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# Stable solutions of semilinear elliptic problems in convex domains

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**Abstract.** In this note we consider semilinear equations $-\Delta u = f(u)$, with zero Dirichlet boundary condition, for smooth and nonnegative $f$, in smooth, bounded, strictly convex domains of $\mathbb{R}^N$. We study positive classical solutions that are semi-stable. A solution $u$ is said to be semi-stable if the linearized operator at $u$ is nonnegative definite. We show that in dimension two, any positive semi-stable solution has a unique, nondegenerate, critical point. This point is necessarily the maximum of $u$. As a consequence, all level curves of $u$ are simple, smooth and closed. Moreover, the nondegeneracy of the critical point implies that the level curves are strictly convex in a neighborhood of the maximum of $u$. Some extensions of this result to higher dimensions are also discussed.

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## 1. Introduction

Let $f$ be a nonnegative $C^\infty$ function on $\mathbb{R}$, and $u \in C^2(\overline{\Omega})$ a classical solution of

$$\begin{cases} -\Delta u = f(u) & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

The linearized operator $L$ of (1) at $u$ is given by

$$L = -\Delta - f'(u).$$

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We say that $u$ is a semi-stable solution of (1) if the linearized operator of (1) at $u$ is nonnegative definite, that is, if

$$(L\phi, \phi) = \int_\Omega |\nabla \phi|^2 - \int_\Omega f'(u)\phi^2 \geq 0, \tag{2}$$

for any $C^\infty$ function $\phi$ with compact support in $\Omega$. Equivalently, $u$ is a semi-stable solution if the first eigenvalue of the operator $L$ in $\Omega$, $\lambda_1(L, \Omega)$, is nonnegative.

Our main result is concerned with the set of critical points of a semi-stable solution of (1) in a strictly convex domain of $\mathbb{R}^2$.

**Theorem 1.** *Let $\Omega$ be a smooth, bounded and convex domain of $\mathbb{R}^2$ whose boundary has positive curvature. Suppose that $f \geq 0$ in $\mathbb{R}$ and that $u$ is a semi-stable solution of* (1) *such that $u > 0$ in $\Omega$. Then $u$ has a unique critical point $x_0$ in $\Omega$. Moreover, $x_0$ is the maximum of $u$ and it is nondegenerate in the sense that the Hessian of $u$ at $x_0$ is negative definite.*

As a consequence, all level curves of $u$ are simple, smooth and closed. Note also that, from the nondegeneracy of the critical point, $u$ is strictly concave in a neighborhood of $x_0$. In particular, the level curves of $u$ are strictly convex in a neighborhood of the critical point. The convexity of all the level curves of any solution $u$ as in Theorem 1 is an open problem, even in dimension 2 (see Remark 4).

In dimension $N \geq 3$, and for strictly convex domains of revolution with respect to an axis, we prove the following analogue of Theorem 1.

**Theorem 2.** *Let $\Omega$ be a smooth, bounded, strictly convex domain of revolution with respect to an axis in $\mathbb{R}^N$, $N \geq 3$. Let $f \in C^\infty(\mathbb{R})$ be a non-negative function. Let $u$ be a classical, semi-stable solution of* (1) *such that $u > 0$ in $\Omega$. Then $u$ has a unique critical point in $\Omega$. Moreover, this unique critical point is the maximum of $u$, and it is nondegenerate.*

The uniqueness of a critical point remains open for general strictly convex domains of $\mathbb{R}^N$. As we will see, the proof of Theorem 2 will be simpler than that of Theorem 1 due to the symmetry of revolution assumed in the second result.

**Remark 3.** Our work was motivated by the following. Consider the positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where $\lambda \geq 0$ is a parameter, and $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$. The function $g$ is continuous, nondecreasing and convex in $[0, \infty)$ and satisfies

$$g(0) > 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{g(u)}{u} = \infty.$$

Typical examples are $g(u) = e^u$ and $g(u) = (1+u)^p$, with $p > 1$. It is well known that there exists a parameter $0 < \lambda^* < \infty$ such that: for $0 < \lambda < \lambda^*$, problem (3)has a classical minimal solution $u_\lambda$; for $\lambda = \lambda^*$, (3)has a weak minimal solution $u_{\lambda^*}$; for $\lambda > \lambda^*$ problem (3)has no weak solution. Moreover, the solutions $u_\lambda$, for $0 < \lambda \leq \lambda^*$, are semi-stable solutions in the sense (2). For all these results, see for instance [BV]. We point out that the same properties are true for the nonlinearity $g(u) = u^q + u^p$, with $0 < q < 1 < p$; see [ABC].

