note if \( q \geq 2 \), the curve \( y = x^{q-1} \) is concave up.
If \( 1 \leq q < 2 \), the curve \( y = x^{q-1} \) is concave down.

Fig. 2.

Show for Fig. 1.

\[ ab \leq \text{Area of Region I} + \text{Area of Region II}. \]

Compute the area of Region I and Region II. Repeat the argument for Fig. 2.

When will \( ab = \text{Region I Area} + \text{Area of Region II} \)?