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Abstract: An estimate is derived for the Jacobi polynomial Cesaro summation kernel of arbitrary positive order. This is used to show that the supremum of the Cesaro summation operators is weak type (p,p) at the lower critical value of p and weak restricted type at the upper critical value. An immediate consequence is the convergence almost everywhere of the Cesaro summation operator for functions in L^p for the lower critical value of p. It is also shown that the supremum of the Cesaro summation operators is not weak type at the upper critical value and not strong restricted type at the lower critical value. For p not between the critical values the growth of the p norm of the Cesaro mean operator is determined.

Key words: Jacobi polynomial series, Cesaro means, Weak type estimates, maximal Cesaro summation operator.

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1. Introduction. The principal results of this paper concern norm estimates for $Tf(x) = \sup_{n\geq 0}|\sigma_n^{(\alpha,\beta),\theta}(f,x)|$, where $\sigma_n^{(\alpha,\beta),\theta}(f,x)$ is the n^{th} Cesaro mean of order $\theta > 0$ for the Jacobi polynomial expansion with parameters $\alpha > -1$ and $\beta > -1$ of $f(x)$. Let $\gamma = \max(\alpha,\beta)$. At the lower critical value of p, $\max(\frac{4\gamma+4}{2\gamma+2\theta+3}, 1)$, we show that T is weak type (p,p). For $\theta < \gamma + \frac{1}{2}$, at the upper critical value of p, $\frac{(4\gamma+4)}{(2\gamma-2\theta+1)}$ we show that T is weak restricted type (p,p). This is also true for $\theta = \gamma + \frac{1}{2}$ provided $2 < p < \infty$. For $\theta > \gamma + \frac{1}{2}$ we also show that T is strong type (∞,∞). These, of course, imply that T is strong type (p,p) for p between these values. We also show that at the lower critical value T is not strong restricted type (p,p) and at the upper critical value T is not weak type (p,p).

Previous related results include the statement by Askey and Wainger on page 483 of [3] that $\sigma_n^{(\alpha,\beta),\theta}(f,x)$ is strong type (p,p) if p lies between the critical values, $\alpha \geq -1/2$ and $\beta \geq -1/2$. Bonami and Clerc in theorem (6.4), page 255 of [4], showed that T is bounded for p between the critical values provided $\alpha = \beta > -1/2$. Their statement includes the critical values, but this is clearly a misprint. Their proof does not include these values and Askey and Hirschman in theorem 4c, p. 173 of [2], show the unboundedness of T at these values. This is also shown in theorems (21.1) and (21.2).

Our principal results are obtained from an accurate estimate of the Cesaro kernel obtained here for all $\theta > 0$, $\alpha > -1$ and $\beta > -1$. This is derived in §§3–14; the estimate is stated in §14 in various forms. This estimate has many applications besides the weak type results mentioned above. These include new simple proofs of the summability theorems 9.1.3 and 9.1.4 of [13]. The summability

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results in [2], [3] and [4] are immediate consequences of theorems (1.1), (1.2) and (1.3) with less restriction on the parameters; they can also be proved directly from our kernel estimate without interpolation. Weighted norm inequalities for Cesaro sums of Jacobi series, including ones that cannot be proved by interpolation, can also be proved easily using the kernel estimates.

The main results are proved in §§15–17. As usual it is easier to prove norm inequalities by changing the variables x and y of the Cesaro summation kernel to $\cos s$ and $\cos t$ and taking the function to be 0 on half the interval. This approach is used to obtain norm inequalities for the supremum of the Cesaro sum operators in §15 and §16 at the lower and upper critical values respectively. The final forms, theorems (1.1)–(1.3) are then proved in §17.

In §§18–20 we derive results needed to estimate upper and lower bounds of the p norm of Cesaro means for p not between the critical values. These are used in §21 to prove that $\sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_{p,\infty}$ at the upper critical p and $\sup_{E\subset[-1,1]} \|\sigma_n^{(\alpha,\beta),\theta}(\chi_E,x)\|_p$ at the lower critical p are unbounded functions of n. An obvious question remains as to whether $\sup_{E\subset[-1,1]} \|\sigma_n^{(\alpha,\beta),\theta}(\chi_E,x)\|_p$ is bounded at the upper critical p. We conjecture that it is but observe in §21 why this cannot be proved from an upper bound for the kernel.

Finally, in §22 we use the results of §§18–21 to obtain upper and lower bounds for $\sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p$ for p not between the critical values. This improves a result of Görlich and Markett [8] by giving upper bounds that are constant multiples of the lower bounds and being valid for a larger range of the parameters.

Our main results are the following.

Theorem (1.1). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $\theta > 0$, $p = \max(1, \frac{4\gamma+4}{2\gamma+2\theta+3})$, $a > 0$ and $E(a)$ is the subset of $[-1,1]$ where $\sup_{n\geq 0}|\sigma_n^{(\alpha,\beta),\theta}(f,x)| > a$, then there is a constant c, independent of f and a such that

$$\int_{E(a)}(1-x)^{\alpha}(1+x)^{\beta}\,dx \leq \frac{c}{a^p}\int_{-1}^{1}|f(x)|^p(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

Theorem (1.2). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $0 < \theta < \gamma+1/2$, $p = \frac{4\gamma+4}{2\gamma-2\theta+1}$, $a > 0$, $H \subset [-1,1]$ and $E(a)$ is the subset of $[-1,1]$ where $\sup_{n\geq 0}|\sigma_n^{(\alpha,\beta),\theta}(\chi_H,x)| > a$, then there is a constant c, independent of H and a, such that

$$\int_{E(a)}(1-x)^{\alpha}(1+x)^{\beta}\,dx \leq \frac{c}{a^p}\int_{H}(1-x)^{\alpha}(1+x)^{\beta}\,dx.$$

This is also true if $0 < \theta = \gamma+1/2$ and $2 < p < \infty$.

The following simple result completes theorem (1.2).

Theorem (1.3). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $\theta > 0$ and $\theta > \gamma+1/2$, then there is a constant c, independent of f, such that

$$\|\sup_{n\geq 0}|\sigma_n^{(\alpha,\beta),\theta}(f,x)|\|_{\infty} \leq c\|f(x)\|_{\infty},$$

where $\|\ \ \|_{\infty}$ is the essential supremum on $[-1,1]$.

The following convergence theorem is a consequence of theorem (1.1) and proposition 6.2, p. 95, of [14].

Theorem (1.4). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $\theta > 0$, $p = \max(1, \frac{4\gamma+4}{2\gamma+2\theta+3})$ and $\int_{-1}^{1} |f(x)|^p (1-x)^\alpha (1+x)^\beta \, dx < \infty$, then for almost every x in $[-1,1]$, $\lim_{n\to\infty} \sigma_n^{(\alpha,\beta),\theta}(f,x) = f(x)$.

Given our estimate of the kernel, the proofs of theorems (1.1), (1.2) and (1.3) in §§15–17 are straightforward, being based primarily on Hölder's inequality and Hardy's inequality. An interesting extension of Hölder's inequality, lemma (16.5) is needed to prove theorem (1.2); this lemma is valid when the functions are powers of x times characteristic functions. The proofs are long, however, because the estimate of the kernel used is a sum of eleven parts and each part is treated separately.

More complicated is the derivation of the estimate for the kernel. This is done in §§3–14. The result is stated in theorem (14.1) and in alternate forms in corollaries (14.2) and (14.7). The most troublesome case for arguments (cos s, cos t) with $0 \le t \le s/2$ and $1/n \le s \le \pi/2$ is treated in §§3–8. It might appear that a high order asymptotic formula like (2.15) could be used for this. This fails because essential cancellation is provided by $P_k^{(\alpha,\beta)}(\cos t)$ for k between $1/s$ and $1/t$ while the error term for $P_k^{(\alpha,\beta)}(\cos t)$ is larger than the principal term for k in this range. The procedure used is an inductive one using summation by parts. This is complicated for several reasons. First, the obvious way to sum by parts is to sum one polynomial times a suitable function of k and to difference the rest. This can be done but the result is no easier to estimate than the original. The procedure used here is to sum the product of the polynomials times a suitable function of k and to difference the rest. Unfortunately, there is no simple closed form expression for the sum after the first summation by parts, so asymptotic expansions must be used. In addition, to make the induction work the sum has to be re–expressed using an identity that leads to more terms and another asymptotic expansion. High order asymptotic expansions are needed to obtain error terms small enough that suitable estimates can be obtained without using cancellation. The basic summation by parts procedure is contained in §6.

Another complication of the estimation for $0 \le t \le s/2$ and $1/n \le s \le \pi/2$ is that the summation by parts procedure reduces the order of summation θ in some terms and introduces negative powers of $k+1$ in others. Consequently, the starting point for the induction consists of two parts: one for sums with large negative powers of $k+1$ given in §3 and the other for sums with $-1 \le \theta \le 0$ given in §5. The estimation in §3 is a straightforward estimation of a sum of absolute values, but in §5 the cancellation of terms is still essential and requires another summation by parts result proved in §4. The induction argument is given in §7; this produces a general result which is used to obtain the Cesaro sum estimate for this case in §8.

The case $|s-t| \ge a > 0$ is treated in §§9–12. this is again an inductive proof using summation by parts but simpler than §§3–8. This case is needed for the last estimate in theorem (14.1) as well as filling in the case $\pi/2 \le s \le 3\pi/4$ in the second estimate.

For $s/2 \le t \le s-2/n$ an asymptotic formula (2.15) of Darboux is used. This is done in §13. The proof of theorem (14.1) is completed in §14 by proving the estimate for a few simple cases. The alternate statements, corollaries (14.2) and (14.7) are also proved there.

For readers familiar with a result of Gilbert, theorem 1, p. 497 of [7], the method used here to prove theorems (1.1)–(1.3) may appear unnecessarily complicated. Gilbert's result produces weak and strong type norm inequalities for operators associated with various orthogonal expansions from the same norm inequality for the corresponding operator for trigonometric expansions. His proof is based on the fact that an easily obtained estimate K(x,y) of the difference of the kernels of the two operators has the property that $Tf(x) = \int_0^{\pi} K(x,y)f(y)dy$ is strong type (p,p) for $1 < p < \infty$ on $[-\pi,\pi]$ with weight function 1. This does not work here, however, because, as shown at the ends of §§15 and 16, Tf is not a weak type operator for the weight functions and values of p used.

The upper bounds for $\sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p$ in §18 are also a straightforward derivation from the kernel estimate of theorem (14.1). The lemmas in §§19–20 are modification of results in [2] and the proofs are similar to those in [2]. The theorems in §§21–22 follow easily from the rest of the paper.

2. Facts and definitions. For α and β real and n a nonnegative integer

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{m}\binom{n+\beta}{n-m}\left(\frac{x-1}{2}\right)^{n-m}\left(\frac{x+1}{2}\right)^{m}$$

is the usual Jacobi polynomial. Define

$$\phi_n^{(\alpha,\beta)}(s) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos s)(\sin s/2)^{\alpha+1/2}(\cos s/2)^{\beta+1/2} \tag{2.1}$$

where

$$t_n^{(\alpha,\beta)} = \left[\frac{(2n+\alpha+\beta+1)n!\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right]^{1/2}. \tag{2.2}$$

For $\alpha > -1$, $\beta > -1$, the functions $\phi_n^{(\alpha,\beta)}(s)$ are orthonormal on $[0,\pi]$ by (4.3.3) of [13]. Given a nonnegative integer J and a fixed integer d, the definition (2.2) shows that there are real numbers c_j, $1 \leq j \leq J$, independent of n such that for $n \geq \max(0,-d)$

$$\left| t_{n+d}^{(\alpha,\beta)} - \sum_{j=0}^{J-1} c_j(n+1)^{-j+1/2} \right| \leq c_J(n+1)^{-J+1/2}. \tag{2.3}$$

From (4.1.3) of [13] we have

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{2.4}$$

From theorem 7.32.2 of [13] and (2.4) it follows that if d is a fixed integer, α and β are fixed real numbers and $n \geq \max(0,-d)$, then

$$|P_{n+d}^{(\alpha,\beta)}(x)| \leq c\, E_n^{(\alpha,\beta)}(x), \tag{2.5}$$

where c is independent of n and x and

$$E_n^{(\alpha,\beta)}(x) = \begin{cases} (n+1)^{\alpha} & 1-(n+1)^{-2} \leq x \leq 1 \\ (n+1)^{-1/2}(1-x)^{-\alpha/2-1/4} & 0 \leq x \leq 1-(n+1)^{-2} \\ (n+1)^{-1/2}(1+x)^{-\beta/2-1/4} & -1+(n+1)^{-2} \leq x \leq 0 \\ (n+1)^{\beta} & -1 \leq x \leq -1+(n+1)^{-2}. \end{cases} \tag{2.6}$$

For $\alpha > -1$, $\beta > -1$, the estimates of (2.5)–(2.6) can also be written in the form

$$(2.7)\qquad |\phi_{n+d}^{(\alpha,\beta)}(x)| \le \begin{cases} c[(n+1)x]^{\alpha+1/2} & 0 \le x \le 1/(n+1) \\ c & 1/(n+1) \le x \le \pi-1/(n+1) \\ c[(n+1)(\pi-x)]^{\beta+1/2} & \pi-1/(n+1) \le x \le \pi \end{cases}$$

Given α,β,γ and δ greater than -1, k, u, v and J integers, $J \ge 2$ and x and y, we define

$$(2.8)\qquad Q_k(x,y) = \frac{2k+3}{4}\left[P_{k+u+1}^{(\alpha,\beta)}(x)P_{k+v}^{(\gamma,\delta)}(y) - P_{k+u}^{(\alpha,\beta)}(x)P_{k+v+1}^{(\gamma,\delta)}(y)\right].$$

Then by (3.7) of [9], we have

$$(2.9)\qquad P_{k+u}^{(\alpha,\beta)}(x)P_{k+v}^{(\gamma,\delta)}(y) = \frac{Q_k(x,y) - Q_{k-1}(x,y)}{(k+1)(x-y)} + \frac{A_k+B_k}{x-y}.$$

where A_k is a sum of terms of the form

$$(2.10)\qquad \frac{ax+by+c}{(k+1)^j}\, P_{k+U}^{(\alpha,\beta)}(x)P_{k+V}^{(\gamma,\delta)}(y)$$

with $U = u$ or $u-1$, $V = v$ or $v-1$, $2 \le j \le J-1$ and a,b and c independent of n,x and y and

$$(2.11)\qquad |B_k| \le c(k+1)^{-J}\, E_k^{(\alpha,\beta)}(x)\, E_k^{(\gamma,\delta)}(y).$$

By (6.9)–(6.11) of [9], $Q_k(x,y)$ equals the sum of

$$(2.12)\qquad \frac{-(2k+3)(2k+\alpha+\beta+2u+2)}{8(k+u+1)}\, P_{k+u}^{(\alpha+1,\beta)}(x)\, P_{k+v}^{(\gamma,\delta)}(y)(1-x),$$

$$(2.13)\qquad \frac{(2k+3)(2k+\gamma+\delta+2v+2)}{8(k+v+1)}\, P_{k+u}^{(\alpha,\beta)}(x)\, P_{k+v}^{(\gamma+1,\delta)}(y)(1-y)$$

and

$$\frac{2k+3}{4}\left[\frac{\alpha}{k+u+1} - \frac{\gamma}{k+v+1}\right] P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y). \tag{2.14}$$

We will need an asymptotic formula of Darboux, (39) p. 44 of [6] in the following form. There are functions $u_i(x)$, $1 \le i \le 4$, independent of n and analytic on $[0,\pi]$ and a constant c, independent of n and x such that

$$\left|\phi_n^{(\alpha,\beta)}(x) - [u_1(x) + \frac{u_2(x)}{n \sin x}]\cos nx - [u_3(x) + \frac{u_4(x)}{n \sin x}]\cos(nx + \frac{\pi}{2})\right| \le \frac{c}{[n \sin x]^2}. \tag{2.15}$$

We will need the summation by parts formula

$$\sum_{k=m}^{n} a_k \Delta b_k = a_n b_{n+1} - a_m b_m - \sum_{k=m}^{n-1} b_{k+1} \Delta a_k , \tag{2.16}$$

where $\Delta c_k = c_{k+1} - c_k$ and the following lemma proved by taking $b_k = \sum_{j=k}^{n} - c_k$ in (2.16).

Lemma (2.17). If a_k is nonnegative and increasing and $m \le n$, then

$$\left|\sum_{k=m}^{n} a_k c_k\right| \le a_n \max_{m \le j \le n} \left|\sum_{k=j}^{n} c_k\right|.$$

The n^{th} orthonormalized Cesaro kernel of order θ is defined by

$$K_n^{(\alpha,\beta),\theta}(s,t) = \frac{1}{A_n^\theta} \sum_{k=0}^{n} A_{n-k}^\theta \phi_k^{(\alpha,\beta)}(s) \phi_k^{(\alpha,\beta)}(t), \tag{2.18}$$

where $A_n^\theta = \binom{n+\theta}{n}$ for $n \ge 0$ and $\theta \ge -1$. In particular $A_n^{-1} = 0$ for

$n > 0$ and $A_0^{-1} = 1$. We will use the fact that $A_n^\theta \geq 0$ for $n \geq 0$ and $\theta \geq -1$. For fixed $\theta \geq -1$, the sequence A_n^θ is monotone in n, increasing if $\theta \geq 0$ and decreasing if $\theta \leq 0$. For all θ we have

$$A_n^\theta \leq c(n+1)^\theta \tag{2.19}$$

with c independent of n, and for $\theta > -1$ we have

$$(n+1)^\theta < c\, A_n^\theta \tag{2.20}$$

with c independent of n. Also needed is

$$A_j^\theta - A_{j-1}^\theta = A_j^{\theta-1} \tag{2.21}$$

and the summed version of (2.21)

$$\sum_{k=0}^{n} A_k^\theta = A_n^{\theta+1}. \tag{2.22}$$

By (4.5.2) of [13] we have

$$K_n^{(\alpha,\beta),0}(s,t) = \frac{\omega(s,t)u_n^{(\alpha,\beta)}}{x-y}\, Q_n(x,y) \tag{2.23}$$

where

$$\omega(s,t) = (\sin s/2 \sin t/2)^{\alpha+1/2}(\cos s/2 \cos t/2)^{\beta+1/2}, \tag{2.24}$$

for any $J \geq 0$

(2.25) $$u_n^{(\alpha,\beta)} = 2 + \sum_{j=1-J}^{-1} d_j(n+1)^j + O((n+1)^{-J})$$

with d_j independent of n, $x = \cos s$, $y = \cos t$ and Q_n as defined in (2.8) with $\gamma = \alpha$, $\delta = \beta$ and $u = v = 0$. Using (2.16) with $a_k = A_{n-k}^{\theta}$ and $b_k = K_{k-1}^{(\alpha,\beta),0}(s,t)$, we see that

(2.26) $$K_n^{(\alpha,\beta),\theta}(s,t) = \frac{1}{A_n^{\theta}} \sum_{k=0}^{n} A_{n-k}^{\theta-1} K_k^{(\alpha,\beta),0}(s,t).$$

Although most of our estimates and proofs will use $K_n^{(\alpha,\beta),\theta}(s,t)$ we will make some use of the basic Cesaro kernel

(2.27) $$L_n^{(\alpha,\beta),\theta}(x,y) = \frac{1}{A_n^{\theta}} \sum_{k=0}^{n} \frac{A_{n-k}^{\theta} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y)}{\int_{-1}^{1} P_k^{(\alpha,\beta)}(t)^2 (1-t)^{\alpha}(1+t)^{\beta}\, dt}.$$

With this definition the n^{th} Cesaro mean of f,

(2.28) $$\sigma_n^{(\alpha,\beta),\theta}(f,x) = \int_{-1}^{1} f(y) L_n^{(\alpha,\beta),\theta}(x,y)(1-y)^{\alpha}(1+y)^{\beta}\, dy,$$

and by (4.3.3) of [13] we have the equality

(2.29) $$L_n^{(\alpha,\beta),\theta}(\cos s, \cos t) = \frac{2^{-\alpha-\beta-1} K_n^{(\alpha,\beta),\theta}(s,t)}{[\sin(s/2)\sin(t/2)]^{\alpha+1/2}[\cos(s/2)\cos(t/2)]^{\beta+1/2}}.$$

The following conventions will be used. Norms will be on $[-1,1]$ and weighted so that for $1 \le p < \infty$

$$\|f\|_p = \left[\int_{-1}^{1} |f(x)|^p (1-x)^\alpha (1+x)^\beta \, dx\right]^{1/p} \tag{2.30}$$

and the weak norm

$$\|f\|_{p,\infty} = \left[\sup_{a>0} a^p \int_{E(a)} (1-x)^\alpha (1+x)^\beta \, dx\right]^{1/p}, \tag{2.31}$$

where $E(a)$ is the subset of $[-1,1]$ where $|f(x)| > a$. The function $\chi_E(x)$ is the function equal to 1 if $x \in E$ and 0 if $x \notin E$. The letter c is used for positive constants not necessarily the same at each occurrence. For p satisfying $1 \le p \le \infty$, $p' = p/(p-1)$. An expression of the form $[x]$ will denote the greatest integer less than or equal to x when the brackets have no other function.