For the domains in which our theorems apply and when $u_\lambda$ is classical (for instance when $\lambda < \lambda^*$), we obtain the uniqueness of a critical point of $u_\lambda$. We recall that, in dimension 2, it is not known if there is a convex nonlinearity $g$ as above for which $u_{\lambda^*}$ is singular.

**Remark 4.** Note that the strict convexity of the level sets of a solution $u$ is a stronger property than the uniqueness of a critical point of $u$. The convexity of the level sets of solutions of (1) (when $\Omega$ is convex) has been extensively studied; see, for instance, [Ka] and [Ko]. It is known to be true, for example, for the nonlinearities $f(u) = \lambda u^q$ for any $0 \leq q \leq 1$ and any $N$. However, to our knowledge, the available results on convexity of level sets do not apply to convex, superlinear nonlinearities such as $f(u) = \lambda e^u$, even in dimension 2.

The following is the idea of the proof of Theorem 1. Let us denote the set of critical points of $u$ by

$$K = \{x \in \overline{\Omega} \, : \, \nabla u(x) = 0\}.$$

Note that $K \subset\subset \Omega$ since the normal derivative of $u$ at any point on $\partial\Omega$ is nonzero, by the Hopf boundary lemma. For any $\theta \in \mathbb{R}$, we define the vector $e_\theta = (\cos\theta, \sin\theta)$ and the directional derivative $u_{e_\theta} = \langle \nabla u, e_\theta \rangle$ of $u$. We consider the nodal set of $u_{e_\theta}$ defined by

$$N_\theta = \{x \in \overline{\Omega} \, : \, u_{e_\theta}(x) = 0\}.$$

Note that $N_\theta$ contains all the critical points of $u$. That is, $K \subset N_\theta$ for any $\theta \in \mathbb{R}$.

Using the semi-stability of $u$, we first show that $N_\theta$ is a smooth embedded curve in $\overline{\Omega}$, homeomorphic to the closed interval $[0,1]$, and that intersects $\partial\Omega$ exactly at two points (which are the two end points of $N_\theta$). Next we consider an appropriate vector field, which we construct from the one given by

$$X = \frac{1}{|\nabla u|^2} D^2 u \cdot (-u_2, u_1),$$

that leaves $\overline{\Omega}$ invariant and whose flow defines homeomorphisms between the nodal sets $N_\theta$. Letting $\theta$ increase from 0 to $\pi$, we obtain a homeomorphism between $N_0$ and $N_\pi = N_0$ that reverses the end-points of $N_0$ and leaves any critical point of $u$ fixed. As a consequence, this homeomorphism has exactly one fixed point, and hence $K$ consists of one point. Our proof also gives the nondegeneracy of the unique critical point.

After the conclusion of our investigations, we learned about three interesting articles: [P], [S] and [BCP]. The key idea in the proof described above (that is, to consider the "rotation" of the nodal sets $N_\theta$) already appears in the work of Payne [P] and later in Sperb [S]. Our proof — completely detailed — gives the additional property of nondegeneracy of the critical point.

It is interesting that a similar argument may be applied for some systems, like the one of Ginzburg–Landau, in simply connected domains of $\mathbb{R}^2$. Bauman, Carlson and Phillips [BCP] have shown that, for a certain class of Dirichlet boundary data, every minimizer has exactly one zero which has necessarily winding number $\pm 1$.

## 2. Proof of Theorem 1

*Proof of Theorem* 1**.** Recall that $K$ is the set of critical points of $u$, and that $N_\theta$ is the nodal set of the directional derivative $u_{e_\theta}$. We proceed in two steps.

*Step* 1. Here we prove the following:

(a) $N_\theta$ is a smooth curve embedded in $\overline{\Omega}$ (i.e., without self-intersections), which is homeomorphic to the closed interval $[0,1]$, and such that $N_\theta \cap \partial\Omega$ consists of the two end points of $N_\theta$.

(b) $D^2u(x)$ has rank 2 at any critical point $x$ of $u$.

