The Conformal Plate Buckling Equation

SAGUN CHANILLO
AND
MICHAEL K.-H. KIESSLING
Rutgers, The State University of New Jersey

Abstract

The linear equation $\Delta^2 u = 1$ for the infinitesimal buckling under uniform unit load of a thin elastic plate over $\mathbb{R}^2$ has the particularly interesting nonlinear generalization ${\Delta_g}^2 u = 1$, where $\Delta_g = e^{-2u}\Delta$ is the Laplace-Beltrami operator for the metric $g = e^{2u} g_0$, with g_0 the standard Euclidean metric on $\mathbb{R}^2$. This conformal elliptic PDE of fourth order is equivalent to the nonlinear system of elliptic PDEs of second order $\Delta u(x) + K_g(x)\exp(2u(x)) = 0$ and $\Delta K_g(x) + \exp(2u(x)) = 0$, with $x \in \mathbb{R}^2$, describing a conformally flat surface with a Gauss curvature function K_g that is generated self-consistently through the metric's conformal factor. We study this conformal plate buckling equation under the hypotheses of finite integral curvature $\int K_g \exp(2u)\,\mathrm{d}x = \kappa$, finite area $\int \exp(2u)\,\mathrm{d}x = \alpha$, and the mild compactness condition $K_+ \in L^1(B_1(y))$, uniformly with respect to $y \in \mathbb{R}^2$. We show that asymptotically for $|x| \to \infty$, all solutions behave like $u(x) = -(\kappa/2\pi)\ln|x| + C + o(1)$ and $K(x) = -(\alpha/2\pi)\ln|x| + C + o(1)$, with $\kappa \in (2\pi, 4\pi)$ and $\alpha = \sqrt{2\kappa(4\pi - \kappa)}$. We also show that for each $\kappa \in (2\pi, 4\pi)$, there exists a K^* and a radially symmetric solution pair u, K satisfying $\mathcal{K}(u) = \kappa$ and $\max K = K^*$, which is unique modulo translation of the origin and scaling of x coupled with a translation of u.

1 Introduction

In this paper we study the nonlinear, fourth-order elliptic PDE

$$ {\Delta_g}^2 u(x) = \lambda\,, \quad x \in \mathbb{R}^2\,, \tag{1.1} $$

for a smooth scalar function $u : \mathbb{R}^2 \to \mathbb{R}$, where $\Delta_g = e^{-2u}\Delta_{g_0}$ is the Laplace-Beltrami operator with respect to the conformally flat metric $g = e^{2u} g_0$, with g_0 the Euclidean standard metric of $\mathbb{R}^2$ or $\Delta_{g_0} \equiv \Delta$ the standard Laplacian on $\mathbb{R}^2$, and $\lambda \in \mathbb{R}^+$ a parameter. In the limit of small u, the nonlinear equation (1.1) reduces to the linear equation

$$ \Delta^2 u(x) = \lambda\,, \quad x \in \mathbb{R}^2\,, \tag{1.2} $$

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which is familiar from the linear theory of stationary buckling of a thin, elastic plate under uniform load λ. For this reason, we will call (1.1) the *conformal plate buckling equation*.

For fixed λ, equation (1.1) is invariant under the isometries of Euclidean space $\mathbb{R}^2$ and under the scaling $x \mapsto kx$ combined with the translation $u \mapsto u - \ln k$, where $k > 0$. On the punctured plane, (1.1) is also invariant under the Kelvin transform (inversion) $x \mapsto x/|x|^2$ combined with the map $u(x) \mapsto u(x/|x|^2) - 2\ln|x|$. However, as we shall see, the singularity at the origin is not removable so that invariance under the full Euclidean group of $\mathbb{R}^2$ does not hold.

If we allow λ to change its value under a transformation, then (1.1) is also invariant under the combined transformation $u \mapsto u + u_0$ and $\lambda \mapsto e^{-4u_0}\lambda$. Thus, by choosing the constant $u_0 = \ln \lambda^{1/4}$, we can always achieve that

$$\lambda = 1\,. \tag{1.3}$$

Henceforth we assume (1.3) without loss of generality.

For $\lambda = 1$ the fourth-order equation (1.1) is equivalent to the nonlinear system of second-order elliptic PDEs

$$-\Delta u(x) = K(x)e^{2u(x)}\,, \tag{1.4}$$

$$-\Delta K(x) = e^{2u(x)}\,, \tag{1.5}$$

which describes a conformally flat surface over $\mathbb{R}^2$ with metric $g = e^{2u}g_0$ and Gauss curvature function $K \equiv K_g$ generated in a self-consistent manner. While a considerable literature has accumulated about the celebrated prescribed Gauss curvature problem where K is given and only u has to be found by solving (1.4)—see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 21, 22, 23, 24, 25, 26, 27, 28] and further references therein—the literature on self-consistent Gauss curvature problems is relatively sparse [6, 16, 18, 20]. In particular, we are not aware of any previous study of the self-consistent Gauss curvature problem (1.4)–(1.5), or, equivalently, the conformal plate buckling equation.

We now present our main results for the conformal plate buckling equation, which we state in their equivalent self-consistent Gauss curvature form. We are interested in surfaces over $\mathbb{R}^2$ having finite area

$$\mathcal{A}(u) = \int_{\mathbb{R}^2} e^{2u(x)}\,\mathrm{d}x \tag{1.6}$$

and finite integral curvature

$$\mathcal{K}(u) = \int_{\mathbb{R}^2} K(x)e^{2u(x)}\,\mathrm{d}x\,. \tag{1.7}$$

THEOREM 1.1 *Assume $u \in C^{2,\alpha}$ and $K \in C^{2,\alpha}$ jointly solve* (1.4) *and* (1.5) *for finite integral curvature, $\mathcal{K}(u) = \kappa$, and finite area $\mathcal{A}(u) = \alpha$. In addition, assume* *that* $K_+ \in L^1(B_1(x_0))$ *uniformly with respect to* x_0*, where* $K_+ \equiv \max\{K, 0\}$*. Then, uniformly as* $|x| \to \infty$*, we have*

$$u(x) = -\kappa \frac{1}{2\pi} \ln|x| + u(0) + \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \ln|y| K(y) e^{2u(y)}\, \mathrm{d}y + o(1)\,, \tag{1.8}$$

$$K(x) = -\alpha \frac{1}{2\pi} \ln|x| + K(0) + \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \ln|y| e^{2u(y)}\, \mathrm{d}y + o(1)\,, \tag{1.9}$$

with $\kappa \in (2\pi, 4\pi)$ *and with* $\alpha \in (0, 2^{3/2}\pi)$ *given by*

$$\alpha = \sqrt{2\kappa(4\pi - \kappa)}\,. \tag{1.10}$$

Remark.

(1) Since $\kappa \in (2\pi, 4\pi)$, the map $\kappa \mapsto \alpha$ given in (1.10) is strictly monotonically decreasing, hence invertible, so that alternatively to (1.10) we have

$$\kappa = 2\pi \left(1 + \sqrt{1 - \frac{1}{2}\left(\frac{\alpha}{2\pi}\right)^2}\right). \tag{1.11}$$

(2) The corresponding results for general positive load λ in (1.1) obtain by replacing $\alpha \mapsto \sqrt{\lambda}\,\alpha$ in (1.10) and (1.11). This leaves the bounds on κ unchanged, i.e., $2\pi < \kappa < 4\pi$, while the bounds on α change to $0 < \alpha < \sqrt{2/\lambda}\, 2\pi$.

Our next theorem asserts that the range of integral curvature values $\kappa \in (2\pi, 4\pi)$ displayed in Theorem 1.1 is optimal, and so is then the associated range of values of the area $\alpha \in (0, 2^{3/2}\pi)$.

THEOREM 1.2 *For each* $\kappa \in (2\pi, 4\pi)$ *there exists a value* $K^* > 0$ *such that* (1.4)–(1.5) *are jointly solved by any member of a unique orbit of radially symmetric and decreasing* C^∞ *function pairs* u, K*, each pair having a common center of symmetry* x_* *and satisfying* $\max K = K^*$ *and* $\mathcal{K}(u) = \kappa$*. The group actions for the orbit are the translations of* x_* *and the scalings* $x \mapsto kx$ *coupled with the target space translations* $u \mapsto u - \ln k$*.*

Remark. A radial solution pair u, K is illustrated in Figures 1.1 and 1.2.

We conclude our introduction with two interesting open questions.

PROBLEM 1.3 Is the value K^* in Theorem 1.2 uniquely determined by each $\kappa \in (2\pi, 4\pi)$?

Remark. We can show that there is a surjective map $K^* \mapsto \kappa$ on the interval of admissible K^*; Problem 1.3 asks whether this map is also injective.

PROBLEM 1.4 Given the conditions stated in Theorem 1.1, are all solution pairs u, K of (1.4)–(1.5) radially symmetric?

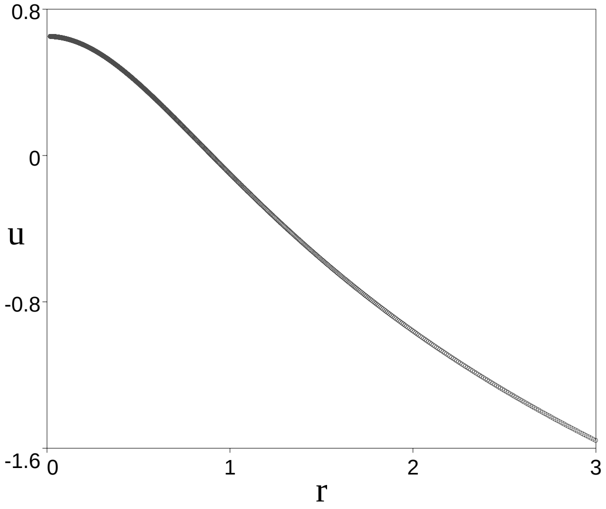

FIGURE 1.1. Graph of a radial solution u of (1.1) as a function of the radial coordinate r. The parameter $\lambda = 1$.

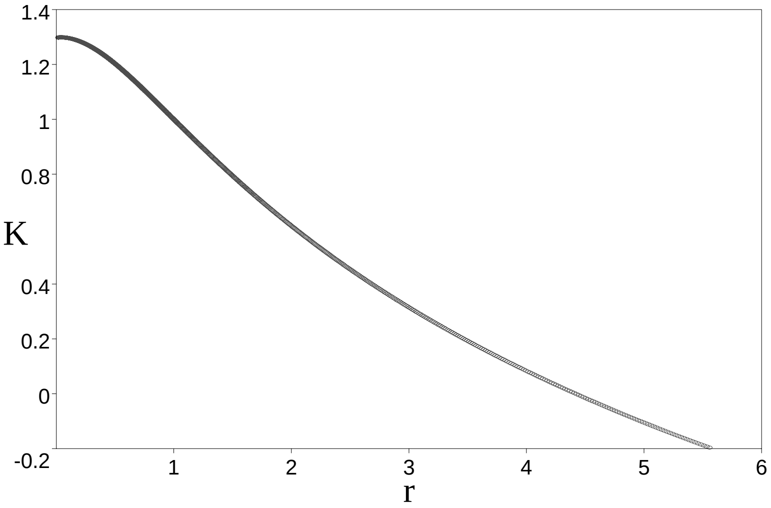

FIGURE 1.2. Graph of the Gauss curvature K for the radial solution u displayed in Figure 1.1.