3. An absolute value estimate for $3(1-y) \le 2(1-x)$. This section consists of the proof of lemma (3.1). This lemma is used to estimate error terms in the proofs of §§4, 5 and 7.

Lemma (3.1) If α,β, γ and δ are greater than -1, $0 \le 3(1-y) \le 2(1-x) \le 2$, $M = [(1-x)^{-1/2}]$, n is an integer, $n \ge M$, $\theta \ge -1$, $b \ge 0$, $t < 0$ and $t < 2b-\gamma-1/2$, then

$$(1-y)^b \sum_{k=M}^{n} (k+1)^t A_{n-k}^\theta \, E_k^{(\alpha,\beta)}(x) \, E_k^{(\gamma,\delta)}(y) \le c \, n^\theta (1-x)^{b-(\alpha+\gamma+t+1)/2}, \tag{3.2}$$

where c is independent of x, y and n.

To prove this if $n < 2M$, observe that in this case (2.6) implies $E_k^{(\alpha,\beta)}(x) \le c\,M^\alpha$ and $E_k^{(\gamma,\delta)}(y) \le c\,M^\gamma$ for $M \le k \le n$. Therefore, the left side of (3.2) has the bound

$$c(1-y)^b(M+1)^{t+\alpha+\gamma}\sum_{k=M}^{n} A_{n-k}^{\theta}.$$

By (2.19) we get the bound $c(1-y)^b(M+1)^{t+\alpha+\gamma+\theta+1}$. Since $0 \le 1-x \le 1$, the definition of M shows that

$$(M+1)/2 \le (1-x)^{-1/2} \le M+1. \tag{3.3}$$

From this we see that $c(M+1)^{t+\alpha+\gamma+\theta+1}(1-y)^b$ is bounded by the right side of (3.2).

To prove lemma (3.1) for $n \ge 2M$, let $N = \min([n/2], [(1-y)^{-1/2}])$ and write the left side of (3.2) as the sum of

$$(1-y)^b \sum_{k=M}^{N} (k+1)^t\ A_{n-k}^{\theta}E_k^{(\alpha,\beta)}(x)E_k^{(\gamma,\delta)}(y), \tag{3.4}$$

$$(1-y)^b \sum_{k=N+1}^{[n/2]} (k+1)^t\ A_{n-k}^{\theta}E_k^{(\alpha,\beta)}(x)E_k^{(\gamma,\delta)}(y), \tag{3.5}$$

and

$$(1-y)^b \sum_{k=1+[n/2]}^{n} (k+1)^t\ A_{n-k}^{\theta}E_k^{(\alpha,\beta)}(x)E_k^{(\gamma,\delta)}(y), \tag{3.6}$$

To estimate (3.4), use (2.6), (2.19) and the fact that $(1-y)^b \le N^{-2b}$ to get the bound

$$c\,n^{\theta}(1-x)^{-\alpha/2-1/4}\sum_{k=M}^{N} (k+1)^{t+\gamma-2b-1/2}.$$

Since $t+\gamma-2b-1/2 < -1$, this has the bound

$$c\, n^{\theta}(1-x)^{-\alpha/2-1/4} M^{t+\gamma-2b+1/2}.$$

By (3.3) we see that this is bounded by the right side of (3.2).

To estimate (3.5), note first that it contains no terms if $[n/2] = N$; we may assume, therefore, that

$$N = [(1-y)^{-1/2}] < [n/2]. \tag{3.7}$$

Now use (2.6) and (2.19) to show that (3.5) is bounded by

$$c\, n^{\theta}(1-x)^{-\alpha/2-1/4}(1-y)^{b-\gamma/2-1/4} \sum_{k=N+1}^{[n/2]} (k+1)^{t-1}. \tag{3.8}$$

Since $t < 0$, the sum has the bound $c(N+2)^{t}$. By (3.7) and the fact that $0 \le (1-y) \le 1$, we have

$$(1-y)^{-1/2} \le N+2 \le 3(1-y)^{-1/2}.$$

Therefore, (3.8) is bounded by

$$c\, n^{\theta}(1-x)^{-\alpha/2-1/4}(1-y)^{b-(\gamma+t+1/2)/2}. \tag{3.9}$$

Finally, since the exponent of $(1-y)$ is positive, we can replace $(1-y)$ by $(1-x)$. This produces the estimate for (3.5).

For (3.6) we use (2.6) to get

$$c(n+1)^{t-1/2}(1-x)^{-\alpha/2-1/4}(1-y)^{b}\, E_n^{(\gamma,\delta)}(y) \sum_{k=[n/2]+1}^{n} A_{n-k}^{\theta}.$$

By (2.22) and (2.19), this is bounded by

$$c(n+1)^{t+\theta+1/2}(1-x)^{-\alpha/2-1/4}(1-y)^{b}E_{n}^{(\gamma,\delta)}(y). \tag{3.10}$$

If $(1-y)^{-1/2} \geq n$, then (3.10) is bounded by

$$c(n+1)^{\theta+t+\gamma-2b+1/2}(1-x)^{-\alpha/2-1/4}.$$

Since $t+\gamma-2b+1/2 < 0$ and $n+1 > (1-x)^{-1/2}$, we can replace $(n+1)^{t+\gamma-2b+1/2}$ by $(1-x)^{b-(t+\gamma+1/2)/2}$ and get the right side of (3.2). If $(1-y)^{-1/2} < n$, then (3.10) is

$$c(n+1)^{t+\theta}(1-x)^{-\alpha/2-1/4}(1-y)^{b-\gamma/2-1/4}.$$

Since $t < 0$, we can replace $(n+1)^{t}$ by $(1-y)^{-t/2}$. This gives (3.9), and, as shown when estimating (3.5), we have (3.9) bounded by the right side of (3.1). This completes the proof of lemma (3.1).

4. A basic estimate for $3(1-y) \leq 2(1-x)$. To obtain Cesaro sum estimates, we will first need a fact about sums of products of polynomials. This is contained in lemma (4.1); the rest of this section is the proof of this lemma.

Lemma (4.1). If α, β, γ and δ are greater than -1, $0 \leq 3(1-y) \leq 2(1-x) \leq 2$, j, n, u and v are integers, t is real and $1/(2\sqrt{1-x}) \leq j \leq n$, then

$$\left|\sum_{k=j}^{n}(k+1)^{t}\ P_{k+u}^{(\alpha,\beta)}(x)P_{k+v}^{(\gamma,\delta)}(y)\right| \tag{4.2}$$

has the bound

$$c(1-x)^{-\alpha/2-3/4} \max_{j\le k\le n} (k+1)^{t-1/2} E_k^{(\gamma,\delta)}(y), \tag{4.3}$$

where c is independent of j, n, x and y.

To prove this, use (2.9) with $J = 2$ to show that (4.2) is bounded by the sum of

$$\left| \sum_{k=j}^{n} \frac{Q_k(x,y) - Q_{k-1}(x,y)}{y-x} (k+1)^{t-1} \right| \tag{4.4}$$

and

$$c \sum_{k=j}^{n} \frac{E_k^{(\alpha,\beta)}(x)\ E_k^{(\gamma,\delta)}(y)}{y-x} (k+1)^{t-2}. \tag{4.5}$$

To estimate (4.5), use (2.6) and the fact that

$$y-x = (1-x) - (1-y) \ge (1-x)/3 \tag{4.6}$$

to get the bound

$$c(1-x)^{-\alpha/2-5/4} \sum_{k=j}^{n} (k+1)^{t-5/2} E_k^{(\gamma,\delta)}(y).$$

This is bounded by

$$c(1-x)^{-\alpha/2-5/4} \left[\max_{j\le k\le n} (k+1)^{t-1/2} E_k^{(\gamma,\delta)}(y) \right] \sum_{k=j}^{n} (k+1)^{-2}.$$

Since $1/j \le 2\sqrt{1-x}$, the sum is less than $c\sqrt{1-x}$ and we get the bound (4.3).

To estimate (4.4), use (2.16) to show that (4.4) is bounded by the sum of

$$\frac{c}{y-x}(n+1)^{t-1} | Q_n(x,y) | + \frac{c}{y-x}(j+1)^{t-1} | Q_{j-1}(x,y) | \tag{4.7}$$

and

$$(4.8) \qquad \frac{c}{y-x}\left|\sum_{k=j}^{n-1} Q_k(x,y)\Delta(k+1)^{t-1}\right|.$$

To estimate (4.7) and (4.8) we shall use the fact that $Q_k(x,y)$ is the sum of (2.12)–(2.14). For $k \geq 1/(2\sqrt{1-x})$, the quantity (2.12) equals

$$(4.9) \quad c(1-x)(k+1)P_{k+u}^{(\alpha+1,\beta)}(x)P_{k+v}^{(\gamma,\delta)}(y) + O\left[(1-x)^{-\alpha/2+1/4}(k+1)^{-1/2}E_k^{(\gamma,\delta)}(y)\right],$$

and the absolute value of (2.12) is bounded by

$$(4.10) \qquad c(1-x)^{-\alpha/2+1/4}(k+1)^{1/2}E_k^{(\gamma,\delta)}(y).$$

The absolute value of the sum of (2.13) and (2.14) is bounded by

$$(4.11) \quad c(k+1)^{1/2}(1-x)^{-\alpha/2-1/4}(1-y)E_k^{(\gamma+1,\delta)}(y) + c(k+1)^{-1/2}(1-x)^{-\alpha/2-1/4}E_k^{(\gamma,\delta)}(y).$$

From (2.6) it follows that

$$(4.12) \qquad E_k^{(\gamma+1,\delta)}(y) \leq \frac{1}{\sqrt{1-y}}\, E_k^{(\gamma,\delta)}(y).$$

Using this and the inequalities $\sqrt{1-y} \leq \sqrt{1-x}$ and $(k+1)^{-1} \leq \sqrt{1-x}$, we see that (4.11) also has the bound (4.10). Therefore,

$$(4.13) \qquad |Q_k(x,y)| \leq c(1-x)^{-\alpha/2+1/4}(k+1)^{1/2}E_k^{(\gamma,\delta)}(y).$$

Using (4.6) and (4.13) in (4.7) immediately gives the estimate (4.3). For (4.8) we consider the cases $j \geq 1/\sqrt{1-y}$ and $n \leq 1/\sqrt{1-y}$; the case $j < 1/\sqrt{1-y} < n$ then follows by adding the estimate for the sum from j to $[1/\sqrt{1-y}\,]$ to the estimate for the sum from $[1/\sqrt{1-y}\,] + 1$ to n.

For the case $j \geq 1/\sqrt{1-y}$, we use (4.6), (4.13) and (2.6) to show that (4.8) is bounded by

$$c \sum_{k=j}^{n-1} (1-x)^{-\alpha/2-3/4}(1-y)^{-\gamma/2-1/4}\Delta(k+1)^{t-1}.$$

This has the bound

$$c(1-x)^{-\alpha/2-3/4}(1-y)^{-\gamma/2-1/4}\left[(n+1)^{t-1}+(j+1)^{t-1}\right]$$

and by (2.6) this is bounded by (4.3).

To estimate (4.8) for $n \leq 1/\sqrt{1-y}$, consider first the case $\gamma+t \neq 1/2$. For this use (4.6) and (4.13) in (4.8); then use (2.6) and the fact that $|\Delta(k+1)^{t-1}| \leq c(k+1)^{t-2}$ to get

$$c \sum_{k=j}^{n-1} (1-x)^{-\alpha/2-3/4}(k+1)^{t+\gamma-3/2}.$$

By (2.6) this is bounded by (4.3). Finally, to estimate (4.8) if $n \leq 1/\sqrt{1-y}$ and $\gamma+t = 1/2$, use (2.6) and the fact that $1-y \leq (k+1)^{-2}$ to show that (4.11) has the bound

$$c(k+1)^{\gamma-1/2}(1-x)^{-\alpha/2-1/4}.$$

Since the error term in (4.9) also has this bound, we see, using (4.6), that (4.8) is bounded by the sum of

$$c\left|\sum_{k=j}^{n} P_{k+u}^{(\alpha+1,\beta)}(x)P_{k+v}^{(\gamma,\delta)}(y)(k+1)\Delta(k+1)^{t-1}\right| \tag{4.14}$$

and

$$c \sum_{k=j}^{n} (k+1)^{\gamma+t-5/2}(1-x)^{-\alpha/2-5/4}. \tag{4.15}$$

For (4.14), write $\Delta(k+1)^{t-1} = c(k+1)^{t-2} + O((k+1)^{t-3})$. This shows that (4.14) is bounded by the sum of

$$c \left| \sum_{k=j}^{n} (k+1)^{t-1} P_{k+u}^{(\alpha+1,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right| \tag{4.16}$$

and (4.15). Since $\gamma+t-1 = -1/2 \neq 1/2$, the case already proved shows that (4.16) is bounded by

$$c(1-x)^{-\alpha/2-5/4} \max_{j \leq k \leq n} (k+1)^{t-3/2}(k+1)^{\gamma}.$$

Since $\gamma+t-3/2 = -1$, the maximum occurs at $k = j$ and the estimate is

$$c(1-x)^{-\alpha/2-5/4}(j+1)^{t+\gamma-3/2}, \tag{4.17}$$

Since $(1-x)^{-1/2} \leq j+1$ we have the bound

$$c(1-x)^{-\alpha/2-3/4}(j+1)^{t+\gamma-1/2}$$

which is bounded by (4.3). For (4.15) the exponent of $k+1$ is -2, and we again get (4.17). Therefore (4.15) also is bounded by (4.3). This completes the proof of lemma (4.1).

5. <u>A kernel estimate for $3(1-y) \leq 2(1-x)$ and $-1 \leq \theta \leq 0$.</u> We state and prove here Lemma (5.1) which is an estimate for a restricted range of θ. This lemma is the basis of the inductive argument in §7 that removes the restriction $\theta \leq 0$.

<u>Lemma (5.1)</u>. If α, β, γ and δ are greater than -1, $0 \leq 3(1-y) \leq 2(1-x) \leq 2$, $M = [(1-x)^{-1/2}]$, n, u and v are integers, $M \leq n$, $t \leq 1$ and $-1 \leq \theta \leq 0$, then

$$\left| \sum_{k=M}^{n} (k+1)^t \, A_{n-k}^{\theta} P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right| \tag{5.2}$$

has the bound

$$c\, n^{\theta}(1-x)^{-(\alpha+\gamma+t+1)/2} + c(1-x)^{-(\alpha+\theta+3/2)/2}[\min(n,(1-y)^{-1/2})]^{\gamma+t-1/2}, \tag{5.3}$$

where c is independent of n, x and y.

It should be noted that in lemma (5.1) the same proofs can be used if $t > 1$; in this case if $n > (1-y)^{-1/2}$ the second term in (5.3) must be replaced by

$$c\, n^{t-1}(1-x)^{-(\alpha+\theta+3/2)}(1-y)^{-(\gamma+1/2)/2}.$$

We will, however, only need an estimate of (5.2) for $t \leq 1$.

To prove lemma (5.1) if $n < 2M$, replace $P_{k+u}^{(\alpha,\beta)}(x)$ and $P_{k+v}^{(\gamma,\delta)}(y)$ by $E_k^{(\alpha,\beta)}(x)$ and $E_k^{(\gamma,\delta)}(y)$ respectively. The estimation is then completed in the same way as the case $n < 2M$ was done in the proof of lemma (3.1). We will, therefore, assume that $n \geq 2M$. The proof consists of estimating separately

$$\left| \sum_{k=M}^{[n/2]} (k+1)^t A_{n-k}^{\theta} P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right|, \tag{5.4}$$

$$\left| \sum_{k=[n/2]+1}^{n-M} (k+1)^t A_{n-k}^{\theta} P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right|, \tag{5.5}$$

and

$$\left| \sum_{k=n-M+1}^{n} (k+1)^t A_{n-k}^{\theta} P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right|. \tag{5.6}$$

To estimate (5.4), use the fact that A_{n-k}^{θ} is an increasing function of k with lemma (2.17) and (2.19) to get the bound

$$c\, n^{\theta} \max_{M \le j \le n/2} \left| \sum_{k=j}^{[n/2]} (k+1)^t P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right|.$$

By lemma (4.1) this is bounded by

$$c\, n^{\theta} (1-x)^{-\alpha/2-3/4} \max_{M \le k \le n} (k+1)^{t-1/2} E_k^{(\gamma,\delta)}(y). \tag{5.7}$$

Now since $0 \le 3(1-y) \le 2$, we have $1/3 \le y \le 1$, and by (2.6)

$$E_k^{(\gamma,\delta)}(y) = (k+1)^{-1/2} \min(k+1, (1-y)^{-1/2})^{\gamma+1/2}. \tag{5.8}$$

Therefore, (5.7) equals

$$c\, n^{\theta} (1-x)^{-\alpha/2-3/4} \max_{M \le k \le n} (k+1)^{t-1} \min(k+1, (1-y)^{-1/2})^{\gamma+1/2}. \tag{5.9}$$

Since $t \le 1$, we also have

$$(k+1)^{t-1} \le \min(k+1,(1-y)^{-1/2})^{t-1}; \tag{5.10}$$

therefore (5.9) has the bound

$$c\, n^{\theta}(1-x)^{-\alpha/2-3/4} \max_{M \le k \le n} \min(k+1,(1-y)^{-1/2})^{\gamma+t-1/2}. \tag{5.11}$$

Now $\min(k+1,(1-y)^{-1/2})$ as a function of k on $[M,n]$ achieves its maximum value at $k = n$ and its minimum at $k = M$. Consequently, the maximum of $\min(k+1,(1-y)^{-1/2})^{\gamma+t-1/2}$ for k in $[M,n]$ occurs at either $k = M$ or $k = n$. If the maximum occurs at $k = n$, then use the fact that $\theta \le 0$ and $n > (1-x)^{-1/2}$ to see that $n^{\theta} \le (1-x)^{-\theta/2}$. This shows that (5.11) is bounded by the second term of (5.3). If the maximum occurs at $k = M$, then since $M+1 \le 2(1-x)^{-1/2} \le 2(1-y)^{-1/2}$, we can replace $\min(M+1,(1-y)^{-1/2})$ by $M+1$ and by (3.3) we can replace $M+1$ with $(1-x)^{-1/2}$. This shows that (5.11) is bounded by the first term of (5.3) in this case and completes the proof that (5.4) is bounded by (5.3).

To estimate (5.5), again use the fact that A^{θ}_{n-k} is an increasing function of k with lemma (2.17) and then use (2.19) to get the bound

$$c\, M^{\theta} \max_{n/2 \le j \le n-M} \left| \sum_{k=j}^{n} (k+1)^{t}\, P^{(\alpha,\beta)}_{k+u}(x) P^{(\gamma,\delta)}_{k+v}(y) \right|.$$

By lemma (4.1) and (3.3) this is bounded by

$$c\, M^{\theta}(1-x)^{-\alpha/2-3/4} \max_{n/2 \le k \le n} (k+1)^{t-1/2}\, E^{(\gamma,\delta)}_{k+v}(y).$$

Using (5.8) and then (5.10) produces the bound

$$c\,M^{\theta}(1-x)^{-\alpha/2-3/4} \max_{n/2\le k\le n} \min(k+1,(1-y)^{-1/2})^{\gamma+t-1/2}.$$

Now replace $k+1$ by n and use (3.3) to replace M by $(1-x)^{-1/2}$. This produces the second term in (5.3) and completes the estimation of (5.5).

For (5.6) use (2.5) and the fact that $A^{\theta}_{n-k} \ge 0$ to get the estimate

$$c\,n^{t-1/2}(1-x)^{-\alpha/2-1/4}\, E_n^{(\gamma,\delta)}(y) \sum_{k=n-M}^{n} A^{\theta}_{n-k}.$$

Now use (2.22) followed by (2.19) and (3.3) to get the bound

$$c(1-x)^{-(\alpha+\theta+3/2)/2} n^{t-1/2} E_n^{(\gamma,\delta)}(y).$$

Using (5.8) to replace $E_n^{(\gamma,\delta)}(y)$ and then (5.10) shows that this is bounded by the second term in (5.3). This completes the proof of lemma (5.1).

6. A reduction lemma. The estimate in §7 is proved by induction using summation by parts. This reduction procedure is based on the following lemma.