We know that $u \in C^\infty(\overline{\Omega})$ by standard regularity theory. Let us fix $\theta \in \mathbb{R}$. Recall that $N_\theta = \{x \in \overline{\Omega} : u_{e_\theta}(x) = \langle \nabla u, (\cos\theta, \sin\theta)\rangle = 0\}$, and consider the set

$$M_\theta = \{x \in N_\theta : \nabla u_{e_\theta}(x) = D^2u(x) \cdot (\cos\theta, \sin\theta) = 0\} \subset N_\theta.$$

Obviously, we have the following:

*Property* 1. Around any $x \in (N_\theta \cap \Omega) \setminus M_\theta$, the nodal set $N_\theta$ is a $C^\infty$ curve.

At any point $x \in \partial\Omega$, let $n$ and $t$ denote, respectively, the exterior normal and the tangent vector to $\partial\Omega$ at $x$, and such that $\{n, t\}$ is an orthonormal, positively oriented basis. Since $\Omega$ is strictly convex, there are exactly two points on $\partial\Omega$ in which the tangent $t$ is parallel to $e_\theta$. It follows, by the Hopf boundary lemma, that $N_\theta \cap \partial\Omega$ consists of exactly two points $\{p_1, p_2\}$. Since $u = 0$ on $\partial\Omega$, we have for each $p_i$

$$\Delta u = u_{nn} + k(p_i)u_n$$

where $k(p_i)$ is the curvature of $\partial\Omega$ at $p_i$, $u_n$ denotes the normal derivative, and $u_{nn}$ the second normal derivative. Denoting by $u_{tt}$ the second tangential derivative, we have at each $p_i$

$$\langle \nabla u_{e_\theta}, e_\theta\rangle = u_{e_\theta e_\theta} = u_{tt} = k(p_i)u_n < 0, \tag{4}$$

by the Hopf boundary lemma. It follows that the points $\{p_1, p_2\}$ do not belong to $M_\theta$ and that, around each $p_i$, $N_\theta$ is a $C^\infty$ curve intersecting $\partial\Omega$ transversally at its end-point $p_i$. We have used the implicit function theorem and $u_{tt}(p_i) \neq 0$. Therefore we have:

*Property* 2. $N_\theta \cap \partial\Omega$ consists of exactly two points $\{p_1, p_2\}$ and, around each $p_i$, $N_\theta$ is a $C^\infty$ curve that intersects $\partial\Omega$ transversally at its end-point $p_i$.

Now we differentiate the equation satisfied by $u$ with respect to the direction $e_\theta$. We obtain that $u_{e_\theta}$ is a solution of the linear equation

$$Lu_{e_\theta} = -\Delta u_{e_\theta} - f'(u)u_{e_\theta} = 0.$$

Note that $f'(u) \in C^\infty$ and that, by the Hopf boundary lemma, $u_{e_\theta}$ is not identically zero in any open subset of $\Omega$. By a well known result (see [CF]), we have that, around any point $x \in \Omega$, $u_{e_\theta}$ behaves locally as a homogeneous harmonic polynomial. Moreover, if $x \in M_\theta$ then this polynomial is of degree bigger than or equal to 2. Hence we have:

*Property* 3. Around any $x \in M_\theta$, $N_\theta$ consists of at least two $C^\infty$ curves intersecting transversally at $x$.

Next we use the semi-stability of the solution $u$. We claim:

*Property* 4. $N_\theta$ cannot "enclose" any subdomain of $\Omega$. More precisely, if $H \subset \Omega$ is a domain then $\partial H \not\subset N_\theta$. Here $\partial H$ denotes the boundary of $H$ as a subset of $\mathbb{R}^2$ (not as a subset of $\Omega$) and we assume $H \neq \varnothing$.

Indeed, if $\partial H \subset N_\theta$ then $|\Omega \setminus H| > 0$, by Property 2. Hence the first eigenvalue of $-\Delta - f'(u)$ in $H$ is positive (recall that the first eigenvalue of this operator in $\Omega$ is nonnegative by our semi-stability assumption). On the other hand, we have that $u_{e_\theta} \not\equiv 0$ in $H$ and

$$\begin{cases} -\Delta u_{e_\theta} - f'(u)u_{e_\theta} = 0 & \text{in} \quad H \\ u_{e_\theta} = 0 & \text{on} \quad \partial H. \end{cases}$$

As a consequence, the first eigenvalue of $L = -\Delta - f'(u)$ in $H$ is nonpositive, a contradiction.