Remark. We tend to believe that the answer to Problem 1.4 is affirmative, but so far a proof has resisted all our attempts.

We now turn to the proofs of our two theorems. Theorem 1.1 will be proved in Section 2 essentially by using harmonic analysis techniques. Theorem 1.2 is proved in Section 3 by applying techniques from scattering theory to an equivalent scattering problem for a Newtonian point particle in $\mathbb{R}^2$ moving in a given potential.

2 Proof of Theorem 1.1

We begin with the observation that standard elliptic regularity theory [17] tells us that if u and K jointly solve (1.4) and (1.5), with $u \in C^{2,\alpha}$, then by (1.5) also $K \in C^{4,\alpha}$, from which it now follows via (1.4) that $u \in C^{4,\alpha}$, whence $u \in C^\infty$ and $K \in C^\infty$ by bootstrapping.

We next state a representation lemma.

LEMMA 2.1 *Together with the hypotheses of Theorem* 1.1*, equations* (1.4) *and* (1.5) *are equivalent to the pair of integral equations*

$$u(x) = u(0) - \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} (\ln|x-y| - \ln|y|) K(y) e^{2u(y)} \, \mathrm{d}y \,, \tag{2.1}$$

$$K(x) = K(0) - \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} (\ln|x-y| - \ln|y|) e^{2u(y)} \, \mathrm{d}y \,. \tag{2.2}$$

PROOF: Clearly, if u, K jointly solve (2.1)–(2.2) and satisfy the other hypotheses of Theorem 1.1, then u, K jointly solve (1.4)–(1.5) under these hypotheses. To prove the converse, let $u \in C^\infty$ satisfy $\int \exp(2u) \mathrm{d}x < \infty$, and let $K \in C^\infty$ solve (1.5). Then K is given by

$$K(x) = H(x) - \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} (\ln|x-y| - \ln|y|) e^{2u(y)} \, \mathrm{d}y \,, \tag{2.3}$$

where $H(x)$ is an entire harmonic function on $\mathbb{R}^2$. Now, by hypothesis, $K_+ \in L^1(B_1(x_0))$, uniformly with respect to $x_0 \in \mathbb{R}^2$. Thus, from (2.3) and $\exp(2u) \in L^1(\mathbb{R}^2)$, we have that $H(x) \le C + C \ln|x|$, whence H is a constant. By inspection of (2.3), it now follows that $H = K(0)$.

We now take into account that our u also solves (1.4) and that $\int K \exp(2u) \mathrm{d}x < \infty$. Then u is given by

$$u(x) = h(x) - \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} (\ln|x-y| - \ln|y|) K(y) e^{2u(y)} \, \mathrm{d}y \,, \tag{2.4}$$

where $h(x)$ is an entire harmonic function on $\mathbb{R}^2$. We now show that $h(x) = u(0)$.

To this effect, having just proven (2.2), we now observe that (2.2) tells us that $K(x) < 0$ for $|x| > R$ (with R sufficiently large, depending on u), whence u is subharmonic for $|x| > R$, and so is u_+, the positive part of u. Thus, for $|y| > 2R$ we have $\|u_+\|_{L^\infty(B_{1/2}(y))} \le C \|u_+\|_{L^1(B_1(y))}$, with C independent of y for $|y| > 2R$. But then, since $u \in C^\infty$, we even have $\|u_+\|_{L^\infty(B_{1/2}(y))} \le C \|u_+\|_{L^1(B_1(y))}$, with C independent of $y \in \mathbb{R}^2$. Furthermore, we have $\|u_+\|_{L^1(B_1(y))} < C$ uniformly with respect to $y \in \mathbb{R}^2$. Namely, setting $\Lambda_y = \operatorname{supp} u_+ \cap B_1(y)$, we have $\|u_+\|_{L^1(B_1(y))} = \|u\|_{L^1(\Lambda_y)} \le \int_{\Lambda_y} \exp(2u) \, \mathrm{d}x \le \int_{\mathbb{R}^2} \exp(2u) \, \mathrm{d}x < \infty$, the last step by our hypothesis. Thus, we conclude that $\|u_+\|_{L^1(B_1(y))} < C$ uniformly with respect to $y \in \mathbb{R}^2$, as claimed. Hence, $\|u_+\|_{L^\infty(B_{1/2}(y))} \leq C$ uniformly with respect to $y \in \mathbb{R}^2$, i.e., $u_+ \in L^\infty(\mathbb{R}^2)$. Finally, from $u_+ \in L^\infty(\mathbb{R}^2)$, together with (2.4) and $K \exp(2u) \in L^1(\mathbb{R}^2)$, we conclude that $h(x) \leq C + C \ln |x|$, whence h is a constant, $h = u(0)$, by inspection of equation (2.4). ☐

COROLLARY 2.2 *Assume u, K jointly solve* (1.4)–(1.5) *and satisfy the other hypotheses of Theorem* 1.1. *Then, uniformly as $|x| \to \infty$, we have*

$$u(x) = -\kappa \frac{1}{2\pi} \ln |x| + u(0) + \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \ln |y| K(y) e^{2u(y)} \, \mathrm{d}y + o(1) \,, \tag{2.5}$$

$$K(x) = -\alpha \frac{1}{2\pi} \ln |x| + K(0) + \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \ln |y| e^{2u(y)} \, \mathrm{d}y + o(1) \,. \tag{2.6}$$

PROOF: By Lemma 2.1, u, K jointly solve (2.1)–(2.2), with $\mathcal{K}(u) = \kappa$ and $\mathcal{A}(u) = \alpha$. By (2.2) and $\mathcal{A}(u) = \alpha$, we immediately have

$$\lim_{|x| \to \infty} \frac{K(x)}{\ln |x|} = -\frac{1}{2\pi} \alpha \,. \tag{2.7}$$

Furthermore, since $\mathcal{K}(u) = \kappa$, we conclude that $\int_{\mathbb{R}^2} \ln(1+|x|) e^{2u(x)} \, \mathrm{d}x < \infty$. With these estimates, our Corollary 2.2 now follows at once from (2.1)–(2.2). ☐

COROLLARY 2.3 *Under the hypotheses of Theorem* 1.1, *the integral curvature is bounded below by*

$$\kappa > 2\pi \,. \tag{2.8}$$

PROOF: Assume $\kappa \leq 2\pi$. It then follows immediately from the asymptotic formula (2.5) that $\int_{\mathbb{R}^2} \exp(2u) \mathrm{d}x = \infty$, in contradiction to our hypothesis that $\mathcal{A}(u) = \alpha$. Hence, the lower bound (2.8) holds. ☐

Our next result is a Pohozaev identity for the system (1.4)–(1.5).

PROPOSITION 2.4 *Under the hypotheses of Theorem* 1.1, *the integral curvature κ and the area α satisfy the identity*

$$\alpha^2 = 2\kappa(4\pi - \kappa) \,. \tag{2.9}$$

PROOF: We multiply (1.4) by $-x \cdot \nabla u(x)$ and (1.5) by $-x \cdot \nabla K(x)$; then we integrate over B_R, apply the usual scheme of integration by parts on the left-hand sides, and get, respectively,

$$R \int\limits_{\partial B_R} \left((\nu \cdot \nabla u(x))^2 - \frac{1}{2} |\nabla u(x)|^2 \right) \mathrm{d}\sigma = -\frac{1}{2} \int\limits_{B_R} K(x) x \cdot \nabla e^{2u(x)} \, \mathrm{d}x \,, \tag{2.10}$$

$$R \int\limits_{\partial B_R} \left((\nu \cdot \nabla K(x))^2 - \frac{1}{2} |\nabla K(x)|^2 \right) \mathrm{d}\sigma = -\int\limits_{B_R} e^{2u(x)} x \cdot \nabla K(x) \mathrm{d}x \,. \tag{2.11}$$

By multiplying (2.10) by 2 and adding the result to (2.11), we obtain

$$R\int_{\partial B_R}\Big(2(\nu\cdot\nabla u(x))^2-|\nabla u(x)|^2+(\nu\cdot\nabla K(x))^2-\frac{1}{2}|\nabla K(x)|^2\Big)\,\mathrm{d}\sigma = -\int_{B_R} x\cdot\nabla\big(K(x)e^{2u(x)}\big)\,\mathrm{d}x\,. \tag{2.12}$$

Integrating next by parts on the right-hand side, using that $\nabla\cdot x = 2$ for $x\in\mathbb{R}^2$ and collecting all perimeter integrals on the left side, we get

$$R\int_{\partial B_R}\Big((\nu\cdot\nabla u(x))^2-\frac{1}{2}|\nabla u(x)|^2+\frac{1}{2}(\nu\cdot\nabla K(x))^2-\frac{1}{4}|\nabla K(x)|^2\Big)\mathrm{d}\sigma + \frac{1}{2}R\int_{\partial B_R}K(x)e^{2u(x)}\,\mathrm{d}\sigma = \int_{B_R}K(x)e^{2u(x)}\,\mathrm{d}x\,. \tag{2.13}$$

We now let $R\to\infty$. Clearly,

$$\int_{B_R}K(x)e^{2u(x)}\,\mathrm{d}x\to\kappa\quad\text{as } R\to\infty\,. \tag{2.14}$$

Furthermore, from Corollary 2.2 we infer right away that

$$R\int_{\partial B_R}K(x)e^{2u(x)}\,\mathrm{d}\sigma\to 0\qquad\text{as } R\to\infty\,, \tag{2.15}$$

$$R\int_{\partial B_R}\Big((\nu\cdot\nabla u(x))^2-\frac{1}{2}|\nabla u(x)|^2\Big)\mathrm{d}\sigma\to\frac{\kappa^2}{4\pi}\quad\text{as } R\to\infty\,, \tag{2.16}$$

$$R\int_{\partial B_R}\Big((\nu\cdot\nabla K(x))^2-\frac{1}{2}|\nabla K(x)|^2\Big)\mathrm{d}\sigma\to\frac{\alpha^2}{4\pi}\quad\text{as } R\to\infty\,. \tag{2.17}$$

Thus, taking the limit $R\to\infty$ in our identity (2.13), we obtain (2.9). Since $\alpha>0$, we see that (2.9) is identical to (1.10). □

COROLLARY 2.5 *The integral curvature is bounded above by*

$$\kappa<4\pi\,. \tag{2.18}$$

The area is bounded above by

$$\alpha<2^{3/2}\pi\,. \tag{2.19}$$

PROOF: The bound (2.18) is an immediate spinoff of (2.9), recalling that, by definition, $\alpha>0$. The bound (2.19) is an immediate consequence of (2.9) and the lower bound $\kappa>2\pi$; see (2.8) in Corollary 2.3. □

This concludes the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section we prove the existence of radial solution pairs u, K of the system (1.4)–(1.5) for all possible values $\kappa \in (2\pi, 4\pi)$ of the integral curvature $\mathcal{K}$ defined in (1.7), having finite area $\mathcal{A} = \alpha$ defined in (1.6) and given by (1.10). Looking only for radial solutions reduces our PDEs for K and u to two ODEs. We transform these ODEs for K and u into a potential-scattering problem for a single Newtonian particle in $\mathbb{R}^2$ that we solve by fixed-point arguments aided with gradient flow techniques. This strategy is adapted from [20], where another self-consistent Gauss curvature problem is considered.