Lemma (6.1). If $\alpha > -1$, $\beta > -1$, $\gamma > -1$, $\delta > -1$, $\theta \ge 0$, t is real, $-1 \le x \le 1$, $-1 \le y \le 1$, J, m, n, u and v are integers, $0 \le m \le n$ and $J \ge 2$, then

$$\sum_{k=m}^{n} (k+1)^{t}\, A^{\theta}_{n-k} P^{(\alpha,\beta)}_{k+u}(x) P^{(\gamma,\delta)}_{k+v}(y) \tag{6.2}$$

can be written as the sum of

$$\frac{Q_{m-1}(x,y)(m+1)^{t-1}\, A_{n-m}^{\theta}}{y - x}, \tag{6.3}$$

a term with absolute value bounded by

$$c \sum_{k=m}^{n} \frac{(k+1)^{t+1-J}}{|x-y|}\, A_{n-k}^{\theta} E_k^{(\alpha,\beta)}(x) E_k^{(\gamma,\delta)}(y) \tag{6.4}$$

and terms of the form

$$(Ax+By+C)\frac{(1-x)^a(1-y)^b}{x-y} \sum_{k=m}^{n'} (k+1)^{t+i}\, A_{n'-k}^{\theta'} P_{k+u'}^{(\alpha',\beta)}(x) P_{k+v'}^{(\gamma',\delta)}(y), \tag{6.5}$$

where $n' = n$ or $n' = n-1$, $|u'-u| \leq 1$, $|v'-v| \leq 1$, the numbers A, B and C are independent of m, n, x and y, i is an integer and a, b, α', γ', θ' and i have a set of values shown on a line of the following table:

a	b	θ'	α'	γ'	
1	0	$\theta-1$	$\alpha+1$	γ	$2-J \leq i \leq 0$
0	1	$\theta-1$	α	$\gamma+1$	$2-J \leq i \leq 0$
0	0	$\theta-1$	α	γ	$2-J \leq i \leq -1$
1	0	θ	$\alpha+1$	γ	$2-J \leq i \leq -1$
0	1	θ	α	$\gamma+1$	$2-J \leq i \leq -1$
0	0	θ	α	γ	$2-J \leq i \leq -2$

To prove this use (2.9) with the J of the hypotheses in (6.2) to show that (6.2) is the sum of

$$\sum_{k=m}^{n} \frac{(k+1)^{t-1}A_{n-k}^{\theta}}{x-y}[Q_k(x,y)-Q_{k-1}(x,y)] \tag{6.6}$$

$$\sum_{k=m}^{n} \frac{(k+1)^{t}A_{n-k}^{\theta}}{x-y} A_k \tag{6.7}$$

and

$$\sum_{k=m}^{n} \frac{(k+1)^{t}A_{n-k}^{\theta}}{x-y} B_k . \tag{6.8}$$

The terms in (6.7) have the form (6.5) with values from the sixth line of the table if $2 \leq j \leq J-2$ and have absolute value bounded by (6.4) if $j = J-1$. The terms in (6.8) also have absolute value bounded by (6.4). To complete the proof we will show that (6.6) equals (6.3) plus terms of the form (6.5) and terms majorized by (6.4).

To estimate (6.6) we will apply (2.16) with $a_k = (k+1)^{t-1}A_{n-k}^{\theta}$ and $b_k = Q_{k-1}(x,y)$. This and (2.21) show that (6.6) equals the sum of

$$\frac{(n+1)^{t-1}Q_n(x,y)A_0^{\theta}}{x-y} - \frac{(m+1)^{t-1}A_{n-m}^{\theta}Q_{m-1}(x,y)}{x-y}, \tag{6.9}$$

$$\sum_{k=m}^{n-1} \frac{Q_k(x,y)(k+1)^{t-1}A_{n-k}^{\theta-1}}{x-y} \tag{6.10}$$

and

$$(6.11) \qquad - \sum_{k=m}^{n-1} \frac{Q_k(x,y)}{x-y}[(k+2)^{t-1}-(k+1)^{t-1}]A_{n-k-1}^{\theta} .$$

The second part of (6.9) is (6.3). For the first part use the fact that $A_0^{\theta} = A_0^{\theta-1}$ and combine it with (6.10) to get

$$(6.12) \qquad \sum_{k=m}^{n} \frac{Q_k(x,y)(k+1)^{t-1}\ A_{n-k}^{\theta-1}}{x-y} .$$

Now replace $Q_k(x,y)$ by the sum of (2.12)–(2.14). The coefficients in (2.12) and (2.13) can be written in the form $\sum_{j=3-J}^{1} d_j(k+1)^j + O((k+1)^{2-J})$, and the coefficient in (2.14) can be written in the form $\sum_{j=3-J}^{0} d_j(k+1)^j + O((k+1)^{2-J})$. The principal term resulting from (2.12) has the form (6.5) with values from the first line of the table. From (2.13) we get (6.5) with values from the second line of the table while (2.14) produces the third line. The error term resulting from (2.12) is bounded by

$$(6.13) \qquad \frac{1-x}{|x-y|} \sum_{k=m}^{n} (k+1)^{t+1-J} E_k^{(\alpha+1,\beta)}(x) E_k^{(\gamma,\delta)}(y) A_{n-k}^{\theta-1}.$$

Now for $k \leq n$ and $\theta \geq 0$ we have $0 \leq A_{n-k}^{\theta-1} \leq A_{n-k}^{\theta}$. This and (4.12) show that (6.13) is bounded by (6.4). Similarly, the error term resulting from (2.13) has the bound

$$\frac{(1-y)}{|x-y|} \sum_{k=m}^{n} (k+1)^{t+1-J} E_k^{(\alpha,\beta)}(x) E_k^{(\gamma+1,\delta)}(y) A_{n-k}^{\theta-1} ;$$

the reasoning used on (6.13) shows this also has the bound (6.4). Finally, the error term resulting from (2.14) has the bound

$$\frac{1}{|x-y|}\sum_{k=m}^{n}(k+1)^{t+1-J}E_k^{(\alpha,\beta)}(x)E_k^{(\gamma,\delta)}(y)A_{n-k}^{\theta-1}$$

which is also bounded by (6.4).

To estimate (6.11), again replace $Q_k(x,y)$ by the sum of (2.12)–(2.14). The product of the coefficients of (2.12) or (2.13) with $(k+2)^{t-1}-(k+1)^{t-1}$ can be written in the form

$$\sum_{j=2-J}^{-1} d_j(k+1)^{t+j} + O((k+1)^{t+1-J});$$

the product of the coefficient of (2.14) with $(k+2)^{t-1} - (k+1)^{t-1}$ can be written in the form

$$\sum_{j=2-J}^{-2} d_j(k+1)^{t+j} + O((k+1)^{t+1-J}). \tag{6.14}$$

The resulting principal terms have the form (6.5) with values from lines 4, 5 and 6 of the table. The error terms are easily seen to have the bound (6.4) by using (4.12).

7. A kernel estimate for $3(1-y) \le 2(1-x)$ and $\theta \ge -1$. This section contains the inductive argument that extends the result of §5 to $\theta > 0$. The lemma to be proved is the following.

Lemma (7.1). If α, β, γ and δ are greater than -1, $0 \le 3(1-y) \le 2(1-x) \le 2$, $M = [(1-x)^{-1/2}]$, n, u and v are integers, $M \le n$, $t \le 1$ and $\theta \ge -1$, then (5.2) has the bound (5.3) with C independent of n, x and y.

The note immediately after the statement of lemma (5.1) about an estimate for (5.2) if $t > 1$ is also valid here. As in §5, this case will not be considered.

To prove lemma (7.1), we will show that if $b \ge 0$, a is real and the hypotheses of lemma (7.1) are satisfied, then

$$(1-x)^a(1-y)^b\left|\sum_{k=M}^{n}(k+1)^t\,A_{n-k}^{\theta}P_{k+u}^{(\alpha,\beta)}(x)P_{k+v}^{(\gamma,\delta)}(y)\right| \tag{7.2}$$

is bounded by the sum of

$$c\,n^{\theta}(1-x)^{a+b-(\alpha+\gamma+t+1)/2} \tag{7.3}$$

and

$$c(1-x)^{a-(\alpha+\theta+3/2)/2}(1-y)^b[\min(n,(1-y)^{-1/2})]^{\gamma+t-1/2} \tag{7.4}$$

with c independent of n, x and y. This is sufficient since taking $a = b = 0$ gives the conclusion of lemma (7.1).

To prove that (7.2) is bounded by the sum of (7.3) and (7.4), let $T = T(t,\gamma,b) = \max([t],[t+\gamma-2b+1/2])$ and $U = U(\theta) = [\theta]$. We will first prove the result if $T \le -1$ or $U \le -1$. If $T \le -1$, then $t < 0$ and

$t+\gamma-2b+1/2<0$. Lemma (3.1) can be applied to show that (7.2) is bounded by (7.3). If $U \le -1$, then $\theta < 0$ and by hypothesis $\theta \ge -1$. The desired estimate for (7.2) then follows from lemma (5.1) and the fact that $(1-y)^b \le (1-x)^b$.

For the proof when $T \ge 0$ and $U \ge 0$, define $K = K(t,\gamma,b,\theta) = T(t,\gamma,b)+U(\theta)$. The estimation of (7.2) will be done by induction on K. If $K \le -1$, then either $T \le -1$ or $U \le -1$ and the inequality has been proved. Therefore, assume that (7.2) is bounded by the sum of (7.3) and (7.4) if $K(t,\gamma,b,\theta) \le I$ where I is an integer and $I \ge -1$, and fix t, γ, β and θ such that $K(t,\gamma,b,\theta) = I+1$. If $T \le -1$ or $U \le -1$, we are done. Therefore, assume that $T \ge 0$ and $U \ge 0$. We will now apply lemma (6.1) to (7.2) with $J = \max(3,[t+\gamma+5/2])$. Note that since $3(1-y) \le 2(1-x)$, we have (4.6) and $1/(y-x)$ can be replaced by $3/(1-x)$ in the estimate produced by lemma (6.1). The result is that (7.2) is bounded by the sum of

$$c(1-x)^{a-1}(1-y)^b(M+1)^{t-1}A^{\theta}_{n-M}|Q_{M-1}(x,y)|, \tag{7.5}$$

$$c(1-x)^{a-1}(1-y)^b \sum_{k=M}^{n} (k+1)^{t-J+1} A^{\theta}_{n-k}E^{(\alpha,\beta)}_k(x)E^{(\gamma,\delta)}_k(y) \tag{7.6}$$

and terms of the form

$$c(1-x)^{a'}(1-y)^{b'}\left|\sum_{k=M}^{n'} (k+1)^{t+j} A^{\theta'}_{n'-k}P^{(\alpha',\beta)}_{k+u'}(x)P^{(\gamma',\delta)}_{k+v'}(y)\right|, \tag{7.7}$$

where $|u-u'| \le 1$, $|v-v'| \le 1$, $n' = n$ or $n' = n-1$, c is independent of x, y and n, j is an integer and a', b', θ', α', γ' and j have a set of values shown on a line in the following table:

a'	b'	θ'	α'	γ'	j	$T' = T(t+j,\gamma',b')$	$U' = U(\theta')$
a	b	$\theta-1$	$\alpha+1$	γ	$2-J \le j \le 0$	$\le T$	$U-1$
$a-1$	$b+1$	$\theta-1$	α	$\gamma+1$	$2-J \le j \le 0$	$\le T$	$U-1$
$a-1$	b	$\theta-1$	α	γ	$2-J \le j \le -1$	$\le T-1$	$U-1$
a	b	θ	$\alpha+1$	γ	$2-J \le j \le -1$	$\le T-1$	U
$a-1$	$b+1$	θ	α	$\gamma+1$	$2-J \le j \le -1$	$\le T-1$	U
$a-1$	b	θ	α	γ	$2-J \le j \le -2$	$\le T-2$	U

To estimate (7.5), use (2.19) and the assumption $\theta \ge 0$ to replace A^{θ}_{n-M} by $c\,n^{\theta}$. Then write $Q_{M-1}(x,y)$ as the sum of (2.12)–(2.14) and use (2.5) to estimate each of the three resulting terms. This shows that (7.5) has the bound

$$(1-x)^{a-1}(1-y)^{b}(M+1)^{t-1}n^{\theta}(M+1)^{\alpha+\gamma}[(M+1)^{2}(1-x)+(M+1)^{2}(1-y)+1].$$

Use (3.3) to replace $M+1$ by $(1-x)^{-1/2}$ and the facts that $b \ge 0$ and $1-y \le 1-x$ to replace $1-y$ by $1-x$. This gives (7.3) and completes the estimation of (7.5).

To estimate (7.6), observe that since $t+\gamma+3/2 < [t+\gamma+5/2] \le J$, and $b \ge 0$ we have $t-J+1 < 2b-\gamma-1/2$. Similarly, since $t+1 < 3 \le J$ we have $t-J+1 < 0$. We can, therefore, apply lemma (3.1); this gives the estimate

$$c\,n^{\theta}(1-x)^{a-1+b-(\alpha+\gamma+t-J+1+1)/2}.$$

Since $J \ge 3$, this is bounded by (7.3).

For the terms (7.7) note that in each case $K' = T' + U' \le I$, $\alpha' > -1$, $\gamma' > -1$ and $b' \ge 0$. Furthermore, since $U \ge 0$, we have $\theta \ge 0$ and $\theta' \ge -1$. Therefore, we can use the inductive hypothesis to estimate each of these. Their sum is bounded by a sum of terms of the form (7.3) and (7.4) for each line in the table with t replaced by $t+j$ and a, b, α, γ and θ replaced by a', b', α', γ' and θ' respectively. We may take j to be the largest value in the line since smaller values of j produce smaller estimates. The resulting modified values of (7.3) are bounded by (7.3) times respectively $n^{-1}(1-x)^{-1/2}$, $n^{-1}(1-x)^{-1/2}$, $n^{-1}(1-x)^{-1/2}$, 1, 1 and 1. Since $n^{-1}(1-x)^{-1/2} \leq M^{-1}(1-x)^{-1/2} \leq 2$, these terms are all bounded by (7.3). The modified versions of (7.4) are bounded by (7.4) times respectively $1,(1-x)^{-1/2}(1-y)\min(n,(1-y)^{-1/2})$, $(1-x)^{-1/2}[\min(n,(1-y)^{-1/2})]^{-1}$, $(1-x)^{-1/2}[\min(n,(1-y)^{-1/2})]^{-1}$, $(1-x)^{-1}(1-y)$ and $(1-x)^{-1}[\min(n,(1-y)^{-1/2})]^{-2}$. Now replace $\min(n,(1-y)^{-1/2})$ by $(1-y)^{-1/2}$ in the second of these factors. Then using the facts that $0 \leq 1-y \leq 1-x$ and $n \geq (1-x)^{-1/2}$ shows that these factors are bounded by 1. Therefore, the modified versions of (7.4) are bounded by (7.4). This completes the proof of lemma (7.1).

8. A Cesaro kernel estimate for $t \leq s/2$. Here we shall prove the following.

Theorem (8.1). If $\alpha > -1$, $\beta > -1$, $0 \leq t \leq s/2 \leq \pi/4$, $s > 2/n$ and $\theta \geq 0$, then $|K_n^{(\alpha,\beta),\theta}(s,t)|$ has the bound

$$\frac{ct^{\alpha+1/2}}{ns^{\alpha+5/2}} + \frac{c[\min(1,nt)]^{\alpha+1/2}}{n^\theta s^{\theta+1}}, \tag{8.2}$$

where c is independent of n, s and t.

To prove this, let $x = \cos s$, $y = \cos t$ and $M = [(1-\cos s)^{-1/2}]$. Since $s \leq \pi/2$, we have

$$1-s^2/2 \leq \cos s \leq 1-s^2/2 + s^4/24 \leq 1+s^2(-1/2+\pi^2/96) \leq 1-s^2/3; \tag{8.3}$$

consequently

$$M \leq (1-\cos s)^{-1/2} \leq \sqrt{3}/s < n\sqrt{3}/2. \tag{8.4}$$

and

(8.5) $$1-y \le t^2/2 \le s^2/8 \le (1-x)/2.$$

By (2.26), $|K_n^{(\alpha,\beta),\theta}(s,t)|$ is bounded by the sum of

(8.6) $$\frac{1}{A_n^\theta}\left|\sum_{k=0}^{M-1} A_{n-k}^{\theta-1}K_k^{(\alpha,\beta),0}(s,t)\right|$$

and

(8.7) $$\frac{1}{A_n^\theta}\left|\sum_{k=M}^{n} A_{n-k}^{\theta-1}K_k^{(\alpha,\beta),0}(s,t)\right|.$$

To estimate (8.6) use the fact obtained from (8.4) that $n-k \ge n(2-\sqrt{3})/2$ for $0 \le k \le M-1$ and (2.19) to show that $A_{n-k}^{\theta-1} \le c\,n^{\theta-1}$. Using this and (2.20) shows that (8.6) has the bound

(8.8) $$\frac{c}{n}\sum_{k=0}^{M-1}\left|K_k^{(\alpha,\beta),0}(s,t)\right|.$$

Now by the definition (2.18)

$$\left|K_k^{(\alpha,\beta),0}(s,t)\right| \le \sum_{j=0}^{k}\left|\phi_j^{(\alpha,\beta)}(s)\phi_j^{(\alpha,\beta)}(t)\right|.$$

By (8.4) and the hypothesis $t \le s/2$, we have both s and t bounded by $\sqrt{3}/M$ Thus, (2.7) implies

$$\left|K_k^{(\alpha,\beta),\theta}(s,t)\right| \le c\sum_{j=0}^{k}(j+1)^{2\alpha+1}(st)^{\alpha+1/2} \le c(k+1)^{2\alpha+2}(st)^{\alpha+1/2}$$

and (8.8) is bounded by

$$\frac{c}{n}(st)^{\alpha+1/2}(M+1)^{2\alpha+3}.$$

Now use (3.3) and (8.3); this shows that (8.5) is bounded by the first term of (8.2).

To estimate (8.7) use (2.23). Since (8.5) is true, we can use (4.6) to show that (8.7) has the bound

$$\frac{\omega(s,t)}{(1-x)A_n^{\theta}}\left|\sum_{k=M}^{n} A_{n-k}^{\theta-1}u_k^{(\alpha,\beta)}Q_k(x,y)\right|. \tag{8.9}$$

Now write $Q_k(x,y)$ as the sum of (2.12)–(2.14), and note that since $\alpha = \gamma$ and $u = v = 0$, the term (2.14) vanishes. Next write the product of $u_k^{(\alpha,\beta)}$ and the coefficients in (2.12) and (2.13) in the form $\sum_{j=1+J}^{1} a_j(k+1)^j + O((k+1)^J)$ where $J = \min(-1, [-\alpha-\frac{3}{2}])$. This shows that (8.9) is bounded by a sum of terms of the form

$$c\,\frac{\omega(s,t)}{A_n^{\theta}}\left|\sum_{k=M}^{n} A_{n-k}^{\theta-1}(k+1)^j P_k^{(\alpha+1,\beta)}(x)P_k^{(\alpha,\beta)}(y)\right| \tag{8.10}$$

and

$$c\,\frac{(1-y)\,\omega(s,t)}{(1-x)\,A_n^{\theta}}\left|\sum_{k=M}^{n} A_{n-k}^{\theta-1}(k+1)^j P_k^{(\alpha,\beta)}(x)P_k^{(\alpha+1,\beta)}(y)\right| \tag{8.11}$$

with j an integer satisfying $1+J \le j \le 1$ plus

$$c\,\frac{\omega(s,t)}{A_n^{\theta}}\sum_{k=M}^{n} A_{n-k}^{\theta-1}(k+1)^J E_k^{(\alpha+1,\beta)}(x)E_k^{(\alpha,\beta)}(y) \tag{8.12}$$

and

$$\text{(8.13)} \qquad \frac{c(1-y)\omega(s,t)}{(1-x)A_n^\theta} \sum_{k=M}^{n} A_{n-k}^{\theta-1}(k+1)^J E_k^{(\alpha,\beta)}(x) E_k^{(\alpha+1,\beta)}(y).$$

For (8.10) use lemma (7.1) and (2.20) to get the bound

$$\text{(8.14)} \; c\, n^{-\theta}\omega(s,t)\Big[n^{\theta-1}(1-x)^{-(2\alpha+j+2)/2} + (1-x)^{-(\alpha+\theta+3/2)/2}[\min(n,(1-y)^{-1/2})]^{\alpha+j-1/2}\Big].$$

This increases with j so we replace j by its maximum value of 1. Then use (8.3) and the same result for t to get the estimate

$$\text{(8.15)} \qquad c\, n^{-\theta}(st)^{\alpha+1/2}\Big[n^{\theta-1}s^{-2\alpha-3} + s^{-\alpha-\theta-3/2}\min(n,t^{-1})^{\alpha+1/2}\Big].$$

Since $t^{\alpha+1/2}[\min(n,t^{-1})]^{\alpha+1/2} = [\min(nt,1)]^{\alpha+1/2}$, (8.15) equals (8.2).