Using Properties 1 to 4, we easily deduce that $M_\theta = \varnothing$ and that $N_\theta$ is homeomorphic to the interval $[0, 1]$. We obtain also the claim (a) made at the beginning of the proof. Assertion (b) also follows immediately. Indeed, if $x$ is a critical point of $u$ then $x \in N_\theta$ for any $\theta$. Since $M_\theta = \varnothing$, we deduce $D^2u(x) \cdot (\cos\theta, \sin\theta) \neq 0$ for any $\theta$. Therefore, $D^2u(x)$ has rank 2.

*Step* 2. Here we prove that the set $K$ of critical points of $u$ consists exactly of one point. This, together with assertion (b) from the previous step, will finish the proof of the theorem.

If $x \in \overline{\Omega} \setminus K$, we write

$$\nabla u(x) = |\nabla u(x)|(\cos\varphi(x), \sin\varphi(x))$$

where $\varphi(x) \in \mathbb{R}$. Given any point $x_0 \in \overline{\Omega} \setminus K$, this expression defines (modulo $2\pi\mathbb{Z}$) a $C^\infty$ function $\varphi$ on the intersection of $\overline{\Omega}$ with a sufficiently small ball centered at $x_0$. Recall that, for $x \in \overline{\Omega} \setminus K$, we have $x \in N_\theta$ if and only if $(\cos\theta, \sin\theta)$ is orthogonal to $\nabla u(x)$. We therefore have

$$x \in N_{\varphi(x)-\pi/2} = N_{\varphi(x)+\pi/2}$$

for any $x \in \overline{\Omega} \setminus K$.

Next we compute, locally, an expression for $\nabla\varphi$. We have

$$\cos\varphi = \frac{u_1}{|\nabla u|} = u_1(u_1^2 + u_2^2)^{-1/2};$$

here we denote by $u_i$ and $u_{ij}$ the first and second derivatives, respectively, of $u$ with respect to the coordinate directions. Hence

$$-\varphi_1 \sin\varphi = \frac{1}{|\nabla u|^3}\left\{u_{11}u_2^2 - u_{12}u_1u_2\right\}$$

and

$$-\varphi_2 \sin\varphi = \frac{1}{|\nabla u|^3}\left\{u_{12}u_2^2 - u_{22}u_1u_2\right\}.$$

In case that $\sin\varphi \neq 0$, we conclude

$$\begin{aligned}\nabla\varphi &= \frac{1}{|\nabla u|^2}(-u_{11}u_2 + u_{12}u_1, -u_{12}u_2 + u_{22}u_1)\\ &= \frac{1}{|\nabla u|^2}D^2u \cdot (-u_2, u_1).\end{aligned}$$

The same formula is obtained at the points where $\sin\varphi = 0$, differentiating $\sin\varphi = u_2/|\nabla u|$ in this case.

Even though $\varphi$ was defined only locally, the above expression allows us to define the vector field

$$X = \frac{1}{|\nabla u|^2}D^2u \cdot (-u_2, u_1) \quad \text{in} \quad \overline{\Omega} \setminus K,$$

which coincides with $\nabla\varphi$.

Note that $X(x) \neq 0$ for any $x \in \overline{\Omega} \setminus K$. Indeed, since $x \in N_{\varphi(x)+\pi/2}$ and $M_{\varphi(x)+\pi/2} = \varnothing$, we deduce $X(x) = |\nabla u(x)|^{-1}D^2u(x) \cdot (-\sin\varphi(x), \cos\varphi(x)) \neq 0$.

On the other hand, we know that $D^2u(x)$ has rank 2 at any $x \in K$, and hence also at all points in some neighborhood of $K$. It follows that, for some constant $c > 0$,

$$|X| = \frac{1}{|\nabla u|} \left| D^2u(x) \cdot (-u_2/|\nabla u|, u_1/|\nabla u|) \right| \geq \frac{c}{|\nabla u|}$$

in a neighborhood of $K$. Since $X \neq 0$ outside $K$, we conclude

$$|X| \geq \frac{c}{|\nabla u|} \quad \text{in} \quad \overline{\Omega} \setminus K, \tag{5}$$

for another constant $c > 0$.