Let $\xi = f_\xi(t)$ and $\eta = f_\eta(t)$ be the time-dependent Cartesian coordinates of a point in $\mathbb{R}^2$ that moves according to Newton's equations of motion

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -\frac{\partial V}{\partial \xi}\,, \tag{3.1}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} = -\frac{\partial V}{\partial \eta}\,, \tag{3.2}$$

in a fixed external potential

$$V(\xi, \eta) = \frac{1}{2}\eta e^{2\xi}\,. \tag{3.3}$$

We will sometimes write $\xi(t)$, $\eta(t)$ and $\dot{\xi}(t)$, $\dot{\eta}(t)$ to denote solutions and their respective time derivatives. We seek solutions of (3.1)–(3.3) that satisfy the asymptotic conditions

$$\lim_{t\to-\infty} \xi(t) - t = \xi_{\mathrm{in}}\,, \tag{3.4}$$

$$\lim_{t\to-\infty} \eta(t) = \eta_{\mathrm{in}}\,, \tag{3.5}$$

for suitable real constants ξ_{in} and η_{in} such that

$$\lim_{t\to+\infty} \frac{\xi(t)}{t} = \cos\Theta\,, \tag{3.6}$$

$$\lim_{t\to+\infty} \frac{\eta(t)}{t} = \sin\Theta\,, \tag{3.7}$$

for some $\Theta \in (-\pi, -\frac{\pi}{2})$. Clearly, the asymptotic conditions (3.4)–(3.7) imply that asymptotically in the infinite past and the infinite future the particle performs a linear, unaccelerated motion. These two "asymptotically free motions" are linked by a deflection of the particle off of its initial direction by an angle Θ, which is effected by the external potential V. Our problem thus belongs under the category "potential scattering."

THEOREM 3.1 *For each* $\Theta \in (-\pi, -\frac{\pi}{2})$ *there exists a constant* $\eta_{\mathrm{in}} > 0$ *such that for each* $\xi_{\mathrm{in}} \in \mathbb{R}$ *there exists a unique solution pair* $\xi(t)$, $\eta(t)$ *of* (3.1)–(3.3) *satisfying* (3.4)–(3.7). *Within the family of solutions belonging to the same* η_{in} *we can*

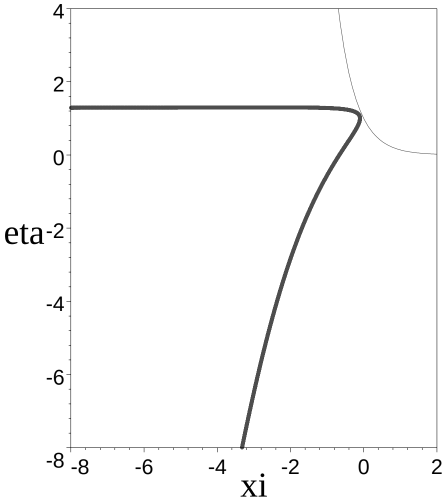

FIGURE 3.1. The thick curve is a trajectory for the scattering problem (3.1)–(3.7). The thin curve is the locus of singular points (see text).

switch from one solution to another by means of the transformation $\xi_{\text{in}} \to \xi'_{\text{in}}$ *combined with a corresponding time translation* $t \to t + \xi_{\text{in}} - \xi'_{\text{in}}$. *This transformation leaves* Θ *unchanged.*

Remark. A trajectory solving the scattering problem (3.1)–(3.7) is displayed in Figure 3.1.

Before we prove Theorem 3.1, we first show that our Theorem 1.2 is a corollary of Theorem 3.1.

PROOF OF THEOREM 1.2: Let $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, denote the motion of a Newtonian point particle in $\mathbb{R}^2$ according to (3.1)–(3.2) with V given in (3.3), having asymptotic behavior given by (3.4)–(3.7). By Theorem 3.1, such a motion exists. Inserting (3.3) into (3.1) and (3.2), the equations of motion read explicitly

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} = -\eta e^{2\xi}\,, \tag{3.8}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} = -\frac{1}{2}e^{2\xi}\,. \tag{3.9}$$

We now set $t = \ln r$ for $r > 0$, define

$$\overline{u}(r) = f_\xi(\ln r) - \ln r - \frac{1}{4}\ln 2\,, \tag{3.10}$$

$$\overline{K}(r) = \sqrt{2}\, f_\eta(\ln r)\,, \tag{3.11}$$

and find that for $r > 0$, the functions $\overline{u}(r)$ and $\overline{K}(r)$ satisfy

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\overline{u}(r)\right) = \overline{K}(r)e^{2\overline{u}(r)}\,, \tag{3.12}$$

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\overline{K}(r)\right) = e^{2\overline{u}(r)}\,. \tag{3.13}$$

Moreover, we can set $\overline{K}(0) = \sqrt{2}\,\eta_{\text{in}}$ and $\overline{u}(0) = \xi_{\text{in}} - \frac{1}{4}\ln 2$. Identifying $r = |x - x_*|$ for $x \in \mathbb{R}^2$, with x_* the arbitrary center of symmetry, we recognize that (3.12) is (1.4), and (3.13) is (1.5), for radially symmetric $K(x) = \overline{K}(|x - x_*|)$ and $u(x) = \overline{u}(|x - x_*|)$. Furthermore, from (3.9) it follows that $K(x)$ is decreasing away from x_*, and from (3.5) we have $K^* = \sqrt{2}\,\eta_{\text{in}}$. From (3.7) it follows that $K(x) \sim -\sqrt{2}\,\sin\Theta\ln|x|$ as $|x| \to \infty$, as claimed. We have the identification $2\pi\sqrt{2}\,\sin\Theta = \alpha$, so that from (3.7) and (3.12) it follows that $\mathcal{K}(u) = (4\pi + \sqrt{16\pi^2 - 2\alpha^2})/2 \in (2\pi, 4\pi)$, as demanded by (1.11). Finally, translations $t \mapsto t + t_0$ combined with an associated translation $\xi \to \xi + \xi_0$ correspond to scalings $r \mapsto kr$ combined with translations $u \mapsto u - \ln k$, which together with the arbitrariness of x_* proves that u, K is unique modulo the conformal transformations listed in Theorem 1.2. □

It remains to prove Theorem 3.1.

PROOF OF THEOREM 3.1: We begin by listing the symmetries of the ODE system (3.8)–(3.9), which are:

- invariance under time translations $t \to t + t_0$,
- invariance under time reversal $t \to -t$, and
- invariance under homologous transformations $\xi \to \xi + \xi_h$, $t \to e^{-\xi_h}t$.

By E. Noether's theorem, invariance under time translations is associated with the conservation law for the total (kinetic plus potential) energy E of the Newtonian unit mass point, where

$$2E = \dot{\xi}^2 + \dot{\eta}^2 + \eta e^{2\xi}\,. \tag{3.14}$$

Under the homologous transformations $\xi \to \xi + \xi_h$ and $t \to e^{-\xi_h}t$, the conserved quantity E transforms as $E \to e^{2\xi_h}E$. Hence, to obtain all solutions of (3.8)–(3.9), it suffices to obtain all solutions for three generic values of E, say $E = E_+ > 0$, $E = 0$, and $E = E_- < 0$. For the motion of interest to us, the asymptotic conditions (3.4) and (3.5) give

$$E = \frac{1}{2}\,. \tag{3.15}$$

LEMMA 3.2 *A solution* $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, *of equations* (3.8)–(3.9) *satisfying* (3.4)–(3.7) *is restricted to the region* $\{(\xi,\eta) \in \mathbb{R}^2 : \eta < e^{-2\xi}\}$.

PROOF: Since kinetic energy is nonnegative, (3.15) cannot be achieved in the "$E = \frac{1}{2}$ forbidden zone" where $\eta > e^{-2\xi}$. Hence, a solution $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, of the Newtonian equations of motion (3.8)–(3.9) satisfying (3.4)–(3.7) is confined to the region $\{(\xi, \eta) \in \mathbb{R}^2 : \eta \leq e^{-2\xi}\}$. It remains to show that a solution cannot have a point in common with the boundary $\{\eta = e^{-2\xi}\}$ of the $E = \frac{1}{2}$ forbidden zone.

The boundary $\eta = e^{-2\xi}$ of the $E = \frac{1}{2}$ forbidden zone consists of all points $(\xi, \eta) \in \mathbb{R}^2$ for which $E = \frac{1}{2}$ is achieved if and only if $\dot{\xi} = 0 = \dot{\eta}$. Recall that a *singular point* on a trajectory is a point at which both $\dot{\xi} = 0$ and $\dot{\eta} = 0$; hence, the boundary of the $E = \frac{1}{2}$ forbidden zone consists of all the possible singular points. A trajectory that contains (at least one) singular point is called a *singular trajectory*. Thus, a singular trajectory has at least one point in common with the boundary of the $E = \frac{1}{2}$ forbidden zone. On the other hand, it follows immediately from (3.9) that there can be at most one singular point on a singular trajectory; hence a singular trajectory has exactly one singular point. By the time-translation invariance of (3.8)–(3.9), we can assume that this point is reached at $t = 0$. By the time-reversal invariance of (3.8)–(3.9), it now follows that on a singular trajectory the forward motion with respect to $t = 0$ is identical to the backward motion with respect to $t = 0$. This in turn implies that the asymptotic conditions are symmetric under time reversal as well. But then by (3.4) and (3.6), we conclude that $\cos\Theta = 1$, which implies $\sin\Theta = 0$, which contradicts the condition that $\Theta \in (-\pi, -\frac{\pi}{2})$. Hence, the motion on a singular trajectory cannot satisfy all our asymptotic conditions. Put differently, a solution to our equations of motion that does satisfy all asymptotic conditions cannot be singular. Lemma 3.2 is proved. □

LEMMA 3.3 *Let* $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, *solve* (3.8)–(3.9) *for the asymptotic conditions* (3.4)–(3.5). *Then the map* $f = f_\xi \circ f_\eta^{-1}$ *is well defined on the set* $f_\eta(\mathbb{R})$, *and we have* $\xi = f(\eta)$. *Furthermore, there exists a unique* $\eta_\sim < \eta_{\text{in}}$ *such that* f *is strictly convex for* $\eta < \eta_\sim$ *and strictly concave for* $\eta > \eta_\sim$.