For (8.11) again use lemma (7.1), (2.20) and (4.6). This gives the bound

$$\frac{c(1-y)\omega(s,t)}{(1-x)n^\theta}\Big[n^{\theta-1}(1-x)^{-(2\alpha+2+j)/2} + (1-x)^{-(\alpha+\theta+1/2)/2}[\min(n,(1-y)^{-1/2})]^{\alpha+j+1/2}\Big].$$

Again replace j by 1 and use (8.3) and the analogue of (8.3) for t to get the bound

$$\frac{c\,t^2(st)^{\alpha+1/2}}{s^2\, n^\theta}\Big[n^{\theta-1}s^{-2\alpha-3} + s^{-\alpha-\theta-1/2}\min(n,t^{-1})^{\alpha+3/2}\Big].$$

In the first part use the fact that $t^2/s^2 \le 1$ to show it is bounded by the first term in (8.2). In the second part use the fact that $t^2\min(n,t^{-1}) \le s$; with this the second part is the same as the second part of (8.15). This completes the proof for (8.11).

For (8.12) we will use lemma (3.1); the definition of J insures that $J < 0$ and $J < -\alpha-1/2$. This gives the bound

$$\frac{c\,\omega(s,t)}{n^{\theta}}\, n^{\theta-1}(1-x)^{-(2\alpha+2+J)}. \tag{8.16}$$

This is less than the first term of (8.14) which was estimated before. Lemma (3.1) is also applied to (8.13). The result is $(1-y)/(1-x)$ times (8.16) and is, therefore, less than (8.16). This completes the proof of theorem (8.1).

9. A basic estimate for separated arguments. The lemma proved here is the basis for the inductive argument given in §11 to estimate kernels for separated arguments. Lemma (9.1) is also needed to estimate error terms.

Lemma (9.1). If α, β, γ and δ are greater than -1, $a > 0$, $-1 \le x \le y-a \le 1-a$, n is an integer, and either $t \le -\beta-\gamma-3$ or $\theta \le -\beta-\gamma-5$, then

$$\sum_{k=0}^{n} (k+1)^{t}\,|A_{n-k}^{\theta}|\,E_k^{(\alpha,\beta)}(x)\; E_k^{(\gamma,\delta)}(y) \tag{9.2}$$

has the bound

$$c(n+1)^{\theta} + c(n+1)^{t}E_n^{(\alpha,\beta)}(x)\; E_n^{(\gamma,\delta)}(y) \tag{9.3}$$

with c independent of n, x and y.

To prove lemma (9.1), split (9.2) into sums over $0 \le k \le [n/2]-1$ and $[n/2] \le k \le n$ and use (2.19) and (2.6) to show that (9.2) is bounded by the sum of

$$(9.4)\qquad c(n+1)^{\theta} \sum_{k=0}^{[n/2]-1} (k+1)^{t}\ E_k^{(\alpha,\beta)}(x)E_k^{(\gamma,\delta)}(y),$$

and

$$(9.5)\qquad c(n+1)^{t}\ E_n^{(\alpha,\beta)}(x)E_n^{(\gamma,\delta)}(y) \sum_{k=[n/2]}^{n} (n+1-k)^{\theta}.$$

For (9.4) use the facts obtained from (2.6) and the hypothesis $-1 \le x \le y-a \le 1-a$ that

$$(9.6)\qquad E_k^{(\alpha,\beta)}(x) \le c(k+1)^{\beta+1/2}$$

and

$$(9.7)\qquad E_k^{(\gamma,\delta)}(y) \le c(k+1)^{\gamma+1/2}.$$

These show that (9.4) has the bound

$$(9.8)\qquad c(n+1)^{\theta} \sum_{k=0}^{[n/2]-1} (k+1)^{\beta+\gamma+t+1}.$$

If $t < -\beta-\gamma-3$, then the exponent of $k+1$ in the sum is less than -2 and (9.8) is bounded by $c(n+1)^{\theta}$. If $t \ge -\beta-\gamma-3$, then (9.8) has the bound

$$(9.9)\qquad c(n+1)^{\theta+\beta+\gamma+t+3}.$$

From (2.6) we see that

$$(9.10)\qquad 1 \le (n+1)E_n^{(\alpha,\beta)}(x)$$

and

$$1 \le (n+1)E_n^{(\gamma,\delta)}(y). \tag{9.11}$$

From these we see that (9.9) has the bound

$$c(n+1)^{\theta+\beta+\gamma+5}(n+1)^t \; E_n^{(\alpha,\beta)}(x)E_n^{(\gamma,\delta)}(y).$$

Since in this case $\theta \le -\beta-\gamma-5$, this is bounded by the second term in (9.3). This completes the estimation of (9.4).

For (9.5), if $\theta \le -2$, then (9.5) is bounded by the second term in (9.3). If $\theta > -2$, then the sum in (9.5) has the bound $c(n+1)^{\theta+2}$. Using this, (9.6) and (9.7) shows that (9.5) is bounded by

$$c(n+1)^{\beta+\gamma+t+1+\theta+2}. \tag{9.12}$$

Since $\theta > -2$, we also have $\theta > -\beta-\gamma-5$ and by hypothesis $t \le -\beta-\gamma-3$. Therefore, (9.12) is bounded by $c(n+1)^{\theta}$. This completes the proof of lemma (9.1).

10. A reduction lemma for separated arguments. The lemma of this section is like lemma (6.1) but easier to prove. Like lemma (6.1) the result is valid for any x and y in $[-1,1]$, but it is useful only if x and y are separated.

Lemma (10.1). If $\alpha > -1$, $\beta > -1$, $\gamma > -1$, $\delta > -1$, $\theta \ge 0$, t is real, $-1 \le x \le 1$, $-1 \le y \le 1$, J, m, n, u and v are integers, $0 \le m \le n$ and $J \ge 2$, then (6.2) can be written as the sum of (6.3), terms with absolute value bounded by (6.4), terms of the form

$$\frac{(Ax+By+C)}{x\ -\ y}\sum_{k=m}^{n'}(k+1)^{t+i}\,A^{\theta}_{n'-k}P^{(\alpha,\beta)}_{k+u'}(x)P^{(\gamma,\delta)}_{k+v'}(y), \tag{10.2}$$

where $2-J \le i \le -1$, $n' = n$ or $n-1$, $u' = u$ or $u-1$ and $v' = v-1$, and terms of the form

$$\frac{Ax+By+C}{x\ -\ y}\sum_{k=m}^{n}(k+1)^{t+i}\,A^{\theta-1}_{n-k}\,P^{(\alpha,\beta)}_{k+u'}(x)P^{(\gamma,\delta)}_{k+v'}(y); \tag{10.3}$$

where $i = 0$ or $i = -1$, $u' = u$ or $u+1$ and $v' = v+1+u-u'$.

The proof of lemma (10.1) is similar to the proof of lemma (6.1). The equality (2.9) is used with the J of the hypothesis to write (6.2) as the sum of (6.6)–(6.8). As before, the terms in (6.7) have the form (10.2) if $2 \le j \le J-2$ while (6.8) and the terms in (6.7) with $j = J-1$ have absolute value bounded by (6.4). The term (6.6) is written as the sum of (6.9)–(6.11); the second term of (6.9) is (6.3) while the first term of (6.9) plus (6.10) equals (6.12). For (6.11) and (6.12) replace $Q_n(x,y)$ by its definition (2.8), replace $2k+3$ by $2(k+1)+1$ and in (6.11) replace $(k+2)^{t-1} - (k+1)^{t-1}$ by (6.14). For (6.11) this produces terms of the form (10.2) and terms majorized by (6.4). For (6.12) this produces the terms (10.3). This completes the proof of Lemma (10.1).

11. A kernel estimate for separated arguments. This section contains the inductive argument for estimating kernels with parameters larger than those allowed in §9. The result is stated in lemma (11.1).

Lemma (11.1). If α, β, γ and δ are greater than -1, $a > 0$, $-1 \leq x \leq y-a \leq 1-a$, t and θ are real and n, u and v are integers, then

$$\left| \sum_{k=0}^{n} (k+1)^t A_{n-k}^{\theta} P_{k+u}^{(\alpha,\beta)}(x) P_{k+v}^{(\gamma,\delta)}(y) \right| \tag{11.2}$$

has the bound

$$c(n+1)^{\theta} + c(n+1)^t E_n^{(\alpha,\beta)}(x) E_n^{(\gamma,\delta)}(y) \tag{11.3}$$

with c independent of n, x and y.

To prove lemma (11.1) we will show inductively for each integer K that the estimate is valid if $t+\theta \leq K$. If $K = 2[-\beta-\gamma-4]$ and $t+\theta \leq K$, then either $t \leq -\beta-\gamma-3$ or $\theta \leq -\beta-\gamma-5$. Lemma (9.1) will then prove the result for this K.

To complete this induction, assume that the estimate is valid for $t+\theta \leq K$ and fix t and θ satisfying $t+\theta \leq K+1$. To estimate (11.2), apply lemma (10.1) with $m = 0$ and $J = \max([\beta+\gamma+t+5],2)$. Using the fact that $1/(y-x)$ is bounded, we see that (11.2) is bounded by the sum of

$$c \sum_{k=0}^{n} (k+1)^{t+1-J} A_{n-k}^{\theta} E_k^{(\alpha,\beta)}(x) E_k^{(\gamma,\delta)}(y), \tag{11.4}$$

terms of the form

$$\left| \sum_{k=0}^{n'} (k+1)^{t+i} A_{n'-k}^{\theta} P_{k+u'}^{(\alpha,\beta)}(x) P_{k+v'}^{(\gamma,\delta)}(y) \right|, \tag{11.5}$$

where $2-J \le i \le -1$, $n' = n$ or $n-1$, $|u'-u| \le 1$ and $|v'-v| \le 1$, and terms of the form

$$(11.6) \qquad \left| \sum_{k=0}^{n} (k+1)^{t+i} A_{n-k}^{\theta-1} P_{k+u'}^{(\alpha,\beta)}(x) P_{k+v'}^{(\gamma,\delta)}(y) \right|,$$

where $i = 0$ or $i = -1$, $|u'-u| \le 1$ and $|v-v'| \le 1$.

For (11.4) use the fact that $J \ge \beta+\gamma+t+4$ to show that $t+1-J \le -\beta-\gamma-3$. Therefore, lemma (9.1) can be applied. Since $t+1-J \le t$, the result is bounded by (11.3). For (11.5) we have $t+i+\theta \le t+\theta-1 \le K$ so the inductive hypothesis can be applied. Since $t+i \le t$, the result is bounded by (11.3). Similarly for (11.6) we have $t+i+\theta-1 \le K$. The inductive hypothesis again gives a bound not larger than (11.3). This completes the proof of lemma (11.1).

12. Cesaro kernel estimate for $t \le s-b$. Here we shall prove another basic estimate as follows.

Theorem (12.1). If $\alpha > -1$, $\beta > -1$, $b > 0$, $\theta \ge 0$ and $0 \le t \le s-b \le \pi-b$, then $|K_n^{(\alpha,\beta),\theta}(s,t)|$ has the bound

$$(12.2) \qquad c\,\frac{t^{\alpha+1/2}(\pi-s)^{\beta+1/2}}{n+1} + c\,\frac{[\min(1,nt)]^{\alpha+1/2}[\min(1,n(\pi-s))]^{\beta+1/2}}{(n+1)^{\theta}},$$

where c is independent of n, s and t.

To prove this, use (2.23) in (2.26) to show that $|K_n^{(\alpha,\beta),\theta}(s,t)|$ equals

$$(12.3) \qquad \frac{1}{A_n^{\theta}} \left| \sum_{k=0}^{n} A_{n-k}^{\theta-1} \frac{\omega(s,t) u_k^{(\alpha,\beta)}}{x-y} Q_k(x,y) \right|.$$

Now replace $Q_k(x,y)$ by its definition (2.8). By (2.25) it is possible to write $(2k+3)u_k^{(\alpha,\beta)}$ in the form

$$\sum_{j=1-J}^{1} a_j(k+1)^j + O((k+1)^J)$$

with $J = [\beta+\gamma+4]$. Using this, the fact that $1/(y-x) \le 1/(1-\cos b)$ and (2.5), we see that (12.3) is bounded by the sum of

$$c\,\frac{\omega(s,t)}{A_n^\theta}\left|\sum_{k=0}^{n} A_{n-k}^{\theta-1}(k+1)^{-J}E_k^{(\alpha,\beta)}(x)E_k^{(\alpha,\beta)}(y)\right| \tag{12.4}$$

and terms of the form

$$c\,\frac{\omega(s,t)}{A_n^\theta}\left|\sum_{k=0}^{n} A_{n-k}^{\theta-1}(k+1)^{j}P_{k+u}^{(\alpha,\beta)}(x)P_{k+v}^{(\alpha,\beta)}(y)\right| \tag{12.5}$$

with $1-J \le j \le 1$, $u = 0$ or $u = 1$ and $v = 1-u$. For (12.4), since $y-x \ge 1-\cos b$ and $-J < -\beta-\gamma-3$, we can apply lemma (9.1) to get the estimate

$$c\,\frac{\omega(s,t)}{A_n^\theta}\left[(n+1)^{\theta-1}+(n+1)^{-J}E_n^{(\alpha,\beta)}(x)E_n^{(\alpha,\beta)}(y)\right]. \tag{12.6}$$

By the definition (2.24), (2.19), analogues of (5.8) and the fact that $(n+1)^{-J} \le n+1$, this is bounded by

$$c\,\frac{(\pi-s)^{\beta+1/2}t^{\alpha+1/2}}{(n+1)^\theta}\left[(n+1)^{\theta-1}+[\min((1+x)^{-1/2},n+1)]^{\beta+1/2}[\min((1-y)^{-1/2},n+1)]^{\alpha+1/2}\right]$$

which is bounded by (12.2). For (12.5) use lemma (11.1); this also produces the estimate (12.6) with $-J$ replaced by j. This completes the proof of theorem (12.1).

13. Cesaro kernel estimate for s near t. This section is concerned with estimates for $1/n < |s-t| < s/2$. For this case repeated summations by parts are not needed; what is used is Darboux's formula (2.15).

Theorem (13.1). If $\alpha > -1$, $\beta > -1$, n is an integer $\theta > 0$, $0 \leq 1/n \leq s/2 \leq t \leq s \leq 3\pi/4$ and $n(s-t) > 2$, then

$$|K_n^{(\alpha,\beta),\theta}(s,t)| \leq c\, n^{-\theta}(s-t)^{-\theta-1} + cn^{-1}|s-t|^{-2} \tag{13.2}$$

with c independent of n, s and t.

To prove this, substitute (2.23) into (2.26) and use the fact that $Q_n(x,y)$ is the sum of (2.12)–(2.14). Note also that since $\alpha = \gamma$, $u = 0$ and $v = 0$, the term (2.14) vanishes. Therefore, $K_n^{(\alpha,\beta),\theta}(s,t)$ equals

$$\frac{\omega(s,t)}{(x-y)A_n^\theta} \sum_{k=0}^{n} A_{n-k}^{\theta-1}\left[P_k^{(\alpha,\beta)}(x)P_k^{(\alpha+1,\beta)}(y)(1-y)-P_k^{(\alpha+1,\beta)}(x)P_k^{(\alpha,\beta)}(y)(1-x)\right]V_k,$$

where

$$V_k = u_k^{(\alpha,\beta)} \frac{(2k+3)(2k+\alpha+\beta+2)}{8(k+1)} .$$

Now use the definition (2.1) to show that $|K_n^{(\alpha,\beta),\theta}(s,t)|$ is bounded by the sum of

(13.3)

$$\left|\sum_{k=0}^{[1/s]-1} \frac{A_{n-k}^{\theta-1}\omega_k}{A_n^\theta}\left[\frac{\phi_k^{(\alpha,\beta)}(s)\phi_k^{(\alpha+1,\beta)}(t)\sin(t/2)-\phi_k^{(\alpha+1,\beta)}(s)\phi_k^{(\alpha,\beta)}(t)\sin(s/2)}{\cos s - \cos t}\right]\right|$$

and

(13.4)

$$\left|\sum_{k=[1/s]}^{n} \frac{A_{n-k}^{\theta-1}\omega_k}{A_n^{\theta}}\left[\frac{\phi_k^{(\alpha,\beta)}(s)\phi_k^{(\alpha+1,\beta)}(t)\sin(t/2)-\phi_k^{(\alpha+1,\beta)}(s)\phi_k^{(\alpha,\beta)}(t)\sin(s/2)}{\cos s-\cos t}\right]\right|,$$

where

$$\omega_k = 2u_k^{(\alpha,\beta)}\,\frac{(2k+3)(2k+\alpha+\beta+2)}{8k+1}\left[t_k^{(\alpha,\beta)}t_k^{(\alpha+1,\beta)}\right]^{-1}.$$

Note that by (2.3) and (2.25) there exist constants d_0, d_1 and d_2, independent of k, such that

$$|\omega_k-d_0-d_1(k+1)^{-1}| \le d_2(k+1)^{-2}. \tag{13.5}$$

To prove that (13.3) is bounded by the right side of (13.2), use (2.7) and the fact that $s/2 \le t \le s$ to get the estimate

$$c\sum_{k=0}^{[1/s]-1}\frac{A_{n-k}^{\theta-1}|\omega_k|(k+1)^{2\alpha+2}s^{2\alpha+3}}{A_n^{\theta}(\cos t-\cos s)}. \tag{13.6}$$

Since $n \ge 2/s$, we have from (2.19) that $A_{n-k}^{\theta-1} \le c\,n^{\theta-1}$. From (2.20) we have $1/A_n^{\theta} \le c\,n^{-\theta}$ and from (13.5) we see that $|\omega_k| \le c$. Using these facts and estimating the sum, we have (13.6) bounded by

$$\frac{c}{n(\cos t-\cos s)} \le \frac{c}{n\,s(s-t)} \le \frac{c}{n(s-t)^2}.$$

This completes the proof for (13.3).

For (13.4), use (2.15), (13.5) and the fact that $s/2 \le t \le s$ to show that $\omega_k\phi_k^{(\alpha,\beta)}(s)\phi_k^{(\alpha+1,\beta)}(t)$ and $\omega_k\phi_k^{(\alpha+1,\beta)}(s)\phi_k^{(\alpha,\beta)}(t)$ can be written as a sum of terms of the form $\cos(ks+a)\cos(kt+b)uv$, $\cos(ks+a)\cos(kt+b)uv/(ks)$ and

$\cos(ks+a)\cos(kt+b)uv/(kt)$ and a term with absolute value bounded by $c\,k^{-2}s^{-2}$, where a and u are bounded functions of s independent of k and t, b and v are bounded functions of t independent of k and s and c is independent of k, s and t. From this and (2.20) we see that (13.4) can be estimated by finding upper bounds for

$$\frac{1}{n^\theta(s-t)}\left|\sum_{k=[1/s]}^{n} A_{n-k}^{\theta-1}\cos(ks+a)\cos(kt+b)\right|. \tag{13.7}$$

$$\frac{1}{n^\theta s(s-t)}\left|\sum_{k=[1/s]}^{n} \frac{A_{n-k}^{\theta-1}\cos(ks+a)\cos(kt+b)}{k}\right| \tag{13.8}$$

and

$$\frac{1}{n^\theta s^2(s-t)}\sum_{k=[1/s]}^{n} A_{n-k}^{\theta-1}k^{-2}. \tag{13.9}$$

To estimate (13.7) and (13.8) we will use the following lemma.

<u>Lemma (13.10)</u>. If $\theta \geq 0$, $0 < u < 3\pi/2$, $nu > 2$ and $1 \leq m \leq n/2$, then

$$\left|\sum_{k=m}^{n} A_{n-k}^{\theta-1}\cos(ku+b)\right| \leq c\,u^{-\theta} + c\,n^{\theta-1}u^{-1} \tag{13.11}$$

and

$$\left|\sum_{k=m}^{n} A_{n-k}^{\theta-1}\frac{\cos(ku+b)}{k}\right| \leq c\,n^{-1}u^{-\theta} + c\,n^{\theta-1}(mu)^{-1}. \tag{13.12}$$

To prove these, let $N = \max(m, n+1-[1/u])$ and consider separately the sums from m to $N-1$ and from N to n. For (13.11) the quantity $A_{n-k}^{\theta-1}$ is monotone so by lemma (2.17) if $\theta < 1$ or a reversed version if $\theta \geq 1$, the absolute value of the sum from m to $N-1$ has the bound $c\left[A_{n-m}^{\theta-1} + A_{n-N+1}^{\theta-1}\right]u^{-1}$. By (2.19) this has the required bound. For the sum from N to n replace $\cos(ku+b)$ by 1 and use (2.22); this gives the bound A_{n-N}^{θ} which, by (2.19), is bounded by $c\,u^{-\theta}$.

For (13.12), observe that $k^{-1}A_{n-k}^{\theta-1}$ is decreasing for $1 \leq k \leq n$ if $\theta \geq 1$. If $\theta < 1$, then $k^{-1}A_{n-k}^{\theta-1}$ is either monotone in $1 \leq k \leq n$ or has a minimum for k equal to some k_0 and is decreasing for $1 \leq k \leq k_0$ and increasing for $k_0 \leq k \leq n$. In either case, to estimate the sum from m to $N-1$, we can use lemma (2.17) or a reversed version on the whole sum if $k^{-1}A_{n-k}^{\theta-1}$ is monotone or separately on the sum from m to k_0 and on the sum from k_0+1 to N. This produces the estimate

$$c\,u^{-1}\left[m^{-1}A_{n-m}^{\theta-1} + n^{-1}A_{n-N}^{\theta-1}\right],$$

and (2.19) shows this is bounded by the right side of (13.12). For the sum from N to n, replace $\cos(ku+b)$ by 1. Since $nu > 2$, we have $N > n/2$ and the $1/k$ can be replaced by $2/n$. Then using (2.2) we get the estimate $c\,n^{-1}A_{n-N}^{\theta}$ for this part, and by (2.19) this has the bound $c\,n^{-1}u^{-\theta}$. This completes the proof of lemma (13.10).