By identity (4)on $\partial\Omega$, we have $\langle X, t \rangle = -|\nabla u|^{-1} u_{tt} = k$, the curvature of $\partial\Omega$. Hence

$$\langle X, t \rangle > 0 \quad \text{on} \quad \partial\Omega.$$

Using a partition of the unity, we modify $X$ in a neighborhood of the boundary of $\Omega$ to agree with $t$ on $\partial\Omega$. More precisely, there is an open convex neighborhood $\omega$ of $K$, with $K \subset \omega \subset \overline{\omega} \subset \Omega$, and a smooth vector field $Y$ on $\overline{\Omega} \setminus K$ such that

$$Y = \begin{cases} X & \text{in} \quad \omega \setminus K \\ t & \text{on} \quad \partial\Omega \end{cases}$$

and

$$\langle X, Y \rangle > 0 \quad \text{in} \quad \overline{\Omega} \setminus K.$$

We will consider the flow associated with the vector field

$$Z = \frac{Y}{\langle X, Y \rangle},$$

which is smooth in $\overline{\Omega} \setminus K$ and tangent to $\partial\Omega$. We claim that $Z$ can be extended to be Lipschitz in all $\overline{\Omega}$, by defining

$$Z = 0 \quad \text{in} \quad K.$$

Since $Z = X/|X|^2$ in $\omega \setminus K$, we have

$$|Z| \leq C|\nabla u| \quad \text{in} \quad \omega \tag{6}$$

for some constant $C > 0$, by (5). Hence $Z$ is continuous in $\overline{\Omega}$. It remains to show that, for another positive constant $C$,

$$|Z(x_1) - Z(x_2)| \leq C|x_1 - x_2| \tag{7}$$

for any $x_1$ and $x_2$ in $\omega$. If either $x_1$ or $x_2$ belong to $K$, the Lipschitz condition (7)follows immediately from (6). The same is true in case the segment $S$ joining $x_1$ and $x_2$ intersects $K$. Note that $S \subset \omega$, since we took $\omega$ to be convex. Finally, suppose that this segment is included in $\omega \setminus K$. Then we can differentiate $Z$ along the segment. Denoting by $D$ the full differential (acting on vector fields), we have

$$|DZ| = |D(|X|^{-2}X)| \leq C\frac{|DX|}{|X|^2} \leq C|\nabla u|^2|DX| \quad \text{in} \quad \omega \setminus K,$$

by (5). Now, differentiating $X = |\nabla u|^{-2}D^2u \cdot (-u_2, u_1)$, we see $|DX| \leq C|\nabla u|^{-2}$. We conclude that $|DZ| \leq C$ along the segment $S$, and hence (7).

We consider the flow $\psi_\tau$ at time $\tau$ associated with the Lipschitz vector field $Z$. We have that $\psi_\tau$ is a continuous flow that leaves both $\Omega$ and $\partial\Omega$ invariant, since $Z$ is parallel to $\partial\Omega$. Moreover, any point in $K$ is fixed by the flow.

In any open set of $\overline{\Omega} \setminus K$ where $\varphi$ is well defined, we have that $\nabla\varphi = X$ and therefore

$$\dot{\varphi} = \langle Z, X\rangle \equiv 1,$$

where $\dot{\varphi}$ denotes the derivative of $\varphi$ along the flow. It is now easy to deduce that

$$\psi_\tau(N_\theta) \subset N_{\theta+\tau},$$

for any nodal set $N_\theta$ and any time $\tau$. By reversing time, we see that this inclusion is, in fact, an equality and that $\psi_\tau$ is an homeomorphism from $N_\theta$ onto $N_{\theta+\tau}$. Hence the flow at time $\pi$,

$$\psi_\pi : N_0 \longrightarrow N_\pi = N_0,$$

is a homeomorphism of $N_0$ that interchanges the two end-points of $N_0$. Since $N_0$ is homeomorphic to $[0, 1]$, it follows that $\psi_\pi$ restricted to $N_0$ has exactly one fixed point. This finishes the proof, since $K \subset N_0$ and the points of $K$ are fixed by the flow. □

## 3. Proof of Theorem 2

*Proof of Theorem* 2**.** We assume that the domain $\Omega$ is a domain of revolution formed by taking a strictly convex, planar domain in the $x_1$, $x_N$ plane which is symmetric about the $x_N$ axis and rotating this planar domain about the $x_N$ axis. In the sequel, $x' = (x_1, \cdots, x_{N-1})$ and $r = |x'|$.