PROOF: Integrating (3.9) once, using (3.5), gives

$$\dot{\eta}(t) = -\frac{1}{2}\int_{-\infty}^{t} e^{2\xi(s)}\,\mathrm{d}s\,. \tag{3.16}$$

Clearly, the map $t \mapsto \dot{\eta}(t)$ is strictly negative for all $t > -\infty$; hence, the map $t \mapsto \eta = f_\eta(t)$ is strictly monotonically decreasing and thus invertible, giving $t = f_\eta^{-1}(\eta)$.

Next, let $'$ denote the derivative with respect to η. Along a trajectory $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, that solves (3.8)–(3.9) for the asymptotic conditions (3.4)–(3.5), we then have

$$f''(\eta) = \frac{\mathrm{d}^2\xi}{\mathrm{d}\eta^2} = \frac{1}{\dot{\eta}^3}(\ddot{\xi}\dot{\eta} - \ddot{\eta}\dot{\xi}) = \frac{\exp(2\xi)}{2\dot{\eta}^3}(\dot{\xi} - 2\eta\dot{\eta})\,, \tag{3.17}$$

the middle and right sides evaluated at t, the left side at $\eta = f_\eta(t)$. By (3.16), the map $t \mapsto \dot\eta^3$ is negative and strictly monotonically decreasing.

Next notice that by multiplying (3.9) by 2η and subtracting that result from (3.8), we get

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - 2\eta\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2} = 0\,. \tag{3.18}$$

Upon integrating (3.18) from $-\infty$ to t, using integration by parts, we obtain

$$(\dot\xi - 2\eta\dot\eta)(t) = 1 - 2\int_{-\infty}^{t} \dot\eta^2(s)\,\mathrm{d}s\,. \tag{3.19}$$

Since $t \mapsto \dot\eta^2(t)$ is positive and strictly monotonically increasing, by (3.19) we now conclude that the map $t \mapsto \int_{-\infty}^{t} \dot\eta^2(s)\,\mathrm{d}s$ is strictly monotonically increasing and strictly convex. Therefore there exists a unique $t_\sim$ such that the right-hand side of (3.19) is strictly positive for $t < t_\sim$ and strictly negative for $t > t_\sim$. Setting $\eta_\sim \equiv f_\eta(t_\sim)$, we then conclude that the right-hand side of (3.19) evaluated at $t = f_\eta^{-1}(\eta)$ is strictly positive for $\eta > \eta_\sim$ and strictly negative for $\eta < \eta_\sim$. We thus conclude from (3.17) that along the trajectory $\xi = f(\eta)$ we have

$$f''(\eta)\begin{cases} > 0 & \text{for } \eta < \eta_\sim \\ < 0 & \text{for } \eta > \eta_\sim \end{cases} \tag{3.20}$$

as claimed. □

By the convexity of $\eta \mapsto \xi = f(\eta)$ for $\eta < \eta_\sim$, it follows that a solution $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, of (3.8)–(3.9) and (3.4)–(3.5) that satisfies a linear bound $f(\eta) < A\eta + B$ for some constants $A > 0$ and B necessarily satisfies the asymptotic conditions (3.6)–(3.7) for some $\Theta \in (-\pi, -\frac{\pi}{2})$. Part of our existence proof will concentrate on proving that for η_{in} large enough, such a linear bound on f exists.

On the other hand, such a linear bound on f will fail to exist if η_{in} is negative. Namely, by (3.16) we have $\dot\eta(t) < 0$ for all $t > -\infty$, which implies that $\sup_t \eta(t) = \lim_{t\to-\infty} \eta(t)$. By (3.5) we then have $\sup_t \eta(t) = \eta_{\text{in}}$. Therefore, if $\eta_{\text{in}} \leq 0$, we conclude that $\eta(t) < 0$ for all $t > -\infty$.

Now integrating (3.8) once, using (3.4), we obtain

$$\dot\xi(t) = 1 - \int_{-\infty}^{t} \eta(s)e^{2\xi(s)}\,\mathrm{d}s\,. \tag{3.21}$$

Since $\eta(t) < 0$ for all $t > -\infty$ if $\eta_{\text{in}} \leq 0$, (3.21) now implies that $\dot\xi(t) > 0$ for all $t > -\infty$, which contradicts the asymptotic condition (3.7), which is negative for $\Theta \in (-\pi, -\frac{\pi}{2})$. Hence, we have the following:

PROPOSITION 3.4 *If a solution $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, of* (3.8)–(3.9) *satisfies* (3.4)–(3.7), *with $\Theta \in (-\pi, -\frac{\pi}{2})$, then $\eta_{\text{in}} > 0$.*

Next, let $T = T(\xi_{\text{in}}, \eta_{\text{in}})$ be the instant where the maximal Cauchy development terminates. Then for $t < T$ the system of differential equations (3.8)–(3.9) with asymptotic conditions (3.4)–(3.5) is equivalent to the coupled system of nonlinear integral equations

$$\xi(t) = \xi_{\text{in}} + t - \int_{-\infty}^{t} \int_{-\infty}^{s} \eta(\tilde{s}) e^{2\xi(\tilde{s})} \, \mathrm{d}\tilde{s} \, \mathrm{d}s \,, \tag{3.22}$$

$$\eta(t) = \eta_{\text{in}} - \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{s} e^{2\xi(\tilde{s})} \, \mathrm{d}\tilde{s} \, \mathrm{d}s \,, \tag{3.23}$$

obtained by integrating (3.21) using (3.4) and integrating (3.16) using (3.5). We remark that there do exist solutions that blow up at a finite time $T < \infty$ if η_{in} is below some critical value (in particular, this is the case if $\eta_{\text{in}} < 0$).

To analyze (3.22)–(3.23), we study the coupled iteration sequences

$$\xi^{(n)}(t) = \xi_{\text{in}} + t - \int_{-\infty}^{t} \int_{-\infty}^{s} \eta^{(n)}(\tilde{s}) e^{2\xi^{(n)}(\tilde{s})} \, \mathrm{d}\tilde{s} \, \mathrm{d}s \,, \tag{3.24}$$

$$\eta^{(n+1)}(t) = \eta_{\text{in}} - \frac{1}{2} \int_{-\infty}^{t} \int_{-\infty}^{s} e^{2\xi^{(n)}(\tilde{s})} \, \mathrm{d}\tilde{s} \, \mathrm{d}s \,, \tag{3.25}$$

$n \geq 0$, with the starting function $\eta^{(0)}$ given by

$$\eta^{(0)}(t) \equiv \eta_{\text{in}} \,. \tag{3.26}$$

By inspection one readily checks that, if the iteration sequences (3.24)–(3.25) with starting function (3.26) converge for all $t < T$, then they converge to functions $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, solving (3.22)–(3.23). We have to show that for large enough η_{in}, the sequences converge to functions also satisfying (3.6)–(3.7), in which case $T = \infty$.

LEMMA 3.5 *For $\eta_{\text{in}} > 0$, the maps $n \mapsto \xi^{(n)}$ and $n \mapsto \eta^{(n)}$ defined jointly by the iteration sequences* (3.24)–(3.25) *with starting function* (3.26) *are pointwise increasing and decreasing, respectively, for each fixed $t > -\infty$.*

PROOF: The claim of Lemma 3.5 follows by standard sub- and supersolution arguments. Using (3.26), we see that (3.24) for $n = 0$ reads

$$\xi^{(0)}(t) = \xi_{\text{in}} + t - \eta_{\text{in}} \int_{-\infty}^{t} \int_{-\infty}^{s} e^{2\xi^{(0)}(\tilde{s})} \, \mathrm{d}\tilde{s} \, \mathrm{d}s \,. \tag{3.27}$$

For $\eta_{\text{in}} > 0$ the nonlinear integral equation (3.27) is solved uniquely by

$$\xi^{(0)}(t) = -\ln\cosh\left(t + \xi_{\text{in}} - \ln\left(\frac{2}{\sqrt{\eta_{\text{in}}}}\right)\right) - \ln\sqrt{\eta_{\text{in}}} \,. \tag{3.28}$$

Thus, for all $p > 0$ and $t > -\infty$ the integral $\int_{-\infty}^{t} \int_{-\infty}^{s} |\tilde{s}|^p e^{2\xi^{(0)}(\tilde{s})} \, \mathrm{d}\tilde{s} \, \mathrm{d}s$ exists; in particular, the integral exists for $p = 0$. Therefore (3.25) for $n = 0$ is well defined for all $t > -\infty$, and by integration we find $\eta^{(1)}(t)$ to be given by

$$\begin{aligned}\eta^{(1)}(t) = &-\frac{1}{2\eta_{\rm in}} \ln\cosh\left(t + \xi_{\rm in} - \ln\left(\frac{2}{\sqrt{\eta_{\rm in}}}\right)\right) \\ &- \frac{t}{2\eta_{\rm in}} - \frac{\xi_{\rm in}}{2\eta_{\rm in}} + \eta_{\rm in} - \frac{1}{4\eta_{\rm in}} \ln \eta_{\rm in}\,.\end{aligned} \tag{3.29}$$

Clearly, $\eta^{(1)}(t) \to \eta_{\rm in}$ as $t \to -\infty$, and $\eta^{(1)}(t) \sim -\frac{1}{\eta_{\rm in}} t$ as $t \to +\infty$; moreover, $\eta^{(1)}(t) < \eta_{\rm in} = \eta^{(0)}$ for all t, which is seen by inspection of (3.29) but also follows immediately from (3.25). Hence, (3.24) with $n = 1$ has a well-defined solution $\xi^{(1)}(t)$ for all $t < T^{(1)}$. Moreover, (3.24) implies at once that $\xi^{(1)}(t) > \xi^{(0)}(t)$ for all t for which $\xi^{(1)}$ exists. Hence, we conclude that $\eta^{(2)} < \eta^{(1)}$ and thus by induction $\eta^{(k+1)} < \eta^{(k)}$ and $\xi^{(k+1)} > \xi^{(k)}$ for all $k > 0$. $\square$

LEMMA 3.6 *Let* $\xi^{(n)}(t)$, $\eta^{(n)}(t)$, *solve* (3.24)–(3.26). *Then there exists a* $T_0 = T_0(\xi_{\rm in}, \eta_{\rm in})$, *independent of* n, *satisfying the bound*

$$T_0 > \ln\left(2\sqrt{2\eta_{\rm in}}\right) - \xi_{\rm in} \tag{3.30}$$

such that for all $t < T_0$ *and for all* n, *we have*

$$\eta^{(n)}(t) > 0\,, \tag{3.31}$$

$$\xi^{(n)}(t) < \xi_{\rm in} + t\,. \tag{3.32}$$

PROOF: Clearly, for each n the function $t \mapsto \eta^{(n)}(t)$ is strictly monotonically decreasing and strictly concave. Since $\eta_{\rm in} > 0$, there exists a unique $T_0^{(n)}(\xi_{\rm in}, \eta_{\rm in})$ such that $\eta^{(n)}(T_0^{(n)}) = 0$. Moreover, since the iteration map $n \mapsto \eta^{(n)}(t)$ is decreasing for each t, we conclude that the sequence $n \mapsto T_0^{(n)}(\xi_{\rm in}, \eta_{\rm in})$ is decreasing, too. We need to show that it has a lower bound $T_0 > -\infty$.