Returning to the estimation of (13.7)–(13.9), we write $\cos(ks+a)\cos(kt+b)$ as a sum of cosines using the usual trigonometric identity. Lemma (13.10) then shows that (13.7) has the bound

$$\frac{1}{(s-t)n^\theta}\,[c(s-t)^{-\theta} + cn^{\theta-1}(s-t)^{-1} + c(s+t)^{-\theta} + cn^{\theta-1}(s+t)^{-1}]$$

and (13.9) has the bound

$$\frac{1}{s(s-t)n^\theta}\,[\,c\,n^{-1}(s-t)^{-\theta} + cn^{\theta-1}s(s-t)^{-1} + cn^{-1}(s+t)^{-\theta} + c\,n^{\theta-1}s(s+t)^{-1}].$$

These are easily seen to be bounded by the right side of (13.2). For (13.9) split the sum into sums over $[1/s] \le k \le n/2$ and $n/2 < k \le n$. In the first replace $A_{n-k}^{\theta-1}$ by $n^{\theta-1}$ and estimate the sum. In the second replace k^{-2} by $c\,n^{-2}$ and use (2.22). This gives an estimate of $\frac{c}{ns(s-t)} + \frac{c}{n^2s^2(s-t)}$ for (13.9) which is bounded by $c\,n^{-1}(s-t)^{-2}$. This completes the proof of theorem (13.1).

14. Kernel estimates. Here we state and complete the proof of theorem (14.1), our estimate of the kernel $K_n^{(\alpha,\beta),\theta}(s,t)$. We also give an alternate version in corollary (14.2) and a version for $L_n^{(\alpha,\beta),\theta}(x,y)$ in corollary (14.7). For theorem (14.1) note that since $K_n^{(\alpha,\beta),\theta}(s,t) = K_n^{(\alpha,\beta),\theta}(t,s)$ and $K_n^{(\alpha,\beta),\theta}(\pi-s,\pi-t) = K_n^{(\beta,\alpha),\theta}(s,t)$, we need only state the result for $0 \le t \le s$ and $t \le \pi/2$.

Theorem (14.1). If $\alpha > -1$, $\beta > -1$, $\theta \ge 0$, and $n \ge 1$, then $|K_n^{(\alpha,\beta),\theta}(s,t)|$ has the bounds

$c\,n^{2\alpha+2}(st)^{\alpha+1/2}$	$0 \le s \le 2/n,\ 0 \le t \le 2/n,$
$\frac{c\,t^{\alpha+1/2}}{n\,s^{\alpha+5/2}} + \frac{c(\min(1,nt))^{\alpha+1/2}}{n^\theta s^{\theta+1}}$	$2/n \le s \le 3\pi/4,\ 0 \le t \le s/2,$
$\frac{c}{n(s-t)^2} + \frac{c}{n^\theta(s-t)^{\theta+1}}$	$2/n \le s \le 3\pi/4,\ s/2 \le t \le s-1/n$

$$cn \qquad 2/n \le s \le 3\pi/4,\ s-1/n \le t \le s$$

$$\frac{c\,t^{\alpha+1/2}(\pi-s)^{\beta+1/2}}{n} + \frac{c\min(1,nt)^{\alpha+1/2}\min(1,n(\pi-s))^{\beta+1/2}}{n^{\theta}} \qquad 3\pi/4 \le s \le \pi,\ 0 \le t \le \pi/2,$$

where c is independent of n, s and t.

To prove theorem (14.1) we must obtain estimates for a few simple cases not included in §§3–13. For $0 \le s \le 2/n$ and $0 \le t \le 2/n$ we can use (2.7), (2.19) and (2.20) to show that

$$\left|K_n^{(\alpha,\beta),\theta}(s,t)\right| \le c(st)^{\alpha+1/2}\left[\sum_{k=0}^{[n/2]-1}(k+1)^{2\alpha+1} + \sum_{k=[n/2]}^{n} n^{2\alpha+1-\theta}A_{n-k}^{\theta}\right].$$

In the first sum use the fact that $2\alpha+1 > -1$; in the second use (2.22) and (2.19). This proves the first estimate of theorem (14.1). The second estimate follows from theorem (8.1) if $s \le \pi/2$ and from theorem (12.1) if $\pi/2 \le s \le 3\pi/4$. The third estimate is a consequence of theorem (13.1). For the fourth we use (2.7), (2.19), (2.20) and the fact that $s/2 \le t < 2$ to get the estimate

$$c\sum_{k=0}^{[1/s]-1}(k+1)^{2\alpha+1}s^{2\alpha+1} + c\sum_{k=[1/s]}^{n} n^{-\theta}A_{n-k}^{\theta}.$$

Since $2\alpha+1 > -1$, the first term has the bound $c/s \le cn$, and by (2.22) and (2.19) the second term has the bound cn. The fifth estimate is a consequence of theorem (12.1). This completes the proof of theorem (14.1).

An alternative form of theorem (14.1) can be stated using the functions

$$g(s) = s^{\alpha+1/2}(\pi-s)^{\beta+1/2}$$

and

$$h(s,n) = (s+\tfrac{1}{n})^{\alpha+1/2}(\pi-s+\tfrac{1}{n})^{\beta+1/2}.$$

This version has the advantage that it gives one expression for all pairs (s,t) in $[0,\pi] \times [0,\pi]$. The disadvantage is that the behavior is not apparent until it is rewritten into a form resembling the conclusion of theorem (14.1).

<u>Corollary (14.2).</u>. If α and β are greater than -1, $\theta > 0$, $0 \le s \le \pi$ and $0 \le t \le \pi$, then $|K_n^{(\alpha,\beta),\theta}(s,t)|$ has the bound

$$\frac{c\ g(s)g(t)}{nh(\frac{s+t}{2},n)^2(|s-t|+\frac{1}{n})^2} + \frac{c\ g(s)g(t)}{n^\theta h(s,n)h(t,n)(|s-t|+\frac{1}{n})^{\theta+1}},$$

where c is independent of n, s and t.

To prove corollary (14.2), let $D_n^{(\alpha,\beta),\theta}(s,t)$ be the asserted upper bound. For the case $0 \le t \le \min(s,\pi/2)$ it is easy to verify that

$$|K_n^{(\alpha,\beta),\theta}(s,t)| \le D_n^{(\alpha,\beta),\theta}(s,t) \tag{14.3}$$

by using theorem (14.1) and the definition of $D_n^{(\alpha,\beta)}(s,t)$. For $\pi/2 \le t \le s \le \pi$, we have $0 \le \pi-s \le \min(\pi-t,\pi/2)$; as just shown, therefore,

$$|K_n^{(\beta,\alpha),\theta}(\pi-t,\pi-s)| \le D_n^{(\beta,\alpha)}(\pi-t,\pi-s). \tag{14.4}$$

Now since $K_n^{(\alpha,\beta),\theta}(s,t) = K_n^{(\alpha,\beta),\theta}(t,s)$ and $K_n^{(\alpha,\beta),\theta}(s,t) = K_n^{(\beta,\alpha),\theta}(\pi-s,\pi-t)$, we have

$$K_n^{(\alpha,\beta),\theta}(s,t) = K_n^{(\beta,\alpha),\theta}(\pi-t,\pi-s). \tag{14.5}$$

from the definition of $D_n^{(\alpha,\beta),\theta}(s,t)$ it is easy to see that

$$D_n^{(\beta,\alpha)}(\pi-t,\pi-s) = D_n^{(\alpha,\beta),\theta}(s,t). \tag{14.6}$$

Combining (14.4), (14.5) and (14.6), we get the result for this case. Combining these cases, we have the estimate $0 \le t \le s \le \pi$. Since $K_n^{(\alpha,\beta),\theta}(s,t) = K_n^{(\alpha,\beta),\theta}(t,s)$ and $D_n^{(\alpha,\beta),\theta}(s,t) = D_n^{(\alpha,\beta),\theta}(t,s)$, the asserted bound follows for $0 \le s \le t \le \pi$. This completes the proof of corollary (14.2).

To state our estimate for the basic Cesaro kernel $L_n^{(\alpha,\beta),\theta}(x,y)$ defined in (2.27) we will use the function

$$H(x,n) = (\sqrt{1-x} + \tfrac{1}{n})^{\alpha+1/2}(\sqrt{1+x} + \tfrac{1}{n})^{\beta+1/2}$$

and

$$J(x,y,n) = \frac{|y-x|}{\sqrt{4-(x+y)^2}} + \frac{1}{n}.$$

The resulting estimate is as follows.

<u>Corollary (14.7)</u>. If α and β are greater than -1, $\theta > 0$, $-1 \le x \le 1$ and $-1 \le y \le 1$, then $|L_n^{(\alpha,\beta),\theta}(x,y)|$ has the bound

$$\frac{c}{nH(\frac{x+y}{2},n)^2\ J(x,y,n)^2} + \frac{c}{n^{\theta}H(x,n)H(y,n)\ J(x,y,n)^{1+\theta}},$$

where c is independent of n, x and y.

To prove corollary (14.7) let $s = \cos^{-1} x$ and $t = \cos^{-1} y$. Then because of (2.29) we see that the result can be proved by showing the existence of a positive constant c, independent of s and t such that

$$\frac{1}{c} \le \frac{g(s)}{(\sin \frac{s}{2})^{\alpha+1/2}(\cos \frac{s}{2})^{\beta+1/2}} \le c \tag{14.8}$$

$$\frac{1}{c} \le \frac{h(s,n)}{H(\cos s,n)} \le c \tag{14.9}$$

and

$$\frac{1}{c} \le \frac{|s-t| + \frac{1}{n}}{J(\cos s, \cos t,n)} \le c \tag{14.10}$$

for s and t in $[0,\pi]$. Now (14.8) is obvious from the definition of g(s), and (14.9) follows easily since $\sqrt{1-\cos s} = \sqrt{2} \sin \frac{s}{2}$ and $\sqrt{1+\cos s} = \sqrt{2} \cos \frac{s}{2}$. For (14.10) use the fact that

$$J(\cos s, \cos t,n) = \frac{|\cos t - \cos s|}{\sqrt{(1-\cos s+1-\cos t)(1+\cos s+1+\cos t)}} + \frac{1}{n}.$$

Using standard identities this becomes

$$J(\cos s, \cos t,n) = \frac{|\sin \frac{s-t}{2}|\sin \frac{s+t}{2}}{\sqrt{(\sin^2 \frac{s}{2} + \sin^2 \frac{t}{2})(\cos^2 \frac{s}{2} + \cos^2 \frac{t}{2})}} + \frac{1}{n}.$$

Considering separately the cases of s and t both in $[0,\pi/4]$, s and t both in $[3\pi/4,\pi]$ and one in $[0,3\pi/4]$ with the other in $[\pi/4,\pi]$, it is easy to see that the function multiplying $|\sin \frac{s-t}{2}|$ is bounded above and below by positive constants. This completes the proof of corollary (14.7).

15. A weak type lemma. Here we shall prove lemma (15.1) which is equivalent to theorem (1.1) with the support of f in [0,1]. Lemma (15.1) will be used to prove theorem (1.1) in §17. At the end of this section we also show that the method used to prove theorem 1 of [7] will not prove lemma (15.1).

Lemma (15.1). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $\theta > 0$, $p = \max[1,(4\gamma+4)/(2\gamma+2\theta+3)]$, $g(t)$ is supported on $[0,\pi/2]$, $a > 0$ and $D(a)$ is the subset of $[0,\pi]$ where

$$\sup_n \int_0^{\pi/2} |K_n^{(\alpha,\beta),\theta}(s,t)g(t)|\,dt > a\, s^{\alpha+1/2}(\pi-s)^{\beta+1/2}, \tag{15.2}$$

then

$$\int_{D(a)} s^{2\alpha+1}(\pi-s)^{2\beta+1}\,ds \le c\, a^{-p} \int_0^{\pi/2} |g(t)|^p\, t^{(\alpha+1/2)(2-p)}dt, \tag{15.3}$$

where c is independent of a and g.

To prove this note first that we may assume that $g(t)$ is nonnegative. For this proof we will use the notation

$$H = \int_0^{\pi/2} g(t)^p\, t^{(\alpha+1/2)(2-p)}dt.$$

To prove the lemma we will define nonnegative functions $\ell_{n,i}(s,t)$ such that for (s,t) in $[0,\pi] \times [0,\pi/2]$

$$K_n^{(\alpha,\beta),\theta}(s,t) \le c \sum_{i=1}^{11} \ell_{n,i}(s,t) \tag{15.4}$$

for $n \ge 1$ and

$$|K_0^{(\alpha,\beta),\theta}(s,t)| \le c[\ell_{1,1}(s,t) + \ell_{1,4}(s,t) + \ell_{1,10}(s,t)]. \tag{15.5}$$

Now let D_i be the subset of $[0,\pi]$ where

$$\sup_{n\geq 1} \int_0^{\pi/2} \ell_{n,i}(s,t)g(t)dt > a\, s^{\alpha+1/2}(\pi-s)^{\beta+1/2}. \tag{15.6}$$

We will show that

$$\int_{D_i} s^{2\alpha+1}(\pi-s)^{2\beta+1}\, ds \leq c\, a^{-p}H \tag{15.7}$$

for $1 \leq i \leq 11$. Because (15.4) and (15.5) are true, this will complete the proof of the lemma.

For some values of i we will prove (15.7) directly. For other values of i, however, we will prove the inequality that

$$\int_0^{\pi} \left[\sup_{n\geq 1} \int_0^{\pi/2} \ell_{n,i}(s,t)g(t)dt\right]^p \left[s^{\alpha+1/2}(\pi-s)^{\beta+1/2}\right]^{2-p} ds \tag{15.8}$$

is bounded by cH. This strong type inequality trivially implies (15.7).

The functions $\ell_{n,i}$ are defined as follows on the indicated sets and 0 off those sets. That (15.4) and (15.5) hold is an immediate consequence of theorem (14.1).

$$\ell_{n,1}(s,t) = n^{2\alpha+2}(st)^{\alpha+1/2} \qquad 0 \leq s \leq 2/n,\ 0 \leq t \leq 2/n,$$

$$\ell_{n,2}(s,t) = \frac{s^{\alpha+1/2}}{n^{\theta-\alpha-1/2}t^{\theta+1}} \qquad 0 \leq s \leq 2/n,\ 2/n \leq t \leq \pi/2,$$

$$\ell_{n,3}(s,t) = \frac{s^{\alpha+1/2}}{n\, t^{\alpha+5/2}} \qquad 0 \leq s \leq 2/n,\ 2/n \leq t \leq \pi/2,$$

$$\ell_{n,4}(s,t) = \frac{t^{\alpha+1/2}}{n^{\theta-\alpha-1/2}s^{\theta+1}} \qquad 2/n \leq s \leq 3\pi/4,\ 0 \leq t \leq 2/n,$$

$$\ell_{n,5}(s,t) = \frac{t^{\alpha+1/2}}{n\, s^{\alpha+5/2}} \qquad 2/n \le s \le 3\pi/4,\ 0 \le t \le s/2,$$

$$\ell_{n,6}(s,t) = \frac{1}{n^{\theta} s^{\theta+1}} \qquad 4/n \le s \le 3\pi/4,\ 2/n \le t \le s/2,$$

$$\ell_{n,7}(s,t) =$$

$$\frac{1}{n(|s-t|+1/n)^2} + \frac{1}{n^{\theta}(|s-t|+1/n)^{1+\theta}} \qquad 2/n \le s \le 3\pi/4,\ s/2 \le t \le \min(2s,\pi/2),$$

$$\ell_{n,8}(s,t) = \frac{s^{\alpha+1/2}}{n\, t^{\alpha+5/2}} \qquad 2/n \le s \le \pi/4,\ 2s \le t \le \pi/2,$$

$$\ell_{n,9}(s,t) = \frac{1}{n^{\theta} t^{1+\theta}} \qquad 2/n \le s \le \pi/4,\ 2s \le t \le \pi/2,$$

$$\ell_{n,10}(s,t) = \frac{t^{\alpha+1/2}(\pi-s)^{\beta+1/2}}{n} \qquad 3\pi/4 \le s \le \pi,\ 0 \le t \le \pi/2$$

$$\ell_{n,11}(s,t) = \frac{\min(1,nt)^{\alpha+1/2}\min(1,n(\pi-s))^{\beta+1/2}}{n^{\theta}} \qquad 3\pi/4 \le s \le \pi,\ 0 \le t \le \pi/2.$$

The following simple inequalities will be used in several of the estimations. First, since $2\gamma+2\theta+3 > 2\gamma+2$, the definition of p shows that

$$1 \le p < 2. \tag{15.9}$$

We also have from the definition of p that

$$p(\gamma+\theta+3/2) \ge 2\gamma+2, \tag{15.10}$$

and since $\alpha \le \gamma$ and $p \le 2$, this implies

$$p(\alpha+\theta+3/2) \ge 2\alpha+2. \tag{15.11}$$

To prove (15.7) for $i = 1$ we start with the fact that the left side of (15.6) equals

$$\sup_{n\geq 1} n^{2\alpha+2} s^{\alpha+1/2} \chi_{[0,2/n]}(s) \int_0^{2/n} g(t)t^{\alpha+1/2}\, dt. \tag{15.12}$$

Now using

$$t^{\alpha+1/2}g(t) = [t^{(\alpha+1/2)(2-p)/p}g(t)][t^{(2\alpha+1)(p-1)/p}] \tag{15.13}$$

and Hölder's inequality we have

$$\int_0^{2/n} g(t)t^{\alpha+1/2}\, dt \leq c\, n^{-(2\alpha+2)/p'} H^{1/p}. \tag{15.14}$$

From this we see that (15.12) is bounded by

$$\sup_{n\geq 1} c\, n^{(2\alpha+2)/p} s^{\alpha+1/2} \chi_{[0,2/n]}(s) H^{1/p}.$$

Since $\alpha > -1$,

$$\sup_{n\geq 1} n^{(2\alpha+2)/p}\chi_{[0,2/n]}(s) \leq c\, s^{-(2\alpha+2)/p}\chi_{[0,3\pi/4]}(s)$$

and D_1 is a subset of the set where

$$c\, s^{\alpha+1/2} s^{-(2\alpha+2)/p} \chi_{[0,3\pi/4]}(s) H^{1/p} > a\, s^{\alpha+1/2}.$$

Therefore, D_1 is a subset of $[0,r]$ with

$$r = \min(3\pi/4, c\, a^{-p/(2\alpha+2)} H^{1/(2\alpha+2)}). \tag{15.15}$$

Since (15.7) is immediate if D_i is replaced by $[0,r]$, this completes this part.

For $i = 2$ we will estimate (15.8). For this case (15.8) equals

$$\int_0^2 \sup_{n\geq 1} \chi_{[0,2/n]}(s)\left[\int_{2/n}^{\pi/2} n^{\alpha-\theta+1/2} t^{-\theta-1} g(t)dt\right]^p s^{2\alpha+1} ds.$$

Since $n^{-\theta-1/2} \leq t^{\theta+1/2}$ in the inner integral, this is bounded by

$$\int_0^2 \sup_{n\geq 1} n^{(\alpha+1)p} \chi_{[0,2/n]}(s)\left[\int_{2/n}^{\pi/2} t^{-1/2} g(t)dt\right]^p s^{2\alpha+1} ds.$$

Since the exponent of n is positve, the supremum occurs for $n = [2/s]$. This gives the estimate

$$c \int_0^{\pi/2}\left[\int_s^{\pi/2} t^{-1/2} g(t)dt\right]^p s^{2\alpha+1-p(\alpha+1)} ds.$$

By (15.9) the exponent of s is greater than -1, and Hardy's inequality, Lemma 3.14 on page 196 of [12], gives the bound cH.

For $i = 3$ we have (15.8) equal to

$$\int_0^2 \sup_{n\geq 1} \chi_{[0,2/n]}(s)\left[\int_{2/n}^{\pi/2} n^{-1} t^{-\alpha-5/2} g(t)dt\right]^p s^{2\alpha+1} ds.$$

Replace n^{-1} in the inner integral by t and then replace n by $[2/s]$. This gives the estimate

$$\int_0^2 \left[\int_s^{\pi/2} t^{-\alpha-3/2} g(t)dt\right]^p s^{2\alpha+1} ds.$$

Since $2\alpha+1 > -1$, Hardy's inequality can be used. This gives the estimate cH for this part.