By the results of Gidas, Ni and Nirenberg [GNN], we deduce that the solution $u$ satisfies

$$u(x', x_N) = u(|x'|, x_N) \tag{8}$$

and

$$\frac{\partial u}{\partial r}(x', x_N) < 0 \quad \text{for} \quad x' \neq 0. \tag{9}$$

By (9), the critical points of $u$ lie on the $x_N$ axis. Next, from (8) we see

$$u_{x_N}(x', x_N) = u_{x_N}(|x'|, x_N). \tag{10}$$

From (1), the strict convexity of $\Omega$ and the Hopf boundary point lemma, it follows that on $\partial\Omega$, $u_{x_N}$ vanishes precisely on the $N-2$ dimensional sphere given by

$$S = \{x_N = a\} \cap \partial\Omega,$$

for some $a \in \mathbb{R}$. Moreover, we have that $u_{x_N}$ satisfies the equation

$$Lu_{x_N} = -\Delta u_{x_N} - f'(u)u_{x_N} = 0. \tag{11}$$

We now define the nodal set

$$\mathcal{N} = \{x \in \Omega,\ u_{x_N}(x) = 0\}.$$

It is clear that all critical points of $u$ are contained in $\mathcal{N}$. Furthermore, by (11) and the maximum principle, no point of $\mathcal{N}$ is isolated. Also by (10), $\mathcal{N}$ is rotationally invariant about the $x_N$ axis.

Around any point of $\mathcal{N}$, this set looks locally like the zero-set of some homogeneous harmonic polynomial in $x_1, \ldots, x_N$, by (11) and the results of [CF]. Moreover, by rotational symmetry, $\mathcal{N}$ is the rotation about the $x_N$ axis of a set $\mathcal{N}_2$ contained in the $x_1, x_N$ 2-dimensional plane. By setting $x_2 = \ldots = x_{N-1} = 0$ in the homogeneous and harmonic polynomials above, we see that locally $\mathcal{N}_2$ looks like the zero-set of some homogeneous polynomial in $x_1, x_N$.

Since the linearized operator $L$ is nonnegative definite, the set $\mathcal{N}$ cannot enclose any subdomain of $\Omega$ (see the proof of Theorem 1 for the precise meaning of this). Recall that $\mathcal{N}$ is the rotation of $\mathcal{N}_2$ about the $x_N$ axis. We deduce that $\mathcal{N}_2$ cannot enclose any planar subdomain of $\Omega \cap \{x_2 = \ldots = x_{N-1} = 0\}$.

Arguing as in the proof of Theorem 1, we deduce that $\mathcal{N}_2$ is a smooth curve without self-intersections, homeomorphic to the interval $(0,1)$, and with ends in the two points of $S \cap \{x_2 = \ldots = x_{N-1} = 0\}$. Moreover, $\mathcal{N}_2$ is symmetric with respect the $x_N$ axis. We deduce that $\mathcal{N}_2$ (and hence also $\mathcal{N}$) intersects the $x_N$ axis at exactly one point. Since all the critical points belong to $\mathcal{N} \cap \{x' = 0\}$, we have proven the existence of a unique critical point $p$.

To show that $p$ is nondegenerate, recall that $p$ lies on the $x_N$ axis and that $u$ is rotationally symmetric with respect to this axis. By (8) and (9), we have that $\{u_{x_i} = 0\} = \{x_i = 0\} \cap \Omega$ for all $1 \leq i \leq N-1$. Hence $u_{x_i x_j}(p) = 0$ for any index

$1 \le j \le N$ such that $i \ne j$. That is, $D^2u(p)$ is diagonal. Next note that, from (9), $u_{x_i} < 0$ in $\mathcal{O}_i = \{x_i > 0\} \cap \Omega$ for $1 \le i \le N-1$. Furthermore, in $\mathcal{O}_i$, $u_{x_i}$ satisfies

$$Lu_{x_i} = -\Delta u_{x_i} - f'(u)u_{x_i} = 0.$$

Applying the Hopf boundary point lemma to the function $u_{x_i}$ at $p \in \partial\mathcal{O}_i$, we conclude that $u_{x_i x_i}(p) < 0$ for all $1 \le i \le N-1$.

Finally, recall we have proved that $\mathcal{N} = \{u_{x_N} = 0\}$ is a smooth hypersurface with $p \in \mathcal{N}$. The function $u_{x_N}$ satisfies (11), that is $Lu_{x_N} = 0$. By definition of $\mathcal{N}$, $u_{x_N} > 0$ to one side of $\mathcal{N}$. Thus the Hopf boundary point lemma, applied at $p$ to the function $u_{x_N}$, gives $u_{x_N x_N}(p) < 0$. Hence, $p$ is a nondegenerate maximum. $\square$

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