Now, by what we just said, it follows with (3.24) that for all $t < T_0^{(n)}$ we have the n-independent upper bound (3.32) for $\xi^{(n)}(t)$. This in turn implies that for all $t < T_0^{(n)}$, we have the n-independent lower bound

$$\eta^{(n)}(t) > \eta_{\rm in} - \frac{1}{8} e^{2\xi_{\rm in} + 2t}\,. \tag{3.33}$$

By setting the right-hand side of (3.33) equal to zero, we obtain the n-independent lower bound right-hand side (3.30) valid for all $T_0^{(n)}$; thus the $T_0^{(n)}$ are bounded below independently of n by some T_0 satisfying (3.30). Lemma 3.6 follows. $\square$

COROLLARY 3.7 *The sequence* $n \mapsto (\xi^{(n)}(t), \eta^{(n)}(t))$, *defined by* (3.24)–(3.26) *converges pointwise for all* $t < T$ *(the life span of the maximal Cauchy development) to a solution* $(\xi_*(t), \eta_*(t))$ *of* (3.22)–(3.23). *This is the unique solution to* (3.8)–(3.9) *satisfying* (3.4) *and* (3.5).

PROOF: It follows from Lemma 3.5 that the sequence $n \mapsto (\xi^{(n)}(t), \eta^{(n)}(t))$ defined by (3.24)–(3.26) is pointwise increasing for ξ and decreasing for η. By Lemma 3.6, for all $t < T_0$ the ξ sequence is bounded above and the η sequence bounded below independently of n. Hence, these two sequences converge for $t < T_0$ to solutions $\xi_*(t)$ and $\eta_*(t)$ of (3.22)–(3.23). Furthermore, by our sharp upper and lower bounds on any solution $\xi(t)$ and $\eta(t)$ for $t < \tau \ll T_0$, we can easily show that the fixed-point map defined by (3.22)–(3.23) is a contraction mapping in the space of integrable functions on $(-\infty, \tau)$ equipped with weighted L^1 norm $\|h\| = \int_{-\infty}^{\tau} |h(t)| e^{2t}\, dt$; hence the solutions $\xi_*(t)$ and $\eta_*(t)$ of (3.22)–(3.23) are unique for $t < \tau$. (We skip the details of the contraction mapping proof here because below we re-prove the uniqueness by a different argument that will be needed in the sequel.)

Next, we can now pick any particular $t_0 < \tau$ as a new initial time and solve (3.8)–(3.9) for $t > t_0$ as a regular initial value problem with data $\xi_*(t_0)$ and $\eta_*(t_0)$. Standard ODE results now guarantee that this initial value problem has a unique solution for all $t \in (t_0, T)$, and this solution satisfies (3.22)–(3.23) and can moreover be computed with (3.24)–(3.26). Thus, the solution $(\xi_*(t), \eta_*(t))$ is continued uniquely from $t \in (-\infty, t_0]$ to $t \in (t_0, T)$, and this proves the corollary. □

Having a unique solution to (3.22)–(3.23) for all $t < T$, we also know that $T = T(\xi_{\text{in}}, \eta_{\text{in}})$. We now bootstrap to a sharper upper bound on $\xi(t)$.

LEMMA 3.8 *Let* $(\xi(t), \eta(t))$ *solve* (3.8)–(3.9) *for the asymptotic conditions* (3.4)–(3.5). *Let* $T_{1/2}$ *be defined by* $\eta(T_{1/2}) = \eta_{\text{in}}/2$. *Then, for* $T_{1/2}$ *we have the lower bound*

$$T_{1/2} > \ln(2\sqrt{\eta_{\text{in}}}) - \xi_{\text{in}}\,, \tag{3.34}$$

and for all $t \in (-\infty, T_{1/2})$ *we have the upper bound* $\xi(t) < \hat{\xi}(t)$, *where*

$$\hat{\xi}(t) = -\ln\cosh\left(t + \xi_{\text{in}} - \ln\left(2\sqrt{\frac{2}{\eta_{\text{in}}}}\right)\right) - \ln\sqrt{\frac{\eta_{\text{in}}}{2}}\,. \tag{3.35}$$

PROOF: As for $T_{1/2}$, for all $t < T_{1/2}$ we have the lower bound (3.33) for η. By setting the right-hand side of (3.33) equal to $\eta_{\text{in}}/2$, we obtain the lower bound (3.34).

Since $\eta(t) > \eta_{\text{in}}/2$ for $t < T_{1/2}$, we find from (3.22) that the solution to

$$\hat{\xi}(t) = \xi_{\text{in}} + t - \frac{1}{2}\eta_{\text{in}} \int_{-\infty}^{t} \int_{-\infty}^{s} e^{2\hat{\xi}(\tilde{s})}\, \mathrm{d}\tilde{s}\, \mathrm{d}s \tag{3.36}$$

is a supersolution for $\xi(t)$ for all $t < T_{1/2}$. For $\eta_{\text{in}} > 0$ the nonlinear integral equation (3.36) is solved uniquely by (3.35). □

LEMMA 3.9 *There exists some* $\eta_{\text{in}}^{\text{crit}} > 0$ *such that when* $\eta_{\text{in}} > \eta_{\text{in}}^{\text{crit}}$, *then* $\xi(t)$ *has a maximum at some finite* $T_M < T_0$ (*the same* T_0 *as in Lemma* 3.6). *In that case, at*

$t = T_0$ we have the bounds

$$\xi(T_0) < -\ln\cosh\ln\left(\frac{\eta_{\rm in}}{\sqrt{2}}\right) - \ln\sqrt{\frac{\eta_{\rm in}}{2}}\,, \tag{3.37}$$

$$\dot\xi(T_0) < \frac{-\ln\cosh\ln(\eta_{\rm in}/\sqrt{2}) + \ln\sqrt{2}}{\ln\eta_{\rm in}} < 0\,, \tag{3.38}$$

and

$$\dot\eta(T_0) > -\sqrt{1-\left(\frac{\ln\cosh\ln(\eta_{\rm in}/\sqrt{2}) - \ln\sqrt{2}}{\ln\eta_{\rm in}}\right)^2}\,. \tag{3.39}$$

PROOF OF LEMMA 3.9: The proof exploits the convexity properties of $\xi(t)$ for $t > T_0$. Namely, by (3.8), for all $t > T_0$, $\xi(t)$ is concave (i.e., convex down). Furthermore, for all $t \in (-\infty, T_{1/2})$ (recall that $T_{1/2} < T_0$), $\xi(t)$ satisfies the manifestly concave sandwich bounds $\xi^{(0)}(t) < \xi(t) < \hat\xi(t)$ given by (3.28) and (3.35). Let $T_M^{(0)}$ and $\hat T_M$ be the instants at which $\xi^{(0)}(t)$ and $\hat\xi(t)$ take their respective maxima, and let $\overline{T}_{1/2}$ be given by the right-hand side of (3.34). It is readily seen that $T_M^{(0)} = \ln(2/\sqrt{\eta_{\rm in}}) - \xi_{\rm in}$ and $\hat T_M = \ln(2\sqrt{2/\eta_{\rm in}}) - \xi_{\rm in}$. For $\eta_{\rm in} > \sqrt{2}$ we have the ordering $-\infty < T_M^{(0)} < \hat T_M < \overline{T}_{1/2} < T_{1/2} < T_0$. Furthermore, we have the monotonic behavior that, as $\eta_{\rm in} \nearrow$, we have $T_M^{(0)} \searrow$ and $\hat T_M \searrow$, but $\overline{T}_{1/2} \nearrow$. Now let $\widetilde\eta_{\rm in}^{\rm crit}$ be the unique solution of $\xi^{(0)}(T_M^{(0)}) = \hat\xi(\overline{T}_{1/2})$. After a simple manipulation, we see that $\widetilde\eta_{\rm in}^{\rm crit}$ is given by

$$\widetilde\eta_{\rm in}^{\rm crit} = \sqrt{2}\exp\operatorname{arcosh} 2\,. \tag{3.40}$$

Clearly, $\widetilde\eta_{\rm in}^{\rm crit} > \sqrt{2}$. But then, by the geometry of the concave sandwich bounds and the ordering and monotonic behavior of the various instances of time, we conclude that for all $\eta_{\rm in} > \widetilde\eta_{\rm in}^{\rm crit}$ we have that $\xi(T_M^{(0)}) > \xi(\overline{T}_{1/2})$, and therefore $\xi(t)$ has a unique maximum at some $T_M < \overline{T}_{1/2}$ whenever $\eta_{\rm in} > \widetilde\eta_{\rm in}^{\rm crit}$.

Next, whenever $\eta_{\rm in} > \widetilde\eta_{\rm in}^{\rm crit}$ so that $\xi(t)$ has a maximum for $T_M < T_0$, it follows directly from (3.8) that $\dot\xi(t) < 0$ for all $T_M < t < T_0$. Therefore, we conclude that $\xi(T_0) < \hat\xi(\overline{T}_{1/2})$, and this gives the bound (3.37).

The bound (3.38) follows once again by convexity arguments. Namely, by the concavity of $\xi(t)$ for $t > T_0$, it follows that whenever $\eta_{\rm in} > \widetilde\eta_{\rm in}^{\rm crit}$, we have that $\dot\xi(T_0) < \dot\xi(\overline{T}_{1/2})$. To estimate $\dot\xi(\overline{T}_{1/2})$, we simply compute the slope of the straight line joining the maximum of $\xi^{(0)}$ with $\hat\xi(\overline{T}_{1/2})$. By the convexity of these sandwich bounds on ξ, it follows right away that the slope of that straight line dominates $\dot\xi(\overline{T}_{1/2})$. This is the content of (3.38).