For $i = 4$ we start with the fact that D_4 is the set where

$$\sup_{n\geq 1} n^{\alpha-\theta+1/2} \chi_{[2/n,3\pi/4]}(s)\int_0^{2/n} t^{\alpha+1/2} g(t)dt \geq a\, s^{\alpha+\theta+3/2}.$$

Using (15.14) shows that D_4 is a subset of the set where

$$c\,H^{1/p} \sup_{n\geq 1} n^{\alpha-\theta+1/2-(2\alpha+2)/p'} \chi_{[2/n,3\pi/4]}(s) > a\,s^{\alpha+\theta+3/2}. \tag{15.16}$$

The exponent of n in (15.16) equals $(2\alpha+2)/p-(\alpha+\theta+3/2)$ and by (15.11) this is less than or equal to 0. The sup is, therefore, attained for the least n satisfying $n \geq 2/s$ and D_4 is a subset of the set where

$$c\,H^{1/p}\, s^{(2\alpha+2)/p'-\alpha+\theta-1/2}\, \chi_{[0,3\pi/4]}(s) > a\,s^{\alpha+\theta+3/2}.$$

Simplifying, we see that this last set is the set where

$$c\,H^{1/p}\, \chi_{[0,3\pi/4]}(s) > a\,s^{(2\alpha+2)/p}.$$

This is the set $[0,r]$ with the r of (15.15) and (15.7) follows immediately for $i = 4$.

Next, D_5 is the set where

$$\sup_n c\,n^{-1}s^{-\alpha-5/2}\chi_{[2/n,3\pi/4]}(s) \int_0^{s/2} t^{\alpha+1/2}g(t)dt > a\,s^{\alpha+1/2}.$$

Now use (15.13) and Hölder's inequality on the integral and replace n by $2/s$ to show that D_5 is a subset of the set where

$$c\,s^{-\alpha-3/2}H^{1/p}s^{(2\alpha+2)/p'}\chi_{[0,3\pi/4]}(s) > a\,s^{\alpha+1/2}.$$

From this D_5 is a subset of $[0,r]$ with the r of (15.15) and (15.7) follows.

The set D_6 is a subset of the set where

$$c \sup_{n\geq 1} n^{-\theta}\left[\int_{2/n}^{s/2} g(t)dt\right]\chi_{[4/n,3\pi/4]}(s) > a\, s^{\alpha+\theta+3/2}.$$

Use Hölder's inequality to show that D_6 is a subset of the set where

$$c \sup_{n\geq 1} \chi_{[2/n,3\pi/4]}(s)H^{1/p}\left[\int_{2/n}^{s/2} n^{-\theta p'} t^{(\alpha+1/2)(2-p)/(1-p)}dt\right]^{1/p'} > a\, s^{\alpha+\theta+3/2}.$$

If the exponent of t is less than -1, integrate to get

$$\sup_{n\geq 1} c\, \chi_{[2/n,3\pi/4]}(s)H^{1/p}n^{(2\alpha+2)/p-(\alpha+\theta+3/2)} \geq a\, s^{\alpha+\theta+3/2}.$$

If the exponent of t is not less than -1, replace $n^{-\theta p'}$ by $t^{\theta p'}$ and integrate to get

$$\sup_{n\geq 1} c\, \chi_{[2/n,3\pi/4]}(s)H^{1/p}\, s^{\alpha+\theta+3/2-(2\alpha+2)/p} \geq a\, s^{\alpha+\theta+3/2}.$$

In the first case the exponent of n is nonpositive by (15.11) and the supremum is attained at the least $n \geq 2/s$. In both cases then D_6 is a subset of the set where

$$c\, \chi_{[0,3\pi/4]}(s)H^{1/p}\, s^{-(2\alpha+2)/p} > a.$$

This is the set $[0,r]$ with the r of (15.15) and (15.7) follows.

To prove (15.7) for $i = 7$ define for k a positive integer $I_k = [2^{-k-1}\pi, 2^{-k}\pi]$, $J_k = [2^{-k-3}\pi, 2^{-k+2}\pi]$ and $g_k(t) = g(t)\chi_{I_k}(t)$. Then for a suitable c

$$D_7 \subset \left\{s: \sup_{n\geq 1} \sum_{k=1}^{\infty} \int_{s/2}^{2s} g_k(t)\ell_{n,7}(s,t)dt > c\,a\, s^{\alpha+1/2}\right\}.$$

Since at most three of these integrals are not zero for any given value of s

$$D_7 \subset \bigcup_{k=1}^{\infty} \left\{s: 3 \sup_{n\geq 1} \int_{s/2}^{2s} g_k(t)\ell_{n,7}(s,t)dt > c\,a\, s^{\alpha+1/2}\right\}.$$

By theorem 2, p. 62 of [11], we have

$$D_7 \subset \bigcup_{k=1}^{\infty} \left\{s: Mg_k(s)\chi_{J_k}(s) > c\,a\, s^{\alpha+1/2}\right\}, \tag{15.17}$$

where M denotes the usual Hardy–Littlewood maximal operator. If E_k denotes the k^{th} set in (15.17), then the left side of (15.7) for $i = 7$ has the bound

$$c \sum_{k=1}^{\infty} \int_{E_k} s^{2\alpha+1}ds \leq c \sum_{k=1}^{\infty} 2^{-k(2\alpha+1)}|E_k|. \tag{15.18}$$

By the usual weak type norm inequality for the Hardy–Littlewood maximal function, theorem 1, p. 5 of [11], we have

$$|E_k| \leq c[\,a\,2^{-k(\alpha+1/2)}]^{-p} \int g_k(s)^p ds.$$

Therefore, the right side of (15.18) has the bound

$$c \sum_{k=1}^{\infty} a^{-p}\, 2^{-k(\alpha+1/2)(2-p)} \int_{I_k} g(s)^p ds.$$

This is bounded by

$$c\,a^{-p} \int_0^{3\pi/4} s^{(\alpha+1/2)(2-p)} g(s)^p ds,$$

and (15.7) is proved for $i = 7$.

For $i = 8$ (15.8) is bounded by

$$\int_0^{\pi/4}\left[\int_{2s}^{\pi/2} g(t)t^{-\alpha-5/2}dt\right]^p s^{2\alpha+1} \sup_{n\geq 1} n^{-p}\chi_{[2/n,\pi/4]}(s)ds.$$

Since $\sup_{n\geq 1} n^{-p}\chi_{[2/n,3\pi/4]}(s) \leq s^p$, this has the bound

$$c\int_0^{\pi/4}\left[\int_s^{\pi/2} g(t)t^{-\alpha-5/2}dt\right]^p s^{2\alpha+1+p}ds.$$

Hardy's inequality completes this case.

For $i = 9$ (15.8) has the bound

$$\int_0^{\pi/4}\left[\int_s^{\pi/2} g(t)t^{-1-\theta}dt\right] s^{(\alpha+1/2)(2-p)} \sup_{n\geq 1} n^{-\theta p}\chi_{[2/n,\pi/4]}(s)ds.$$

Now we use the fact that $\sup_{n\geq 1} n^{-\theta p}\chi_{[2/n,\pi/4]}(s) \leq c\, s^{p\theta}$. The resulting exponent of s is $(\alpha+1/2)(2-p) + p\theta$. Since $\alpha > -1$ and $p \leq 2$, this exponent is greater than $(-1/2)(2-p) + \theta p = -1+(\theta+1/2)p > -1$. Hardy's inequality then completes this part.

For $i = 10$ (15.8) is bounded by

$$c\left[\int_{\frac{3\pi}{4}}^{\pi} (\pi-s)^{2\beta+1}ds\right] \left[\int_0^{\pi/2} g(t)t^{\alpha+1/2}dt\right]^p.$$

The first integral is finite since $\beta > -1$. Hölder's inequality shows that the second term is bounded by $H\left[\int_0^{\pi/2} t^{2\alpha+1}dt\right]^{p-1}$. This completes this part.

Finally, for $i = 11$ we will first show that

$$n^{-\theta} \int_0^{\pi/2} \min(1,nt)^{\alpha+1/2} g(t)dt \le cH^{1/p}. \tag{15.19}$$

To do this we start with the fact that the left side of (15.19) is the sum of

$$n^{\alpha-\theta+1/2} \int_0^{1/n} t^{\alpha+1/2} g(t)dt \tag{15.20}$$

and

$$n^{-\theta} \int_{1/n}^{\pi/2} g(t)dt. \tag{15.21}$$

For (15.20) we use (15.14) to get the bound

$$c\, n^{\alpha-\theta+1/2-(2\alpha+2)/p'} H^{1/p}. \tag{15.22}$$

Inequality (15.11) implies that the exponent of n is not positive and we have the bound $cH^{1/p}$ for (15.22).

For (15.21), Hölder's inequality gives the bound

$$n^{-\theta} H^{1/p} \left[\int_{1/n}^{\pi/2} t^{(\alpha+1/2)(2-p)/(1-p)} dt\right]^{1/p'}. \tag{15.23}$$

Inequality (15.11) implies that the exponent of t is bounded below by $-1-p'\theta$. This shows that (15.23) is bounded by $cH^{1/p}$ and completes the proof of (15.19).

Using (15.19), we have (15.8) with $i = 11$ bounded by

$$cH \int_{3\pi/4}^{\pi} \sup_{n\ge1} \min(1,n(\pi-s))^{p(\beta+1/2)} (\pi-s)^{(\beta+1/2)(2-p)} ds.$$

This is bounded by the sum of

$$cH \int_{3\pi/4}^{\pi} (\pi-s)^{(\beta+1/2)(2-p)} \sup_{n\geq 1} \chi_{[3\pi/4,\pi-1/n]}(s)ds \tag{15.24}$$

and

$$cH \int_{3\pi/4}^{\pi} (\pi-s)^{2\beta+1} \sup\ n^{p(\beta+1/2)} \chi_{[\pi-1/n,\pi]}(s)ds. \tag{15.25}$$

Now (15.24) equals

$$cH \int_{3\pi/4}^{\pi} (\pi-s)^{(\beta+1/2)(2-p)} ds. \tag{15.26}$$

If $\beta \geq -1/2$, then since $p < 2$ we have $(\beta+1/2)(2-p) \geq 0$. If $\beta < -1/2$, then $(\beta+1/2)(2-p) > 2\beta+1 > -1$. Therefore the integral in (15.26) is finite and (15.24) has the bound cH.

If $\beta \geq -1/2$, the sup in (15.25) is attained for $n = [\frac{1}{\pi-s}]$. This produces the bound (15.26). If $\beta < -1/2$, the sup in (15.25) is attained for $n = 1$. It follows that (15.25) has the bound cH. This completes the proof of lemma (15.1).

Finally, we comment on why the method used to prove theorem 1 of [7] cannot be used to prove lemma (15.1) if $\alpha \geq \beta$ and $\theta \leq \alpha + \frac{1}{2}$. In the proof of that theorem in [7], the result of lemma (15.1) must be obtained with $K_n^{(\alpha,\beta),\theta}(s,t)$ replaced by various error estimates, one of which is

$$K(s,t) = \begin{cases} \frac{1}{s} & 0 \leq t \leq s \leq \pi/2 \\ 0 & \text{elsewhere} \end{cases}.$$

If we take $g(t) = t^u \chi_{[0,\pi/2]}(t)$ with $-1-\theta < u < -1$, then $D(a) = [0,\pi]$ for all $a > 0$. However, this integrand on the right side of (15.3) is $t^{p(u-\alpha-\frac{1}{2})+2\alpha+1}$ and this exponent is greater than $p(-\theta-\alpha-3/2)+2\alpha+1 = -1$. Therefore, as $a \to \infty$ the left side of (15.3) is a fixed positive number and the right side approaches 0. This shows that lemma (15.1) fails for this error term and, consequently, the method of [7] cannot be used.

16. Lemmas for the upper critical value. Here we prove two basic results. The first, lemma (16.1), is equivalent to theorem (1.2) with the set a subset of [0,1]. The second, lemma (16.4) is equivalent to theorem (1.3) for functions with support in [0,1]. These lemmas will be used to prove theorems (1.2) and (1.3) in §17. At the end of this section we also show that the method used to prove theorem 1 of [7] will not prove lemma (16.1).

Lemma (16.1). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $0 < \theta < \gamma+1/2$, $p = (2\gamma+2)/(\gamma-\theta+1/2)$, $E \subset [0,\pi/2]$, $a > 0$ and $D(a)$ is the set where

$$\sup_n \int_E |K_n^{(\alpha,\beta),\theta}(s,t)|\, t^{\alpha+1/2} dt > a\, s^{\alpha+1/2}(\pi-s)^{\beta+1/2}, \tag{16.2}$$

then

$$\int_{D(a)} s^{2\alpha+1}(\pi-s)^{2\beta+1}\, ds \le c\, a^{-p} \int_E t^{2\alpha+1} dt, \tag{16.3}$$

where c is independent of a and E. This is also true if $0 < \theta = \gamma+1/2$ and $2 < p < \infty$.

Lemma (16.4). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $\theta > \gamma+1/2$, $\theta > 0$ and $g(t)$ is supported in $[0,\pi/2]$, then

$$\left\| \frac{\sup_n \int_0^{\pi/2} |K_n^{(\alpha,\beta),\theta}(s,t)\, g(t)|\, dt}{s^{\alpha+1/2}(\pi-s)^{\beta+1/2}} \right\|_\infty \le c \left\| \frac{g(t)}{t^{\alpha+1/2}} \right\|_\infty ,$$

where $\| \ \|_\infty$ denotes the essential supremum on $[0,\pi]$ and c is independent of g.

Basic to the proof of lemma (16.1) is the following.

Lemma (16.5). If $1 < p < \infty$, $a > -1$ and $E \subset [0,\infty)$, then

$$(16.6) \qquad \left[\int_E x^a dx\right]^p \le 2^p (a+1)^{1-p} \int_E x^{ap+p-1} dx.$$

This will be used as a substitute for Hölder's inequality in the proof of lemma (16.1). The obvious proof of lemma (16.5) using Hölder's inequality fails because that produces a coefficient $\left[\int_E (1/x) dx\right]^{p-1}$ on the right side.

To prove lemma (16.5) observe first that by the monotone convergence theorem it is sufficient to prove (16.6) for bounded E. Then since $a > -1$, the left side is finite and we can choose s such that

$$\int_0^s x^a \chi_E(x) dx = \frac{1}{2}\int_E x^a dx.$$

With this the left side of (16.6) equals

$$2^p \left[\int_0^s x^a \chi_E(x) dx\right]^{p-1} \left[\int_s^\infty x^a \chi_E(x) dx\right] \le 2^p \left[\int_0^s x^a dx\right]^{p-1} \left[\int_s^\infty x^a \chi_E(x) dx\right].$$

Performing the first integration gives the bound

$$2^p(a+1)^{1-p}\int_s^\infty s^{(a+1)(p-1)}x^a\chi_E(x)dx.$$

Since $(a+1)(p-1) > 0$, we can replace s by x and (16.6) follows immediately.

The proof of lemma (16.1) is similar to that for lemma (15.1). We let

$$H = \int_E t^{2\alpha+1}dt$$

and D_i the subset of $[0,\pi]$ where

$$\sup_{n\geq 1}\int_E \ell_{n,i}(s,t)t^{\alpha+1/2}dt > a\, s^{\alpha+1/2}(\pi-s)^{\beta+1/2}; \tag{16.7}$$

the functions $\ell_{n,i}$ are those used in §15. For each i we will prove (15.7). For some parts this will be done by showing that

$$\int_0^\pi \left[\sup_{n\geq 1}\int_E \ell_{n,i}(s,t)t^{\alpha+1/2}dt\right]^p\left[s^{\alpha+1/2}(\pi-s)^{\beta+1/2}\right]^{2-p}ds \tag{16.8}$$

is bounded by cH.

The following inequalities will be used. First since $\gamma+1/2-\theta < \gamma+1$, we have

$$2 < p < \infty. \tag{16.9}$$

From the definition of p, including the case $\theta = \gamma+1/2$

$$p(\gamma-\theta+1/2) \leq 2\gamma+2. \tag{16.10}$$

Since $\alpha \le \gamma$, $\beta \le \gamma$ and $p > 2$, (16.10) implies

$$p(\alpha-\theta+1/2) \le 2\alpha+2. \tag{16.11}$$

and

$$p(\beta-\theta+1/2) \le 2\beta+2. \tag{16.12}$$

For $i = 1$ the proof used for $i = 1$ in the proof of lemma (15.1) can be used if $g(t)$ is replaced by $t^{\alpha+1/2}\chi_E(t)$.

Next, D_2 is a subset of the set where

$$\sup_{n\ge1}\left|\int_{2/n}^{\pi/2} \frac{t^{\alpha-\theta-1/2}}{n^{\theta-\alpha-1/2}}\, \chi_E(t)dt\right| \chi_{[0,2/n]}(s) > c\,a.$$

Now multiply the numerator and denominator in the integral by $n^{(2\alpha+2)/p}$. By (16.11) the resulting exponent of n in the denominator is nonnegative and replacing that n with $1/t$ increases the left side. Therefore, D_2 is a subset of the set where

$$\sup_{n\ge1} n^{(2\alpha+2)/p}\left[\int_{2/n}^{\pi/2} t^{-1+(2\alpha+2)/p}\chi_E(t)dt\right]\chi_{[0,2/n]}(s) > c\,a.$$

Since the left side is 0 for $n > 2/s$, we have D_2 a subset of the set where

$$c\,\chi_{[0,2]}(s) \int_0^{\pi/2} t^{-1+(2\alpha+2)/p}\chi_E(t)dt > a\,s^{(2\alpha+2)/p}. \tag{16.13}$$

Since $-1+(2\alpha+2)/p > -1$, we can apply lemma (16.5) to see that D_2 is a subset of the set where

$$c\,\chi_{[0,2]}(s)\left[\int_E t^{2\alpha+1}dt\right]^{1/p} > a\,s^{(2\alpha+2)/p}.$$

Therefore D_2 is a subset of $[0,r]$ with

$$r = \min(3\pi/4, c[a^{-p}H]^{1/(2\alpha+2)})$$

and (15.7) follows immediately.

For $i = 3$ we have (16.8) equal to

$$\int_0^2 \sup_{n\geq 1} \chi_{[0,2/n]}(s)\left[\int_{2/n}^{\pi/2} n^{-1}t^{-2}\chi_E(t)dt\right]^p s^{2\alpha+1}ds.$$

Replacing n^{-1} by t in the inner integral gives the bound

$$\int_0^2 \sup_{n\geq 1} \chi_{[0,2/n]}(s)\left[\int_{2/n}^{\pi/2} t^{-1}\chi_E(t)dt\right]^p s^{2\alpha+1}ds.$$

which is bounded by

$$\int_0^2\left[\int_s^{\pi/2} t^{-1}\chi_E(t)dt\right]^p s^{2\alpha+1}ds.$$

Hardy's inequality completes this part.

For $i = 4$ (16.8) is bounded by

$$\int_0^{3\pi/4}\left[\sup_{n\geq 1}\int_0^{2/n}\frac{t^{2\alpha+1}\chi_E(t)dt}{n^{\theta-\alpha-1/2}}\right]^p \chi_{[2/n,3\pi/4]}(s)\ s^{(\alpha+1/2)(2-p)-p(\theta+1)}ds.$$

Now if $\theta-\alpha-1/2 \leq 0$, replace n by $2/t$; otherwise replace n by $1/s$. This gives the estimate

$$c\int_0^{3\pi/4}\left[\sup_{n\geq 1}\int_0^{2/n} t^{2\alpha+1+r}\chi_E(t)dt\right]^p \chi_{[2/n,3\pi/4]}(s)\ s^{2\alpha+1-p(2\alpha+2+r)}ds,$$

where $r = \min(\theta-\alpha-1/2,0)$. The supremum occurs for the least n such that

$2/n \le s$. Therefore, we have the estimate

$$c\int_0^{3\pi/4}\left[\int_0^s t^{2\alpha+1+r}\chi_E(t)dt\right]^p s^{2\alpha+1-p(2\alpha+2+r)}ds.$$

If $r = 0$, the exponent of s is -1 for $p = 1$. If $r \neq 0$, the exponent is $-2-2\theta$ for $p = 2$. Since $\alpha > 1$ and $\theta \geq 0$, we have $2\alpha+2+r > 0$ and the exponent of s is a decreasing function of p. Therefore, since $p > 2$, that exponent is less than -1 and Hardy's inequality completes this case.

For $i = 5$ (16.8) has the bound

$$\int_0^{3\pi/4} \sup_{n\geq 1}\left|\int_0^{s/2} \frac{t^{2\alpha+1}\chi_E(t)dt}{n}\right|^p \chi_{[2/n,3\pi/4]}(s)\ s^{(\alpha+1/2)(2-p)-p(\alpha+5/2)}ds.$$

Replacing n by $2/s$ gives the bound

$$\int_0^{3\pi/4}\left|\int_0^s t^{2\alpha+1}\chi_E(t)dt\right|^p s^{2\alpha+1-p(2\alpha+2)}ds.$$

Since $p > 2$ the exponent of s is less than -1 and Hardy's inequality completes this part.