Finally, at $t = T_0$ we have $\eta(T_0) = 0$, so that by the energy law (3.15) we have that $\dot\xi(T_0)^2 + \dot\eta(T_0)^2 = 1$. But $\dot\eta(t) < 0$ for all t; hence at $t = T_0$ we have $\dot\eta(T_0) = -(1-\dot\xi(T_0)^2)^{1/2}$. With (3.38) we now obtain (3.39). Finally, from the way it is constructed, it is manifestly clear that $\widetilde\eta_{\rm in}^{\rm crit}$ is an upper estimate for $\eta_{\rm in}^{\rm crit}$. $\square$

We now turn to the time zone $t \geq T_0$ and derive an asymptotically linear upper bound for $\xi(t)$ and an asymptotically linear lower bound for $\eta(t)$, valid whenever $\eta_{\text{in}} > \widetilde{\eta}_{\text{in}}^{\text{crit}}$. Thus, $\eta_{\text{in}} > \widetilde{\eta}_{\text{in}}^{\text{crit}}$, and let $\epsilon \ll 1$. For $t \geq T_0$, define two maps F_ϵ and G_ϵ from $C^0 \times C^0$ to C^0 by

$$F_\epsilon(X,Y)(t) = X(t) - \epsilon\left(X(t) - \dot{X}(T_0)(t-T_0) - X(T_0) + \int_{T_0}^{t}\int_{T_0}^{s} Y(s')e^{2X(s')}\,\mathrm{d}s'\,\mathrm{d}s\right), \tag{3.41}$$

$$G_\epsilon(X,Y)(t) = Y(t) - \epsilon\left(Y(t) - \dot{Y}(T_0)(t-T_0) + \int_{T_0}^{t}\int_{T_0}^{s} \frac{1}{2}e^{2X(s')}\,\mathrm{d}s'\,\mathrm{d}s\right), \tag{3.42}$$

where $t \mapsto X(t)$ and $t \mapsto Y(t)$ are any two continuous functions that satisfy the initial bounds $X(T_0) <$ right-hand side of (3.37), $\dot{X}(T_0) <$ right-hand side of (3.38), $Y(T_0) = 0$, and right-hand side of (3.39) $< \dot{Y}(T_0) < 0$.

Now consider the coupled iteration sequences

$$X^{(n+1)}(t) = F_\epsilon(X^{(n)}, Y^{(n)}), \tag{3.43}$$

$$Y^{(n+1)}(t) = G_\epsilon(X^{(n)}, Y^{(n)}), \tag{3.44}$$

with the starting functions

$$X^{(0)}(t) = \dot{X}(T_0)(t-T_0) + X(T_0), \quad t \geq T_0, \tag{3.45}$$

$$Y^{(0)}(t) = \dot{Y}(T_0)(t-T_0), \qquad t \geq T_0. \tag{3.46}$$

LEMMA 3.10 *The maps $n \mapsto X^{(n)}$ and $n \mapsto Y^{(n)}$ defined jointly by the iteration sequences* (3.43)–(3.44) *with* (3.41)–(3.42) *and starting functions* (3.45)–(3.46) *are increasing and decreasing pointwise for all $t > T_0$, respectively.*

PROOF: We prove Lemma 3.10 by induction.

First, obviously we have $Y^{(1)}(t) < Y^{(0)}(t)$ for all $t > T_0$. Since furthermore $\dot{Y}(T_0) < 0$ by (3.16), we also have $Y^{(0)}(t) < 0$ for all $t > T_0$, and therefore $X^{(1)}(t) > X^{(0)}(t)$ for all $t > T_0$.

Next, assume that for some n we have $X^{(n)} > X^{(n-1)}$ and $Y^{(n)} < Y^{(n-1)} < 0$. Then we find for all $t > T_0$ that

$$\begin{aligned} &X^{(n+1)}(t) - X^{(n)}(t) \\ &\quad = F_\epsilon(X^{(n)}, Y^{(n)})(t) - F_\epsilon(X^{(n-1)}, Y^{(n-1)})(t) \\ &\quad = (1-\epsilon)(X^{(n)} - X^{(n-1)})(t) \\ &\qquad - \epsilon\int_{T_0}^{t}\int_{T_0}^{s}(Y^{(n)}(s')e^{2X^{(n)}(s')} - Y^{(n-1)}(s')e^{2X^{(n-1)}(s')})\,\mathrm{d}s'\,\mathrm{d}s \end{aligned}$$

$$\geq -\epsilon \int_{T_0}^{t} \int_{T_0}^{s} (Y^{(n)}(s')e^{2X^{(n)}(s')} - Y^{(n-1)}(s')e^{2X^{(n-1)}(s')})\, \mathrm{d}s'\, \mathrm{d}s$$

$$\geq -\epsilon \int_{T_0}^{t} \int_{T_0}^{s} Y^{(n-1)}(s')(e^{2X^{(n)}(s')} - e^{2X^{(n-1)}(s')})\, \mathrm{d}s'\, \mathrm{d}s$$

$$\geq 0\,. \tag{3.47}$$

Here we have used (3.43) in the first equality, and (3.41) and (3.45) in the second. The first inequality in (3.47) is the induction hypothesis $X^{(n)} > X^{(n-1)}$. The second inequality is a consequence of the induction hypothesis $Y^{(n)} < Y^{(n-1)}$. The last inequality is once again a consequence of the induction hypothesis $X^{(n)} > X^{(n-1)}$ but now together with $Y^{(n-1)} < 0$. Hence it follows that $n \mapsto X^{(n)}(t)$ is increasing, pointwise for each $t > T_0$. Similarly, by using first (3.44) and next (3.42) and (3.46), then the induction hypothesis $Y^{(n)} < Y^{(n-1)}$, then the induction hypothesis $X^{(n)} > X^{(n-1)}$ together with the negative sign in front of the integral, we find for all $t > T_0$ that

$$\begin{aligned} Y^{(n+1)}(t) - Y^{(n)}(t) &= G_\epsilon(X^{(n)}, Y^{(n)})(t) - G_\epsilon(X^{(n-1)}, Y^{(n-1)})(t) \\ &= (1-\epsilon)(Y^{(n)} - Y^{(n-1)})(t) \\ &\quad - \epsilon \int_{T_0}^{t} \int_{T_0}^{s} \frac{1}{2}\big(e^{2X^{(n)}(s')} - e^{2X^{(n-1)}(s')}\big)\, \mathrm{d}s'\, \mathrm{d}s \\ &\leq -\epsilon \int_{T_0}^{t} \int_{T_0}^{s} \frac{1}{2}\big(e^{2X^{(n)}(s')} - e^{2X^{(n-1)}(s')}\big)\, \mathrm{d}s'\, \mathrm{d}s \\ &\leq 0\,, \end{aligned} \tag{3.48}$$

and it follows that $n \mapsto Y^{(n)}(t)$ is decreasing for each $t > T_0$. □

PROPOSITION 3.11 *The joint iteration sequences* (3.43)–(3.44) *with initial data* (3.45)–(3.46) *converge in the limit $n \to \infty$ to asymptotically linear solutions of* (3.8)–(3.9) *that satisfy* (3.6)–(3.7).

PROOF: The initial data $X^{(0)}(t)$ and $Y^{(0)}(t)$ are linear functions of t with $t > T_0$. We now show first that a linear upper bound on $X^{(n)}(t)$ together with a linear lower bound on $Y^{(n)}(t)$ implies corresponding linear bounds on $X^{(n+1)}(t)$ and $Y^{(n+1)}(t)$. We then show that these bounds converge with $n \to \infty$ to uniform linear bounds for all $X^{(n)}$ and $Y^{(n)}$. These uniform linear bounds together with the monotonicity of the coupled iteration sequences (3.43)–(3.44) stated in Lemma 3.10 imply that the sequences (3.43)–(3.44) converge. By inspection of (3.43)–(3.44), we see at once that the limit functions are solutions of (3.8)–(3.9) for $t \geq T_0$, with initial data satisfying the stipulated bounds. Therefore the conclusion holds in particular when the initial data are obtained from $\xi(t)$, $\eta(t)$ as $t \to T_0^-$, and then the solutions $X(t)$, $Y(t)$ for $t > T_0$ coincide with the motion on that trajectory for all t.

Moreover, the convexity of the trajectories for t large enough (see Lemma 3.3) now immediately implies that the trajectories are asymptotically straight, with the motion on them asymptotically linear, satisfying (3.6) and (3.7), as claimed.

It thus remains to prove the uniform linear bounds on $X^{(n)}$ and $Y^{(n)}$. We begin with the observation that, if for some n the iterates $X^{(n)}$ and $Y^{(n)}$ satisfy the linear bounds

$$X^{(n)}(t) < \mu_n \times (t - T_0) + X(T_0)\,, \tag{3.49}$$

$$0 > Y^{(n)}(t) > \nu_n \times (t - T_0)\,, \tag{3.50}$$

with some positive constants μ_n and ν_n, then the iterates $X^{(n+1)}$ and $Y^{(n+1)}$ satisfy the linear bounds

$$X^{(n+1)}(t) < \mu_{n+1} \times (t - T_0) + X(T_0)\,, \tag{3.51}$$

$$0 > Y^{(n+1)}(t) > \nu_{n+1} \times (t - T_0)\,, \tag{3.52}$$

with

$$\mu_{n+1} = \mu_n + \epsilon\left(\dot{X}(T_0) - \delta\frac{\nu_n}{\mu_n^2} - \mu_n\right), \tag{3.53}$$

$$\nu_{n+1} = \nu_n + \epsilon\left(\dot{Y}(T_0) + \delta\frac{1}{\mu_n} - \nu_n\right). \tag{3.54}$$

Indeed, by the positivity of exp and by (3.49), we have

$$\frac{1}{2}\int_{T_0}^{t}\int_{T_0}^{s} e^{2X^{(n)}(s')}\,\mathrm{d}s'\,\mathrm{d}s < \frac{1}{2}\int_{T_0}^{t}\int_{T_0}^{\infty} e^{2X^{(n)}(s')}\,\mathrm{d}s'\,\mathrm{d}s < -\delta\frac{1}{\mu_n}(t - T_0)\,, \tag{3.55}$$

while by the negativity of $Y^{(n)}$ together with the positivity of exp, and then by (3.50), we have

$$\begin{aligned}\int_{T_0}^{t}\int_{T_0}^{s} Y^{(n)}(s')e^{2X^{(n)}(s')}\,\mathrm{d}s'\,\mathrm{d}s &> \int_{T_0}^{t}\int_{T_0}^{\infty} Y^{(n)}(s')e^{2X^{(n)}(s')}\,\mathrm{d}s'\,\mathrm{d}s \\ &> \delta\frac{\nu_n}{\mu_n^2}(t - T_0)\,,\end{aligned} \tag{3.56}$$

where

$$4\delta = \exp(2X(T_0))\,. \tag{3.57}$$

With these estimates the joint iteration maps (3.43)–(3.44), with F_ϵ and G_ϵ given by (3.41)–(3.42), now give (3.51)–(3.52) with (3.53)–(3.54) when (3.49)–(3.50) hold.

Hence, to obtain a linear upper bound on $X(t)$ and a linear lower bound on $Y(t)$, we need to study the coupled recurrence relations (3.53)–(3.54), starting with initial data

$$\mu_0 = \dot{X}(T_0) < 0\,, \tag{3.58}$$

$$\nu_0 = \dot{Y}(T_0) < 0\,, \tag{3.59}$$

satisfying

$$\mu_0^2 + \nu_0^2 = 1\,. \tag{3.60}$$

The last constraint follows from (3.14)–(3.15). The recurrence relations are valid from $n = 0$ on upward as long as $Y^{(n)} < 0$. We need to show that for some legitimate μ_0 and ν_0 the recurrence relations converge to limits μ_∞ and ν_∞ in the desired region of the μ, ν-plane.