For $i = 6$ the bound for (16.8) is

$$\int_0^{3\pi/4}\left[\sup_{n\geq 1}\int_{2/n}^{s/2} \frac{\chi_E(t)t^{\alpha+1/2}}{n^\theta s^{\theta+1}}dt\right]^p \chi_{[4/n,3\pi/4]}(s)\ s^{(\alpha+1/2)(2-p)}ds.$$

We can replace the lower limit of integration in the inner integral by 0 and then replace the other n with $4/s$ to get the bound

$$\int_0^{3\pi/4}\left[\int_0^s \chi_E(t)t^{\alpha+1/2}dt\right]^p s^{2\alpha+1-p(\alpha+3/2)}ds.$$

Since $p > 2$, the exponent of s is less than -1, and Hardy's inequality completes this part.

For $i = 7$ and $i = 8$ the reasoning in the proof of lemma (15.1) for these parts can be used with $g(t)$ taken to be $t^{\alpha+1/2}\chi_E(t)$.

For $i = 9$ we see that D_9 is a subset of the set where

$$\sup_{n\geq 1}\left[\int_{2/s}^{\pi/2} \frac{t^{\alpha+1/2}}{n^{\theta}\; t^{1+\theta}}\, \chi_E(t)dt\right]\chi_{[2/n,\pi,4]}(s) > c\, a\; s^{\alpha+1/2}.$$

We can replace n by $[2/s]$ to see that D_9 is a subset of the set where

$$\chi_{[0,\pi/4]}(s)\int_s^{\pi/2} t^{\alpha-\theta-1/2}\chi_E(t)dt > c\, a\; s^{\alpha-\theta+1/2}.$$

Because of (16.11) we can multiply the integrand by $(t/s)^{\theta-\alpha-1/2+(2\alpha+2)/p}$ to show that D_9 is a subset of the set where (16.13) holds. The estimation is then completed as in the case $i = 2$.

For $i = 10$ the reasoning in lemma (15.1) applies with $g(t)$ replaced by $t^{\alpha+1/2}\chi_E(t)$. For $i = 11$ we first prove that

$$\int_s^{\pi/2} \min(1,nt)^{\alpha+1/2} t^{\alpha+1/2}\chi_E(t)dt \leq cH^{1/p}. \tag{16.14}$$

To do this split the integral into integrals over $[0,1/n]$ and $[1/n,\pi/2]$. Hölder's inequality shows the first integral is bounded by

$$cH^{1/p} n^{\alpha+1/2-(2\alpha+2)/p'}.$$

From the fact that $1 < p' < 2$ and $\alpha > -1$ it follows that the exponent of n is negative and, therefore, this part is bounded by $cH^{1/p}$. Hölder's inequality applied to the second part gives

$$H^{1/p}\left[\int_{1/n}^{\pi/2} t^{(\alpha+1/2)(p-2)/(p-1)}dt\right]^{1/p'}.$$

Since $\alpha > -1$ and $2 < p < \infty$, the exponent of t is greater than $-1/2$, and this part also has the bound $cH^{1/p}$.

Using (16.14) we see that D_{11} is a subset of the set where

$$\sup_{n\geq 1} cH^{1/p}\ n^{-\theta}\min(1,n(\pi-s))^{\beta+1/2}\chi_{[3\pi/4,\pi]}(s) > a(\pi-s)^{\beta+1/2}.$$

This is a subset of the union of the sets where

$$\text{(16.15)} \qquad \sup_{n\geq 1} cH^{1/p}\ n^{-\theta}\chi_{[3\pi/4,\pi-1/n]}(s) > a(\pi-s)^{\beta+1/2}$$

and

$$\text{(16.16)} \qquad \sup_{n\geq 1} cH^{1/p}\ n^{\beta-\theta+1/2}\chi_{[\pi-1/n,\pi]}(s) > a.$$

For (16.15) the supremum occurs for the least n satisfying $n \geq 1/(\pi-s)$ and the set is a subset of the set where

$$\text{(16.17)} \qquad cH^{1/p}\ \chi_{[3\pi/4,\pi]}(s) > a(\pi-s)^{\beta-\theta+1/2}.$$

From (16.12) we see this is a subset of the set where

$$cH^{1/p}\ \chi_{[3\pi/4,\pi]}(s) > a(\pi-s)^{(2\beta+2)/p}.$$

This is the set $[r,\pi]$ with

$$r = \max(3\pi/4,\pi-(c\,a^{-p}H)^{1/(2\beta+2)})$$

and (15.7) follows.

For (16.16) if $\beta-\theta+1/2 \geq 0$ the supremum occurs for $n = [1/(\pi-s)]$. Then this set is a subset of the set satisfying (16.17) which was estimated before. If $\beta-\theta+1/2 < 0$, the supremum occurs when $n = 1$ and the set is a subset of the set where

$$cH^{1/p} \chi_{[\pi-1,\pi]}(s) > a.$$

If $a \geq cH^{1/p}$, this set is empty and (15.7) is trivial. If $a < cH^{1/p}$, this set is $[\pi-1,\pi]$, $a^{-p}H > c$ and (15.7) follows. This completes the proof of lemma (16.1).

To prove lemma (16.4) we will prove the equivalent inequality

$$\left\| \frac{\sup_n \int_0^{\pi/2} |K_n^{(\alpha,\beta),\theta}(s,t) t^{\alpha+1/2} g(t)| dt}{s^{\alpha+1/2}(\pi-s)^{\beta+1/2}} \right\|_\infty \leq c\|g(t)\|_\infty.$$

For this it is sufficient to prove that

$$\frac{\int_0^{\pi/2} |K_n^{(\alpha,\beta),\theta}(s,t)| t^{\alpha+1/2} dt}{s^{\alpha+1/2}(\pi-s)^{\beta+1/2}} \chi_{[0,\pi]}(s) \leq c$$

with c independent of n and s. Because of (15.4) and (15.5) it is sufficent to show that

$$\int_0^{\pi/2} \ell_{n,i}(s,t) t^{\alpha+1/2} dt \leq c\, s^{\alpha+1/2}(\pi-s)^{\beta+1/2} \tag{16.18}$$

for $0 < s < \pi$.

Inequality (16.18) is easily proved for $1 \leq i \leq 10$ by inserting the definition of $\ell_{n,i}$, performing the integration and using the fact that $\theta > \alpha+1/2$. For

$i = 11$ performing the integration gives the bound

$$c\,n^{-\theta}\min(1,n(\pi-s))^{\beta+1/2}\chi_{[3\pi/4,\pi]}(s). \tag{16.19}$$

If $0 < \pi-s \leq 1/n$, (16.19) is $c\,n^{\beta-\theta+1/2}(\pi-s)^{\beta+1/2}$ and (16.18) follows since $\theta > \beta+1/2$. If $1/n < \pi-s < \pi/4$, (16.19) is $c\,n^{-\theta} \leq c(\pi-s)^{\theta}$ since $\theta > 0$, and (16.18) follows from $\theta > \beta+1/2$.

Finally, we show that the methods of [7] cannot prove lemma (16.1) if $\alpha \geq \beta$. If this could be done, it would require that lemma (16.1) be true with $K_n^{(\alpha,\beta),\theta}(s,t)$ replaced by the error term

$$K(s,t) = \begin{cases} \frac{1}{t} & 0 \leq s \leq t \leq \pi/2 \\ 0 & \text{elsewhere} \end{cases}.$$

If we take $E = [0,1]$ and $a > 1/(\alpha+1/2)$, then $D(a)$ contains the set $[0, c\,a^{-1/(\alpha+1/2)}]$ for a suitable constant c and (16.3) would require that

$$a^{-(2\alpha+2)/(\alpha+1/2)} \leq c\,a^{-p}.$$

For this to be true for arbitrarily large a we must have $-(2\alpha+2)/(\alpha+1/2) \leq -p$ or $p \leq (2\alpha+2)/(\alpha+1/2)$. Since the p of lemma (16.1) does not satisfy this inequality, lemma (16.1) fails for this error term. Consequently, the method of [7] cannot be used.

17. Proofs of theorems (1.1)–(1.3). To prove theorem (1.1) we make the change of variables $x = \cos s$ and $y = \cos t$ and let $g(t) = f(\cos t)(\sin t/2)^{\alpha+1/2}(\cos t/2)^{\beta+1/2}$. Then by (2.29) the set $E(a)$ is the set of all values of $\cos s$ for s in the set $D_1(a)$ where

$$\sup_{n\geq 0}\left|\int_0^\pi K_n^{(\alpha,\beta),\theta}(s,t)g(t)dt\right| > a(\sin s/2)^{\alpha+1/2}(\cos s/2)^{\beta+1/2}. \tag{17.1}$$

Furthermore, the conclusion of theorem (1.1) is equivalent to

$$\begin{aligned}\int_{D_1(a)}&(\sin s/2)^{2\alpha+1}(\cos s/2)^{2\beta+1}ds\\ &\leq c\,a^{-p}\int_0^\pi |g(t)|^p\left[(\sin s/2)^{\alpha+1/2}(\cos s/2)^{\beta+1/2}\right]^{2-p}ds.\end{aligned} \tag{17.2}$$

For suitable c, $D_1(a)$ is a subset of the set $D(a)$ where

$$\sup_{n\geq 0}\int_0^\pi |K_n^{(\alpha,\beta),\theta}(s,t)g(t)|\,dt > c\,a\,s^{\alpha+1/2}(\pi-s)^{\beta+1/2}, \tag{17.3}$$

and it is sufficient to prove that

$$\int_{D(a)} s^{2\alpha+1}(\pi-s)^{2\beta+1}ds \leq c\,a^{-p}\int_0^\pi |g(s)|^p[s^{\alpha+1/2}(\pi-s)^{\beta+1/2}]^{2-p}ds. \tag{17.4}$$

For g with support in $[0,\pi/2]$, (17.4) was proved in lemma (15.1). For g with support in $[\pi/2,\pi]$ the change of variables $s = \pi-u$, $t = \pi-v$ and the fact that

$$K_n^{(\alpha,\beta),\theta}(\pi-u,\ \pi-v) = K_n^{(\beta,\alpha),\theta}(u,v) \tag{17.5}$$

show that (17.4) is equivalent to

$$(17.6) \quad \int_{D_2(a)} u^{2\beta+1}(\pi-u)^{2\alpha+1}du \leq c\, a^{-p} \int_0^{\pi} |g(\pi-u)|^p[u^{\beta+1/2}(\pi-u)^{\alpha+1/2}]^{2-p}du.$$

where $D_2(a)$ is the set where

$$(17.7) \qquad \sup_{n\geq 0} \int_0^{\pi} |K_n^{(\beta,\alpha),\theta}(u,v)g(\pi-v)|\,dv > a\, u^{\beta+1/2}(\pi-u)^{\alpha+1/2}.$$

Since $g(\pi-t)$ has support in $[0,\pi/2]$, inequality (17.6) follows from lemma (15.1). That (17.4) holds for general g follows from these two cases.

The proof of theorem (1.2) is similar. Let G be the subset of $[0,\pi]$ such that $\cos t \in H$ and let $g(t) = \chi_G(t)(\sin t/2)^{\alpha+1/2}(\cos t/2)^{\beta+1/2}$. Then $E(a)$ is the set of values of $\cos s$ for s in the set $D_1(a)$ where (17.1) holds and the conclusion of theorem (1.2) is equivalent to (17.2). For suitable c, $D_1(a)$ is a subset of the set $D(a)$ where (17.3) holds and it is sufficient to prove (17.4). If $E \subset [0,1]$, then $G \subset [0,\pi/2]$ and (17.4) follows from lemma (16.1). If $E \subset [-1,0]$ we again get (17.4) by changing variables and using lemma (16.1). As before, the general case follows from these two cases.

For theorem (1.3) we let $g(t) = f(\cos t)(\sin t/2)^{\alpha+1/2}(\cos t/2)^{\beta+1/2}$ and change variables in the conclusion to get

$$\left\| \frac{\sup_n \int_0^{\pi} K_n^{(\alpha,\beta),\theta}(s,t)g(t)\,dt}{(\sin s/2)^{\alpha+1/2}(\cos s/2)^{\beta+1/2}} \right\|_{\infty} \leq c \left\| \frac{g(s)}{(\sin s/2)^{\alpha+1/2}(\cos s/2)^{\beta+1/2}} \right\|_{\infty},$$

where $\|\ \|_{\infty}$ is taken over $[0,\pi]$. For f with support in $[0,1]$ this follows from lemma (16.4). As before for f with support in $[-1,0]$ a change of variables proves the result and the general case follows from the two special cases.

18. Norm estimates for p not between the critical values. From theorems (1.1)–(1.3) it follows that for p between the critical values $\|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p \le c\|f\|_p$ with c independent of n. For other values of p this is not true; we obtain upper bounds as a function of n in lemma (18.11). This lemma is the upper bound part of theorem (22.2). It is also needed to obtain the lower bound in §20. Throughout this section and §§20–22 we will use the notation

$$(18.1) \qquad p_1(\theta) = \frac{2\alpha+2}{\alpha+\theta+3/2},$$

$$(18.2) \qquad p_2(\theta) = \frac{2\alpha+2}{\alpha-\theta+1/2}$$

and

$$(18.3) \quad G(n,p,\theta) = \begin{cases} (n+1)^{(2\alpha+2)/p-(\alpha+\theta+3/2)} & 1 \le p < p_1(\theta),\ \theta \le \alpha+1/2 \\ [\log(n+1)]^{1/p} & p = p_1(\theta),\ \theta \le \alpha+1/2 \\ 1 & p_1(\theta) < p < p_2(\theta),\ 0 \le \alpha+1/2, \\ [\log(n+1)]^{1/p'} & p = p_2(\theta),\ \theta \le \alpha+1/2 \\ (n+1)^{(\alpha-\theta+1/2)-(2\alpha+2)/p} & p_2(\theta) < p \le \infty,\ \theta \le \alpha+1/2 \\ 1 & 1 \le p \le \infty,\ \theta > \alpha+1/2 \end{cases}$$

Lemma (18.4). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta)$, $\theta > 0$ and $1 \le p \le 2$, then

$$(18.5) \qquad \int_0^\pi \left[\int_0^{\pi/2} |K_n^{(\alpha,\beta),\theta}(s,t)g(t)|\,dt\right]^p [s^{\alpha+1/2}(\pi-s)^{\beta+1/2}]^{2-p}ds$$

is bounded by

$$(18.6) \qquad c\,G(n,p,\theta)^p \int_0^{\pi/2} |g(t)|^p\, t^{(\alpha+1/2)(2-p)}dt,$$

with c independent of n and g.

To prove this let $\ell_{n,i}(s,t)$ denote the functions defined in the proof of lemma (15.1), and as in the proof of lemma (15.1), let H denote the integral in (18.6). We may assume that g(t) is nonnegative and that it is 0 outside $[0,\pi/2]$. Because of (15.4) and (15.5) it is sufficient to show that

$$(18.7) \qquad \int_0^{\pi}\left[\int_0^{\pi/2}\ell_{n,i}(s,t)g(t)dt\right]^p\left[s^{\alpha+1/2}(\pi-s)^{\beta+1/2}\right]^{2-p}ds$$

is bounded by $cG(n,p.\theta)^pH$ for $1 \leq i \leq 11$ and $n \geq 1$.

Now (18.7) is bounded by (15.8). For $i = 3$, 8, 9 and 10, (15.8) was shown in the proof of lemma (15.1) to be bounded by cH for $p = p_1(\theta)$, and these estimates are valid for $1 \leq p \leq 2$. It is sufficient, therefore, to consider the cases $i = 1$, 2, 4, 5, 6, 7 and 11.

For $i = 1$ (18.7) is bounded by

$$c\, n^{(2\alpha+2)p}\int_0^{2/n}\left[\int_0^{\pi/2}t^{\alpha+1/2}g(t)dt\right]^p s^{2\alpha+1}ds.$$

Now use (15.14) to estimate the inner integral and the estimate cH is immediate for this part.

For $i = 2$ (18.7) is bounded by

$$c\, n^{p(\alpha-\theta+1/2)}\int_0^{2/n}\left[\int_{2/n}^{\pi/2}t^{-\theta-1}g(t)dt\right]^p s^{2\alpha+1}ds.$$

Now evaluate the outer integral and use Hölder's inequality on the inner integral to get the bound

$$c\, n^{p(\alpha-\theta+1/2)-2\alpha-2}\; H\left[\int_{2/n}^{\pi/2}t^{\alpha-\theta-1/2-(\alpha+\theta+3/2)/(p-1)}dt\right]^{p-1}.$$

The exponent of t is less than -1; evaluating the integral completes this part.

For $i = 4$ we have

$$c\,n^{p(\alpha-\theta+1/2)}\int_{2/n}^{3\pi/4}\left[\int_0^s t^{\alpha+1/2}g(t)dt\right]^p s^{2\alpha+1-p(\alpha+\theta+3/2)}ds.$$

The inner integral is estimated using (15.14) and the outer integral is computed directly. The result is $cHG(n,p,\theta)^p$.

For $i = 5$ (18.7) is bounded by the sum of

$$c\,n^{-p}\int_{2/n}^{3\pi/4}\left[\int_0^{2/n} t^{\alpha+1/2}g(t)dt\right]^p s^{2\alpha+1-p(2\alpha+3)}ds \tag{18.8}$$

and

$$c\,n^{-p}\int_{2/n}^{3\pi/4}\left[\int_{2/n}^{s} t^{\alpha+1/2}g(t)dt\right]^p s^{2\alpha+1-p(2\alpha+3)}ds\ . \tag{18.9}$$

For (18.8) use (15.14) on the inner integral and evaluate the outer integral to get an estimate of cH. For (18.9) since $p \geq 1$, the exponent of s is less than or equal to -2. Hardy's inequality then gives a bound of

$$c\int_{2/n}^{3\pi/4} n^{-p}g(t)^p t^{p(\alpha+1/2)+p+2\alpha+1-p(2\alpha+3)}ds.$$

Replacing n^{-p} by t^p completes this part.

For $i = 6$ we get

$$c\,n^{-\theta p}\int_{4/n}^{3\pi/4}\left[\int_{2/n}^{s/2} g(t)dt\right]^p s^{2\alpha+1-p(\alpha+\theta+3/2)}ds. \tag{18.10}$$

If $p > p_1(\theta)$, the exponent of s is less than -1. Hardy's inequality then gives the bound

$$c \int_{2/n}^{3\pi/4} g(t)^p \, n^{-\theta p} \, t^{2\alpha+1-p(\alpha+\theta+1/2)} dt.$$

Replacing $n^{-\theta p}$ by $t^{\theta p}$ completes this case.

If $1 \leq p \leq p_1(\theta)$ use Hölder's inequality to show that (18.10) is bounded by

$$c\, n^{-\theta p} \int_{4/n}^{3\pi/4} H \left[\int_{2/n}^{s/2} t^{(\alpha+1/2)(2-p)/(1-p)} dt\right]^{p-1} s^{2\alpha+1-p(\alpha+\theta+3/2)} ds.$$

The exponent of t is less than -1. The exponent of s is -1 if $p = p_1(\theta)$ and greater than -1 if $1 \leq p < p_1(\theta)$. Computing the integrals completes this case.

To estimate (18.7) with $i = 7$ let a be a number satisfying $1/p < a < 1+1/p$ and $a < \theta+1/p$. Applying Hölder's inequality to the inner integral gives a bound of

$$c \int_{2/n}^{\pi} \left[\int_{s/2}^{2s} \frac{g(t)^p}{(\frac{1}{n} + |s-t|)^{ap}} dt\right] \times$$

$$\times \left[\int_{s/2}^{2s} [\ell_{n,7}(s,t)]^{p'} [\tfrac{1}{n} + |s-t|]^{ap'} dt\right]^{p-1} s^{(\alpha+1/2)(2-p)} ds.$$

Substituting the value of $\ell_{n,7}$ we get a bound of

$$c\, n^{1-ap} \int_{2/n}^{\pi} \left[\int_{s/2}^{2s} \frac{g(t)^p}{(\frac{1}{n} + |s-t|)^{ap}} dt\right] s^{(\alpha+1/2)(2-p)} ds.$$

Now interchange the order of the integration and compute the inner integral to complete this part.

For $i = 11$ the procedure is similar to that used for the case $i = 1$ in the proof of lemma (15.1). In place of (15.19) we must show that

$$n^{-\theta}\int_0^{\pi/2} \min(1,nt)^{\alpha+1/2} g(t)dt \le c\, G(n,p,\theta)H^{1/p}.$$

The proof is like that of (15.19). We have (15.22) bounded by $c\,G(n,p,\theta)H^{1/p}$ for $1 \le p \le 2$. If $p \ge \frac{2\alpha+2}{\alpha+3/2}$, the exponent of t in (15.23) is bounded below by -1 and (15.23) has the bound $c\,H^{1/p}$. If $p < \frac{2\alpha+2}{\alpha+3/2}$, the integral in (15.23) can be evaluated to get the bound $c\,G(n,p,\theta)H^{1/p}$. The rest of the proof is the same except that each estimate must be multiplied by $G(n,p,\theta)$.