By inspection we recognize equations (3.53)–(3.54) as the forward Euler approximation to a gradient flow with time step ϵ, defined as follows. We conveniently introduce a new, fictitious "time" variable $\tau \in \mathbb{R}^+$ and a τ-dependent point $(\mu(\tau), \nu(\tau)) \in \mathbb{R}^2$, and we let Grad denote gradient with respect to (μ, ν). We also define

$$W(\mu, \nu) = \frac{1}{2}\big((\mu - \mu_0)^2 + (\nu - \nu_0)^2\big) - \delta\frac{\nu}{\mu}\,. \tag{3.61}$$

Then the gradient flow in question is given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mu, \nu)(\tau) = -\mathrm{Grad}\, W(\mu, \nu)(\tau)\,, \tag{3.62}$$

$$(\mu, \nu)(0) = (\mu_0, \nu_0)\,, \tag{3.63}$$

with initial data (μ_0, ν_0) in the set

$$\mathbb{S}^1_{-,-} = \mathbb{S}^1 \cap \mathbb{R}^2_{-,-}\,, \tag{3.64}$$

where $\mathbb{R}^2_{-,-}$ is the southwestern quadrant of μ, ν-space,

$$\mathbb{R}^2_{-,-} = \{(\mu, \nu) \in \mathbb{R}^2 : \mu < 0,\ \nu < 0\}\,. \tag{3.65}$$

If the gradient flow converges to a stable fixed point when starting at the initial datum (3.63), then for ϵ small enough the iteration (3.53)–(3.54), starting at (3.58)–(3.59), will likewise converge to the same stable fixed point of (3.62). If that fixed point is in $\mathbb{R}^2_{-,-}$ and the flow from (μ_0, ν_0) does not leave $\mathbb{R}^2_{-,-}$, then the proposition is proved. It therefore suffices to inspect the gradient flow (3.62) for stable fixed points in $\mathbb{R}^2_{-,-}$.

Stable fixed points of the gradient flow (3.62) are critical points of W that locally minimize W. Clearly, the paraboloidal part of W, given by $((\mu - \mu_0)^2 + (\nu - \nu_0)^2)/2$, has a unique minimum at (μ_0, ν_0), and an elementary perturbation argument shows that for each (μ_0, ν_0) in the admitted set of initial data there exists a $\delta_0(\mu_0, \nu_0) > 0$ such that, if $\delta < \delta_0$, then $W(\mu, \nu)$ still has a unique minimum at $(\mu_M, \nu_M)(\delta)$ in $\mathbb{R}^2_{-,-}$, with $\mu_M > \mu_0$ and $\nu_M < \nu_0$. Moreover, the map $\mu_0 \mapsto \delta_0$ is strictly monotonically decreasing. On the other hand, the exponential map $X(T_0) \mapsto \delta$ given in (3.57) tells us that $\delta \to 0$ rapidly when $X(T_0) \to -\infty$. Also, $\mu_0 \to -1$ as $X(T_0) \to -\infty$.

Because of (3.37), for η_{in} large enough we have $X(T_0) \ll -1$, so that we have $\delta \ll 1$ exponentially small, given in (3.57). Moreover, we have $(\mu_0, \nu_0) \in \mathbb{S}^1$ with two negative components that satisfy the asymptotic bounds (3.38)–(3.39) so that (μ_0, ν_0) is exponentially close to the point $(-1, 0)$. Therefore, for large negative $X(T_0)$, we surely have $\delta < \delta_0$. It follows that $W(\mu, \nu)$ then has a unique minimum in the southwestern quadrant, very close to μ_0, ν_0 itself. Moreover, along the line $\nu = \nu_0$ the ν component of the gradient flow is given by $\delta/\mu < 0$ for $\mu < 0$. Therefore, the gradient flow (3.62) with initial datum (3.63) satisfying (3.64) remains in $\mathbb{R}^2_{-,-}$ and converges to (μ_M, ν_M). The existence proof is complete. $\square$

We have thus shown that for sufficiently large $\eta_{\text{in}} > 0$ there exists a solution with the correct scattering asymptotics (3.4)–(3.7). We next re-prove our uniqueness statement of Corollary 3.7 by a different argument that will recur in the sequel.

THEOREM 3.12 *The solutions* $(\xi(t), \eta(t))$ *to* (3.8)–(3.9) *with asymptotic data* ξ_{in}, η_{in} *in* (3.4)–(3.5) *are unique.*

PROOF: Let $(\xi_1(t), \eta_1(t))$ and $(\xi_2(t), \eta_2(t))$ be two pairs of functions that solve (3.8)–(3.9) with identical data (3.4)–(3.5). We now define $w_\xi(t) = \xi_1(t) - \xi_2(t)$ and $w_\eta(t) = \eta_1(t) - \eta_2(t)$ and set $\mathbf{u} = (w_\xi, \dot{w}_\xi, w_\eta, \dot{w}_\eta)^{\mathsf{T}}$. Note that

$$\lim_{t\to-\infty} \mathbf{u}(t) = 0\,. \tag{3.66}$$

Next, by the mean value theorem we conclude that, since w_ξ and w_η satisfy the differential equations

$$\frac{\mathrm{d}^2 w_\xi}{\mathrm{d}t^2} = -\eta_1 e^{2\xi_1} + \eta_2 e^{2\xi_2}\,, \tag{3.67}$$

$$\frac{\mathrm{d}^2 w_\eta}{\mathrm{d}t^2} = -\frac{1}{2} e^{2\xi_1} + \frac{1}{2} e^{2\xi_2}\,, \tag{3.68}$$

there exists a $\phi(t) \in (\min(\xi_1(t), \xi_2(t)), \max(\xi_1(t), \xi_2(t)))$ such that we can rewrite the ODEs for w_ξ and w_η as

$$\frac{\mathrm{d}^2 w_\xi}{\mathrm{d}t^2} = -w_\eta e^{2\xi_1} - 2 w_\xi \eta_2 e^{2\phi}\,, \tag{3.69}$$

$$\frac{\mathrm{d}^2 w_\eta}{\mathrm{d}t^2} = -w_\xi e^{2\phi}\,. \tag{3.70}$$

We remark that (3.69)–(3.70) are linear equations for w_ξ and w_η. We now rewrite (3.69)–(3.70) into the first-order system $\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$, where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2\eta_2 e^{2\phi} & 0 & -e^{2\xi_1} & 0 \\ 0 & 0 & 0 & 1 \\ -e^{2\phi} & 0 & 0 & 0 \end{pmatrix} \tag{3.71}$$

is the coefficient matrix. Notice that $\det \mathbf{A} = -\exp(2\phi + 2\xi) < 0$, whence $\mathbf{A}$ is invertible. More specifically, the characteristic polynomial of $\mathbf{A}$ is readily found to be

$$P(\lambda) = \lambda^4 + 2\eta_2 e^{2\phi}\lambda^2 - e^{2\phi + 2\xi_1}\,. \tag{3.72}$$

Solving for the roots of λ^2, we find two real values

$$\lambda^2 = \big(- \eta_2 \pm \sqrt{\eta_2{}^2 + e^{2\xi_1}} \big) e^{\phi} \,, \tag{3.73}$$

one positive and the other negative. Hence, there are two real and two purely imaginary eigenvalues λ of $\mathbf{A}$. Now, in view of (3.66) the purely imaginary roots do not contribute to the solutions with our scattering data. Next, the real roots are

$$\lambda_{\pm}^{R} = \pm \sqrt{-\eta_2 + \sqrt{\eta_2{}^2 + e^{2\xi_1}}}\, e^{\phi/2} \,, \tag{3.74}$$

one negative and the other positive for all $t \in \mathbb{R}$. Thus, $\phi(t) \sim -|t|$ for $t \to -\infty$; by letting $t \to -\infty$ we see that the real roots converge to zero exponentially fast. Hence the nontrivial orbits of $\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$ coming from the real roots converge to some $\mathbf{u}^{\sharp} \neq \mathbf{0}$ outside some ball in $\mathbb{R}^4$ centered at the origin. Therefore, the only vector solution compatible with the asymptotic conditions (3.66) is $\mathbf{u} \equiv \mathbf{0}$, viz. $w_\xi(t) \equiv 0 \equiv w_\eta(t)$. Uniqueness is proved. □

We remark that Theorem 3.12, like Corollary 3.7, claims uniqueness not only for the scattering solutions for which there is a $\Theta \in (-\pi, -\frac{\pi}{2})$, but also for arbitrary solutions to (3.8)–(3.9). We now return to those scattering solutions and show that there exist scattering solutions for the whole range of deflection angles $\Theta \in (-\pi, -\frac{\pi}{2})$.

THEOREM 3.13 *For every* $\Theta \in (-\pi, -\frac{\pi}{2})$ *there is a choice of parameters* $\eta_{\text{in}} > 0$ *and* ξ_{in} *such that there exists a solution* $(\xi(t), \eta(t))$ *to* (3.8)–(3.9) *with scattering data* (3.4)–(3.7).

PROOF: We argue via continuity.

DEFINITION 3.14 We define S to be the set $(\xi_{\text{in}}, \eta_{\text{in}}, \Theta) \in \mathbb{R}^3$ for which there exists a joint solution $\xi = f_\xi(t)$, $\eta = f_\eta(t)$, of (3.8)–(3.9) satisfying the asymptotic conditions (3.4)–(3.7).

Let $\mathbb{R}^+ = (0, \infty)$ and set $\mathbb{W} = \mathbb{R} \times \mathbb{R}^+ \times (-\pi, -\frac{\pi}{2})$. We will show that S is relatively open and closed in $\mathbb{W}$. Clearly, by our existence proof, S is nonempty; thus, S is a connected nonempty set, and it follows that the projection of S onto the third component is $(-\pi, -\frac{\pi}{2})$. To show that S is open, we will apply the implicit function theorem to our ODEs (3.8)–(3.9) and fix $s_0 \in S$, and we have a solution $\xi_0(t)$, $\eta_0(t)$ with scattering data s_0.