Corollary (18.11). If $\alpha \ge \beta > -1$ and $\theta > 0$, then $\|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p \le c\, G(n,p,\theta)\|f\|_p$ with c independent of f and n.

For $1 \le p \le 2$ this is done as in the proof of theorem (1.1) by splitting the integral defining $\sigma_n^{(\alpha,\beta),\theta}(f,x)$ at 0, changing variables, using the fact that $K_n^{(\alpha,\beta),\theta}(\pi-u,\pi-v) = K_n^{(\beta,\alpha),\theta}(u,v)$ and applying lemma (18.4). For $p > 2$ it follows from the case $1 \le p < 2$. by duality.

19. A polynomial norm inequality. The main result of this section, lemma (19.4), is a modification of lemma 4b of [2]. Lemma (19.4) has a weak type p norm on the right in place of an ordinary p norm. The proof follows that given in [2] but a different interpolation theorem is needed to produce the weak type norm. This is lemma (19.1); its proof is like that for a simple case of the Marcinkiewicz interpolation theorem.

Lemma (19.1). If $\alpha > -1$, $\beta > -1$, T is a linear operator on L^1 of $[-1,1]$ with weight $(1-x)^{\alpha}(1+x)^{\beta}$, $\|Tf\|_{\infty} \le A\|f\|_1$, $\|Tf\|_{\infty} \le B\|f\|_{\infty}$ and $1 < p < \infty$, then

$$\|Tf\|_{\infty} \le c\, A^{1/p}B^{1/p'}\|f\|_{p,\infty}$$

with c independent of A, B, f and T.

To prove this let a be positive, define

$$f_a(x) = \begin{cases} f(x) & |f(x)| < a \\ 0 & |f(x)| \ge a \end{cases}$$

and let $f^a(x) = f(x) - f_a(x)$. Then

$$\|Tf\|_{\infty} \le \|T(f_a)\|_{\infty} + \|T(f^a)\|_{\infty}.$$

By the hypothesis the right side is bounded by

$$Ba + A\int_{|f(x)|>a} |f(x)|(1-x)^{\alpha}(1+x)^{\beta}dx$$

$$\le Ba + A\sum_{n=0}^{\infty} 2^{n+1}a\int_{|f(x)|>2^n a}(1-x)^{\alpha}(1+x)^{\beta}dx.$$

The right side is bounded by

$$Ba + A\sum_{n=0}^{\infty} 2^{n+1}a\,\frac{\|f(x)\|_{p,\infty}^p}{(2^n a)^p} \le Ba + Aca^{1-p}\|f(x)\|_{p,\infty}^p .$$

Taking $a = A^{1/p}B^{-1/p}\|f(x)\|_{p,\infty}$ completes the proof of this lemma.

Lemma (19.2). If $\alpha > -1$, $\beta > -1$, then there are positive integers r and N and linear operators T_n for L^1 on $[-1,1]$ with weight $(1-x)^{\alpha}(1+x)^{\beta}$ such that for $n \geq N$ T_nf is a polynomial of degree rn, $T_nf = f$ if f is a polynomial of degree n, $\|T_nf\|_\infty \leq c\|f\|_\infty$ and $\|T_nf\|_1 \leq c\|f\|_1$ with c independent of n and f.

This is a combination of theorem 1, p. 467 of [10], plus the fact that for $\theta > \max(\alpha,\beta) + 1/2$ we have $\|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p \leq c\|f(x)\|_p$ for $p = 1$ and $p = \infty$. These norm inequalities are included in lemma (18.11).

Lemma (19.3). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta) \geq -1/2$ and $f(x)$ is an n^{th} degree polynomial, then

$$\|f(x)\|_\infty \leq c(n+1)^{2\gamma+2}\|f\|_1$$

with c independent of n and f.

To prove this we start with the fact that

$$f(x) = \sum_{k=0}^{n} \frac{\int_{-1}^{1} f(y)P_k^{(\alpha,\beta)}(y)(1-y)^{\alpha}(1+y)^{\beta}dy}{\int_{-1}^{1}[P_k^{(\alpha,\beta)}(y)]^2(1-y)^{\alpha}(1+y)^{\beta}dy} P_k^{(\alpha,\beta)}(x).$$

By (4.3.3) of [13] and (2.5) we have

$$|f(x)| \leq \sum_{k=0}^{n} \frac{c(k+1)^{\gamma}\|f\|_1}{(k+1)^{-1}} (k+1)^{\gamma},$$

and the conclusion is immediate.

Lemma (19.4). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta) \geq -1/2$ $1 < p < \infty$ and $h(x)$ is a polynomial of degree n, then

$$\|h(x)\|_\infty \leq c(n+1)^{(2\gamma+2)/p}\|h\|_{p,\infty}$$

with c independent of h and n.

Let r, N and T_n be as in lemma (19.2); we may assume $n \geq N$. By lemmas (19.3) and (19.2)

$$\|T_nf(x)\|_\infty \leq c(nr+1)^{2\gamma+2}\|T_nf(x)\|_1 \leq c(nr+1)^{2\gamma+2}\|f(x)\|_1$$

and by lemma (19.2)

$$\|T_nf(x)\|_\infty \leq c\|f(x)\|_\infty \ .$$

Lemma (19.1) then shows that

$$\|T_nf(x)\|_\infty \leq c(n+1)^{(2\gamma+2)/p}\|f(x)\|_{p,\infty}$$

for any f in L^1. Since $T_nh(x) = h(x)$ for h a polynomial of degree n, we have the conclusion of the lemma.

20. A lower bound for a norm of the kernel. The main result here is lemma (20.2) which gives a lower bound for $\|L_n^{(\alpha,\beta),\theta}(1,x)\|_p$. This along with lemma (19.4) will be used in §§21–22 to obtain lower bounds for the L^p norm of the operator $\sigma_n^{(\alpha,\beta),\theta}$.

Lemma (20.1). If $\alpha \geq \beta > -1$, $\alpha \geq -1/2$, $\theta > 0$ and $1 \leq p \leq \infty$, then

$$\|L_n^{(\alpha,\beta),\theta}(1,x)\|_p \leq c\, G(n,p',\theta)(n+1)^{(2\alpha+2)/p'},$$

with c independent of n.

By the converse of Hölder's inequality

$$\|L_n^{(\alpha,\beta),\theta}(1,x)\|_p = \sup_{\|f\|_{p'}=1} \left| \int_{-1}^{1} L_n^{(\alpha,\beta),\theta}(1,x) f(x)(1-x)^{\alpha}(1+x)^{\beta} dx \right| .$$

The right side equals

$$\sup_{\|f\|_{p'}=1} \left| \sigma_n^{(\alpha,\beta),\theta}(f,1) \right| \leq \sup_{\|f\|_{p'}=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_{\infty} .$$

By lemma (19.4) the right side of this has the bound

$$\sup_{\|f\|_{p'}=1} c(n+1)^{(2\alpha+2)/p'} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_{p'} ,$$

for $1 < p < \infty$. If $p = \infty$, this follows from lemma (19.3); for $p = 1$ it is trivial. Corollary (18.11) then completes the proof.

Lemma (20.2). If $-1 < \beta \leq \alpha$, $0 < \theta \leq \alpha+1/2$ and $1 \leq p \leq p_1(\theta)$, then

$$(n+1)^{(2\alpha+2)/p'} G(n,p',\theta) \leq c\|L_n^{(\alpha,\beta),\theta}(1,x)\|_p \tag{20.3}$$

with c independent of n.

The proof is essentially that on pages 174–5 of [2] for the case $\alpha = \beta$ and $p = p_1(\theta)$. We use the fact from (9.41.13), p. 261 of [13], that $L_n^{(\alpha,\beta),\theta}(1,x)$ equals the sum of

$$Q = \frac{n!\,\Gamma(\theta+1)\Gamma(n+\alpha+\beta+\theta+2)\,\Gamma(2n+\alpha+\beta+\theta+3)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(n+\theta+1)\,\Gamma(n+\beta+1)\,\Gamma(2n+\alpha+\beta+2\theta+3)}\; P_n^{(\alpha+\theta+1,\beta)}(x)$$

and

$$R = \sum_{j=1}^{\infty} A(j) L_n^{(\alpha,\beta),\theta+j}(1,x),$$

where

$$A(j) = (-1)^{j+1}\binom{\theta}{j} \prod_{i=1}^{j} \frac{n+\theta+i}{2n+1+\alpha+\beta+2+\theta+i}\,.$$

By use of (8.21.17), p. 197 of [13], and Stirling's formula we have

$$\|Q\|_p \geq c \begin{cases} (n+1)^{\alpha-\theta+1/2} & 1 \leq p < p_1(\theta) \\ (n+1)^{\alpha-\theta+1/2}(\log n)^{1/p} & p = p_1(\theta) \end{cases}.$$

Using the definition (18.3) of G, we can write this as

$$\|Q\|_p \geq c\, G(n,p'\theta)(n+1)^{(2\alpha+2)/p'}, \quad 1 \leq p \leq p_1(\theta). \tag{20.4}$$

To estimate $\|R\|$ we use the fact that

$$L_n^{(\alpha,\beta),\theta+j}(1,x) = \sum_{k=0}^{n} a_k\, L_k^{(\alpha,\beta),\theta+1}(1,x),$$

where

$$a_k = \frac{\begin{bmatrix} k+\theta+1 \\ k \end{bmatrix} \begin{bmatrix} n-k+j-2 \\ n-k \end{bmatrix}}{\begin{bmatrix} n+\theta+j \\ n \end{bmatrix}}\,.$$

Since $a_k \geq 0$ for $j \geq 1$ and $\sum_{k=1}^{n} a_k = 1$, we have by Minkowski's inequality

$$\|R\|_p \leq \sum_{j=1}^{\infty} |A(j)| \sup_{k \leq n} \|L_k^{(\alpha,\beta),\theta+1}(1,x)\|_p .$$

Now since each factor in the product in the definition of $A(j)$ has absolute value less than 1, we have $|A_j| \leq |\binom{\theta}{j}|$. Since $\sum_{j=1}^{\infty} |\binom{\theta}{j}|$ converges for $\theta > 0$, we can use lemma (20.1) to get

$$(20.5) \qquad \|R\|_p \leq c \sup_{k \leq n} \|L_k^{(\alpha,\beta),\theta+1}(1,x)\|_p \leq c\, G(n,p',\theta+1)(n+1)^{(2\alpha+2)/p'}.$$

From the definition of G $\lim_{n\to\infty} G(n,p',\theta+1)/G(n,p',\theta) = 0$. Therefore, (20.4) and (20.5) imply (20.3) for sufficiently large n. Since $\|L_n^{(\alpha,\beta),\theta+1}(1,x)\|_p > 0$ for all n, adjusting c in (20.3) will make (20.3) true for all n.

21. Some limitations of the basic results. Here we show that $Tf(x) = \sup_n |\sigma_n^{(\alpha,\beta),\theta}(f,x)|$ is not a weak type operator at the upper critical index p by showing that $\sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_{p,\infty}$ is an unbounded function of n. As a corollary we also get the fact that $\sup_{E\subset[-1,1]} \|\sigma_n^{(\alpha,\beta),\theta}(\chi_E,x)\|_p/\|\chi_E\|_p$ is an unbounded function of n if p is the lower critical index. It follows from this that T is not a restricted strong type operator for this value of p. The question of whether T is restricted strong type at the upper critical index is not resolved here. At the end of this section we do show, however, that this cannot be decided using our upper bounds for the kernel. The basic result for this section is the following theorem.

Theorem (21.1). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta) > -1/2$, $0 < \theta < \gamma+1/2$ and $p = (2\gamma+2)/(\gamma-\theta+1/2)$, then

$$\sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_{p,\infty} \geq c(\log n)^{1/p'}$$

with $c > 0$ and independent of n. If $\theta = \gamma+1/2$, $\sup_{\|f\|_\infty=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_\infty$ $\geq c \log n$.

To prove this we use lemma (20.2) to get

$$(\log n)^{1/p'} \leq c(n+1)^{-(2\gamma+2)/p}\|L_n^{(\alpha,\beta),\theta}(1,x)\|_{p'} .$$

By the converse of Hölder's inequality the right side equals

$$c(n+1)^{-(2\gamma+2)/p} \sup_{\|f\|_p=1}\left|\int_{-1}^{1} L_n^{(\alpha,\beta),\theta}(1,x)f(x)(1-x)^{\alpha}(1+x)^{\beta}dx\right|$$

which equals

$$c(n+1)^{-(2\gamma+2)/p} \sup_{\|f\|_p=1} |\sigma_n^{(\alpha,\beta),\theta}(f,1)|.$$

This completes the proof if $\theta = \gamma+1/2$. If $\theta < \gamma+1/2$, then since $\sigma_n^{(\alpha,\beta),\theta}(f,x)$ is a polynomial of degree n, lemma (19.4) shows this is bounded by

$$c \sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_{p,\infty} .$$

This completes the proof of theorem (21.1).

Theorem (21.2). If $\alpha > -1$, $\beta > -1$, $\gamma = \max(\alpha,\beta) > -1/2$, $0 < \theta < \gamma+1/2$ and $p = (2\gamma+2)/(\gamma+\theta+3/2)$, then

(21.3) $$\sup_{E\subset[-1,1]} \frac{\|\sigma_n^{(\alpha,\beta),\theta}(\chi_E,x)\|_p}{\|\chi_E\|_p} \geq (\log n)^{1/p}$$

with $c > 0$ and independent of n. If $\theta = \gamma+1/2$, then

$\sup_{\|f\|_1=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_1 \geq c \log n.$

For $\theta < \gamma+1/2$ this follows from theorem (21.1) by the following duality argument. The left side of (21.3) equals

$$\sup_{E\subset[-1,1]} \sup_{\|f\|_{p'}=1} \frac{|\int_{-1}^1 f(x)\sigma_n^{(\alpha,\beta),\theta}(\chi_E,x)(1-x)^{\alpha}(1+x)^{\beta}dx|}{\|\chi_E\|_p} .$$

This is

(21.4) $$\sup_{\|f\|_{p'}=1} \sup_{E\subset[-1,1]} \frac{|\int_{-1}^1 \chi_E(x)\sigma_n^{(\alpha,\beta),\theta}(f,x)(1-x)^{\alpha}(1+x)^{\beta}dx|}{\|\chi_E\|_p} .$$

By theorem (21.1) we can, given n, choose g with $\|g\|_{p'} = 1$ and

(21.5) $$\|\sigma_n^{(\alpha,\beta),\theta}(g,x)\|_{p',\infty} \geq c(\log n)^{1/p}$$

with c independent of n. With the notation

$$\mu(E) = \int_E (1-x)^{\alpha}(1+x)^{\beta}dx$$

we see from (21.5) that there is an $a > 0$ such that if A is the set where $|\sigma_n^{(\alpha,\beta),\theta}(g,x)| > a$, then $a^{p'}\mu(A) \geq c(\log n)^{p'/p}$. Let A_1 be the subset of A where $\sigma_n^{(\alpha,\beta),\theta}(g,x) > 0$ and let $A_2 = A \cap A_1^c$. If $\mu(A_1) \geq \mu(A)/2$, let $D = A_1$; otherwise let $D = A_2$. Then $a^{p'}\mu(D) \geq c(\log n)^{p'/p}$ and

(21.6) $$a \geq c\,\mu(D)^{-1/p'}(\log n)^{1/p}.$$

Replacing f by g and E by D, we see that (21.4) is bounded below by

$$\frac{\int_{-1}^{1}\chi_D(x)a(1-x)^{\alpha}(1+x)^{\beta}dx}{[\mu(D)]^{1/p}}.$$

Using (21.6) to replace a then completes the proof of theorem (21.2) for $\theta < \gamma+1/2$. For $\theta = \gamma+1/2$ a standard duality argument proves the result from theorem (21.1).

Now we will show, as mentioned at the beginning of this section, that the upper bounds for the kernel obtained in this paper cannot be used to show that $Tf(x) = \sup_n |\sigma_n^{(\alpha,\beta),\theta}(f,x)|$ is strong restricted type at the upper critical index. To do this we change variables to show that T being of restricted strong type is equivalent to the statement that for any subset E of $[0,\pi]$

$$\int_0^{\pi} \sup_n \left|\int_E K_n^{(\alpha,\beta),\theta}(s,t)(\sin t/2)^{\alpha+1/2}(\cos t/2)^{\beta+1/2}dt\right|^p \times$$

$$\times [(\sin s/2)^{\alpha+1/2}(\cos s/2)^{\beta+1/2}]^{2-p}ds$$

is bounded by

$$c\int_E (\sin s/2)^{2\alpha+1}(\cos s/2)^{2\beta+1}ds.$$

If this could be proved from our upper bounds for $\alpha \geq \beta$, we would have

$$\int_0^{\pi/4}\left[\sup_n \chi_{[2/n,\pi/4]}(s)\int_{2s}^{\pi/2}\ell_{n,9}(s,t)t^{\alpha+1/2}\chi_E(t)dt\right]^p s^{(\alpha+1/2)(2-p)}ds \leq c\int_E s^{2\alpha+1}ds$$

for any subset E of $[0,\pi/2]$ where $\ell_{n,9}$ is the function used in the proof of lemma (15.1). Taking $E = [\pi/4,\pi/2]$, substituting the value of $\ell_{n,9}$ and reducing the interval of integration for the outer integral we would have

$$\int_0^{\pi/8}\left[\sup_n \chi_{[2/n,\pi/4]}(s)n^{-\theta}\int_{\pi/4}^{\pi/2} t^{\alpha-\theta-1/2}dt\right]^p s^{(\alpha+1/2)(2-p)}ds < \infty.$$

The supremum is attained at the least n satisfying $n \geq 2/s$. Using this fact and performing the inner integration, we see that the integral on the left is bounded below by $c\int_0^{\pi/8} ds/s$. This contradiction completes the demonstration that our upper bounds are not sufficient for this problem. Since our upper bound can easily be shown to be as small as possible for certain simple cases such as $\alpha = \beta = 1/2$, another approach is needed. Presumably, an asymptotic expansion for the kernel could be used for this purpose.

22. Growth of Cesaro means. We obtain here the exact growth rate of the L^p norm of the Cesaro mean operator for p outside the critical region. Görlich and Markett obtained the following result in [8].

Theorem (22.1). If $\alpha \geq \beta \geq -1/2$ and $0 < \theta \leq \alpha+1/2$, then

$$\|\sigma_n^{(\alpha,\beta),\theta}\|_p \leq \begin{cases} B(n)\ n^{(2\alpha+2)/p-(2\alpha+3)/2-\theta} & ,\ p \in [1,p_1] \\ B(n)\ n^{(2\alpha+1)/2-(2\alpha+2)/p-\theta} & ,\ p \in [p_2,\infty] \end{cases},$$

where $\|\sigma_n^{(\alpha,\beta),\theta}\|_p$ denotes the operator norm from L^p_w to L^p_w, $p_1 = (4\alpha+4)/(2\alpha+3+2\theta)$, $p_2 = (4\alpha+4)/(2\alpha+1-2\theta)$ and for any $t > 0$, $\lim_{n\to\infty} n^{-t}B(n) = 0$.

They also obtained lower bounds for $\|\sigma_n^{(\alpha,\beta),\theta}\|_p$ of the same form except that $B(n)$ was replaced by a constant. We will prove the following result.

Theorem (22.2). If $\alpha \geq \beta > -1$, $1 \leq p \leq \infty$ and $\theta > 0$, then there is a constant c such that $G(n,p,\theta) \leq c\|\sigma_n^{(\alpha,\beta),\theta}\|_p \leq c^2G(n,p,\theta)$, where G is as defined in (18.3).

The second inequality was proved in corollary (18.11). In the regions where $G(n,p,\theta) = 1$ the first inequaltiy follows immediately by taking $f(x) = 1$ since then $\sigma_n^{(\alpha,\beta),\theta}(f,x) = 1$ for all n and θ. The result for $1 \leq p \leq p_1(\theta)$ is the dual of the result for $p_2(\theta) \leq p \leq \infty$; therefore it is sufficient to consider the case $p_2(\theta) \leq p \leq \infty$ and $0 < \theta \leq \alpha+1/2$. To prove the lower bound for this case use lemma (20.2) to get

$$G(n,p,\theta) \leq c(n+1)^{-(2\alpha+2)/p}\|L_n^{(\alpha,\beta),\theta}(1,x)\|_{p'} .$$

The right side equals

$$c(n+1)^{-(2\alpha+2)/p} \sup_{\|f\|_p=1} |\sigma_n^{(\alpha,\beta),\theta}(f,1)|.$$

For $p < \infty$ we use lemma (19.4) and the fact that $\|h\|_{p,\infty} \leq \|h\|_p$ to get the bound

$$c \sup_{\|f\|_p=1} \|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p \;;$$

for $p = \infty$ this bound is trivial. This completes the proof.

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