To apply the implicit function theorem, we next consider the linearized part of $\xi = \xi_0 + \xi_1 + \cdots$ and $\eta = \eta_0 + \eta_1 + \cdots$, with ξ_1 and η_1 small, satisfying

$$\lim_{|t|\to\infty} \xi_1(t) = 0 = \lim_{|t|\to\infty} \eta_1(t) \tag{3.75}$$

and satisfying the linearized equations of motion

$$\frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2} = -\eta_1 e^{2\xi_0} - 2\xi_1\eta e^{2\xi_0}\,, \tag{3.76}$$

$$\frac{\mathrm{d}^2\eta_1}{\mathrm{d}t^2} = -\xi_1 e^{2\xi_0}\,. \tag{3.77}$$

Rewriting these second-order equations as a first-order system for $\mathbf{v}^{\mathsf{T}} = (\xi_1, \eta_1)$, we are led to $\dot{\mathbf{v}} = \mathbf{M}\mathbf{v}$, with coefficient matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2\eta_0 e^{2\xi_0} & 0 & -e^{2\xi_0} & 0 \\ 0 & 0 & 0 & 1 \\ -e^{2\xi_0} & 0 & 0 & 0 \end{pmatrix} \tag{3.78}$$

and with $\mathbf{v}(t) \to \mathbf{0}$ as $|t| \to \infty$. Clearly, similarly to the proof of Theorem 3.12, we have $\det \mathbf{M} = -\exp(4\xi_0) < 0$, and the characteristic polynomial is

$$P(\lambda) = \lambda^4 + 2\eta_0 e^{2\xi_0}\lambda^2 - e^{4\xi_0}\,, \tag{3.79}$$

giving two real and two purely imaginary eigenvalues λ of $\mathbf{M}$ for all $t \in \mathbb{R}$. Thus, by the condition that $\mathbf{v}(t) \to \mathbf{0}$ for $|t| \to \infty$, we conclude that $\mathbf{u}(t) = \mathbf{0}$ identically. Therefore, the implicit function theorem applies, and we may conclude that there is a neighborhood about s_0 in $\mathbb{W}$ for which one finds solutions to (3.8)–(3.9) satisfying the asymptotic conditions (3.4)–(3.7). Hence, S is an open set.

To show that S is relatively closed, consider a sequence $s_n \in S$ such that $s_n \to s_* \in \mathbb{W}$. We have $s_n = (\xi_{\mathrm{in},n}, \eta_{\mathrm{in},n}, \Theta_n)$ and $s^* = (\xi^*_{\mathrm{in}}, \eta^*_{\mathrm{in}}, \Theta^*)$. Note that we have solutions of

$$\frac{\mathrm{d}^2\xi_n}{\mathrm{d}t^2} = -\eta_n e^{2\xi_n}\,, \tag{3.80}$$

$$\frac{\mathrm{d}^2\eta_n}{\mathrm{d}t^2} = -\frac{1}{2} e^{2\xi_n}\,, \tag{3.81}$$

satisfying the scattering data for s_n, by Definition 3.14 of S.

Because s_n belongs to a bounded set with compact closure in $\mathbb{W}$, by (3.22) and (3.23) the asymptotic behavior of $(\xi_n(t), \eta_n(t))$ in (3.4)–(3.5) is uniform and independent of the solution (ξ_n, η_n). Similarly, we have uniformity in (3.6)–(3.7). That means, the error term is uniform in n if s_n remains in a set with compact closure in $\mathbb{W}$. Similarly, by differentiating (3.22) and (3.23) once and using (3.14) and the uniformity in $(\eta_n(t), \xi_n(t))$, we may conclude the same uniformity for the derivatives. This allows us to conclude compactness at infinity.

First, we show that

$$\sup_{t,n} \sqrt{\dot{\xi}_n^2(t) + \dot{\eta}_n^2(t)} \le c\,. \tag{3.82}$$

To see that (3.82) holds, recall that $\dot{\eta}_n$ is strictly monotonically decreasing, by (3.9). Since by hypothesis $\lim_{t\to\infty} \dot{\eta}_n(t) = \sin\Theta_n$ and $\lim_{t\to-\infty} \dot{\eta}_n(t) = 0$, we have that $|\dot{\eta}_n| \leq |\sin\Theta_n|$, but also $\dot{\eta}_n < 0$ and therefore η_n is strictly monotonically decreasing. Furthermore, as long as $\eta \geq 0$, we have that $\dot{\xi}_n$ is strictly monotonically decreasing, by (3.8), and when $\eta_n = 0$ at $t = T_0$, we have $\dot{\xi}_n^2 + \dot{\eta}_n^2 = 1$, by (3.14) and (3.15). Thus, since also $\lim_{t\to-\infty} \dot{\xi}_n(t) = 1$, we conclude that $|\dot{\xi}_n| \leq 1$ for $t \in (-\infty, T_0]$. On the other hand, for $t > T_0$ we have $\eta_n < 0$ by the strict monotonic decrease of η_n, and thus by (3.8) we now have that $\dot{\xi}_n$ is strictly monotonically increasing for $t > T_0$. But then, since $\lim_{t\to\infty} \dot{\xi}_n(t) = \cos\Theta_n$, we conclude that $|\dot{\xi}_n| \leq 1$ for $t \in (T_0, \infty)$ as well. Thus, (3.82) is established.

Next we show that there is a point t_n, with $|t_n| \leq c'$ independently of n and some C independent of n, such that

$$|\xi_n(t_n)| + |\eta_n(t_n)| \leq C\,. \tag{3.83}$$

Thus, pick $t_n = \ln(2\sqrt{\eta_{\mathrm{in},n}}) - \xi_{\mathrm{in},n}$. Then, by (3.32), we have $\xi_n(t_n) \leq \xi_{\mathrm{in},n} + t_n$. We proved in Lemma 3.5 (see also the proof of Lemma 3.9) that for $t < T_{1/2}$ we have $\xi(t) > \xi^{(0)}(t)$, with $\xi^{(0)}$ given in (3.28), and this thus holds for any ξ_n with a corresponding $T_{1/2,n}$. Thus, since $t_n < T_{1/2,n}$, by (3.34), we have

$$\xi_n(t_n) > -\ln\cosh\left(t_n + \xi_{\mathrm{in},n} - \ln\left(2/\sqrt{\eta_{\mathrm{in},n}}\right)\right) - \ln\sqrt{\eta_{\mathrm{in},n}}\,, \tag{3.84}$$

and the bounds for ξ_n are established. Since s_n belongs to a set with compact closure, it follows that there exists a c' independent of n such that $|t_n| < c'$.

Next, we know that η_n is a decreasing function, bounded above by $\eta_n < \eta_{\mathrm{in}}$. By Lemma 3.8, since $T_{1/2,n} > t_n$, we see that $\eta_n(t_n) \geq \eta_n(T_{1/2,n})$. Thus, $|\eta_n(t_n)|$ is bounded above independently of n, too, and this finishes the proof of (3.83).

Next, using (3.82) and (3.83), we conclude that $\|(\xi_n, \eta_n)\|_{L^\infty(I)} \leq C(I)$, where I is any bounded subinterval of $\mathbb{R}$. Thus, by using (3.82) and the Ascoli theorem, we conclude that (ξ_n, η_n) converges uniformly on bounded subintervals of $\mathbb{R}$ to continuous functions (ξ^*, η^*).

Using now (3.8)–(3.9), this uniform convergence now implies that the second derivatives $(\ddot{\xi}_n, \ddot{\eta}_n)$ are uniformly bounded on compact subintervals of $\mathbb{R}$. Since we also have (3.82), by Ascoli's theorem again, the first derivatives $(\dot{\xi}_n, \dot{\eta}_n)$ converge uniformly to $(\dot{\xi}^*, \dot{\eta}^*)$ bounded on compact subintervals of $\mathbb{R}$. Therefore, in the sense of distributions,

$$\frac{\mathrm{d}^2\xi^*}{\mathrm{d}t^2} = -\eta^* e^{2\xi^*}\,, \tag{3.85}$$

$$\frac{\mathrm{d}^2\eta^*}{\mathrm{d}t^2} = -\frac{1}{2}e^{2\xi^*}\,. \tag{3.86}$$

We readily establish the following limits:

$$\lim_{t\to\infty} t^{-1}\xi^*(t) = \cos\Theta^*\,, \qquad \lim_{t\to\infty} t^{-1}\eta^*(t) = \sin\Theta^*\,,$$
$$\lim_{t\to-\infty} \xi^*(t) - t = \xi^*_{\mathrm{in}}\,, \qquad \lim_{t\to-\infty} \eta^*(t) = \eta^*_{\mathrm{in}}\,.$$

Thus $(\xi^*(t), \eta^*(t))$ satisfies the asymptotic conditions (3.4)–(3.7); hence, (ξ^*, η^*) is a solution, and therefore S is open and relatively closed in $\mathbb{W}$.

Since $\mathbb{W}$ is connected and $S \neq \emptyset$, we conclude that S is a connected set in $\mathbb{W}$. To finish the proof, we need to show that the projection of S onto the third component of $\mathbb{W}$ is indeed the full interval $(-\pi, -\frac{\pi}{2})$. Since S is connected and open, and since the projection map is continuous and open, the projection of S into $(-\pi, -\frac{\pi}{2})$ is an interval, say $(\vartheta_1, \vartheta_2)$, with $-\pi < \vartheta_1$ and $\vartheta_2 < -\frac{\pi}{2}$. Thus, for instance, as $\Theta_j \to \vartheta_1$, either $\eta_{\text{in},j} \to 0$ or $\eta_{\text{in},j} \to \infty$. Let $\eta_j(t_j) \to 0$. Assuming that $\eta_{\text{in},j} \to \infty$ as $\Theta_j \to \vartheta_1$, from (3.22) we conclude that $\xi_j(t_j) \to -\infty$, which now contradicts the condition that $\Theta_j \to \vartheta_1 \in (-\pi, -\frac{\pi}{2})$. Assuming that $\eta_{\text{in},j} \to 0$ as $\Theta_j \to \vartheta_1$, we again arrive at the contradiction by Lemma 3.4.

The other cases are $\xi_{\text{in},j} \to \pm\infty$ for fixed η_{in}. Assume first that $\xi_{\text{in},j} \to -\infty$. Then by (3.34) we see that $T_{1/2} \to +\infty$ for fixed η_{in}, which means that $\eta(t) > \eta_{\text{in}}/2$ for all $t \in \mathbb{R}$, which is impossible. Finally, assume that $\xi_{\text{in},j} \to +\infty$ for fixed η_{in}. Then, since $\xi^{(0)}$ is a subsolution for ξ, we have that

$$\eta(t) < \eta_{\text{in}} - \frac{1}{2}\int_{-\infty}^{t}\int_{-\infty}^{s} e^{2\xi^{(0)}(\tilde{s})}\, d\tilde{s}\, ds \tag{3.87}$$

for all t. Using (3.28), we obtain

$$\eta(t) < \eta_{\text{in}} - e^{2\xi_{\text{in}}} F(t)\,, \tag{3.88}$$

where $F(t)$ is a monotonically increasing, positive function, and $F(t) \to 0$ exponentially fast as $t \to -\infty$. Next, let $T_{0,j}$ be defined by $\eta(T_{0,j}) = 0$. Clearly, we now conclude from (3.88) and the properties of F that $T_{0,j} \to -\infty$ as $\xi_{\text{in},j} \to +\infty$. But then we conclude that $-1 \leq \dot{\eta}(t) < 0.5 \sin\vartheta_1$ for all $t > T_{0,j}$, with $T_{0,j} \to -\infty$ as $\xi_{\text{in},j} \to +\infty$, in contradiction to (3.5).

This concludes our proof of Theorem 3.13. □

The proof of Theorem 3.1 is complete. □

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SAGUN CHANILLO
Rutgers University
Department of Mathematics
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: `chanillo@math.rutgers.edu`

MICHAEL K.-H. KIESSLING
Rutgers University
Department of Mathematics
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
E-mail: `miki@math.rutgers.edu`

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