manuscipta math. 77, 415 - 433 (1992)

manuscripta mathematica

CONTINUITY OF SOLUTIONS OF UNIFORMLY ELLIPTIC EQUATIONS IN $\mathbb{R}^2$

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1. Introduction

In this paper we prove that the Green's function of the second order uniformly elliptic operator in $\mathbb{R}^2$ belongs to the space BMO. In fact we prove a little bit more. Using this result, we generalize some of the results in Wente [W], Brezis-Coron [BC], Bethuel and Ghidaglia [BG] and Brezis and Merle [BM].

Throughout the paper we assume $\{a_{ij}(x)\} \in L^\infty(\mathbb{R}^2)$, $a_{ij}(x) \equiv a_{ji}(x)(1 \leq i, j \leq 2)$ and, for some $\lambda > 0$, we have the ellipticity condition,

$$\lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2, \forall x, \xi \in \mathbb{R}^2. \tag{0.1}$$

Let $L = \partial_i(a_{ij}(\cdot)\partial_j)$ be a uniformly elliptic operator and $G_x(\cdot)$ denote the global Green's function of $-L$ with pole at $x \in \mathbb{R}^2$ and the normalization $\inf_{B_2(x)} G_x = 0$, namely, for fixed $x \in \mathbb{R}^2$,

$$-LG_x(y) = \delta_x(y). \tag{0.2}$$

Where $\delta_x(\cdot)$ is the Dirac function at x. Here we have used the existence and uniqueness upto constants of the global Green's function which can be found in the Appendix of [KN].

Throughout the paper we use $C_1 = C_1(\lambda) > 0$ to denote some constant depending only on λ and we let $C_0 > 0$ be some universal constant. The values of these constants may change from line to line.

In Section 1, we prove the following theorem.

†Partially supported by NSF grant DMS-9202051
*Partially supported by NSF grant DMS-9104293

THEOREM 0.1. *Under the hypothesis (0.1), and for every $1 \le p < 2$, there exists some positive constant $C = C(\lambda, p)$, such that, for all balls $B = B_R(x_0) \subset \mathbb{R}^2$,*

$$R\Big(\frac{1}{|B|}\int_B |\nabla G_x(y)|^p\,dy\Big)^{1/p} \le C.$$

As will be explained in Section 1, one can conclude from Theorem 0.1 that G_x belongs to the space BMO for any $x \in \mathbb{R}^2$ and $\|G_x\|_{BMO}$ is bounded above by a constant depending only on λ.

In Section 2, we establish the following result.

THEOREM 0.2. *Let L be a uniformly elliptic operator satisfying (0.1), $\Omega \subset \mathbb{R}^2$ be any bounded domain with C^1 boundary. For $u, v \in H^1(\Omega)$, let $\phi \in W_0^{1,1}(\Omega)$ be the unique solution to the problem,*

$$\begin{aligned} -L\phi &= u_{x_1}v_{x_2} - u_{x_2}v_{x_1} && \text{in}\, \Omega \\ \phi &= 0 && \text{on}\, \partial\Omega, \end{aligned} \tag{0.3}$$

then $\phi \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ and

$$\|\phi\|_{L^\infty(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} \le C_1\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}. \tag{0.4}$$

We emphasize that C_1 is independent of Ω.

Remark 0.1: When $a_{ij}(x) \equiv \delta_{ij}$, namely, $L = \Delta$, the result is due to Wente [We] for $\Omega = \mathbb{R}^2$, and Brezis and Coron [BC] for Ω a bounded simply-connected domain. When $a_{ij}(\cdot) \in C^1(\overline{\Omega})$, the result is due to Bethuel and Ghidaglia. After we submitted this paper, we were informed that Bethuel and Ghidaglia have also proved Theorem 0.2 in [BG] by other methods.

In Section 3 we first consider

$$\begin{aligned} -Lu &= f(x) && \text{in}\, \Omega \\ u &= 0 && \text{on}\, \partial\Omega, \end{aligned} \tag{0.5}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $f \in L^1(\Omega)$ and L is defined as above.

We establish the following result which is the starting point of the rest of the results in this section.

THEOREM 0.3. *Under the hypothesis (0.1), there exists $C_1 = C_1(\lambda) > 0$, $C_2 = C_2(\lambda) > 0$, such that the solution u of (0.5) satisfies*

$$\int_\Omega \exp(C_1|u(x)|/\|f\|_1)\,dx \le C_2(diam\,\Omega)^2, \tag{0.6}$$

where $\|f\|_1 = \int_\Omega |f(x)|\,dx$.

Remark 0.2: It is proved by Brezis and Merle [BM] that for $L = \Delta$, one can actually take $C_1 = 4\pi - \delta, C_2 = 4\pi^2/\delta$ in (0.6) for any $\delta > 0$.

1.

In this section we prove that the Green's function of the uniformly elliptic operator in $\mathbb{R}^2$ belongs to the space BMO. In fact we prove a little bit more.

Definition 1.1: $f \in L^1_{loc}(\mathbb{R}^n)$ is said to be of bounded mean oscillation on $\mathbb{R}^n$ (abbreviated as BMO) if there exists a constant $M > 0$, such that, for every ball $B \subset \mathbb{R}^n$, we have

$$\frac{1}{|B|}\int_B |f(x) - f_B|\,dx \le M,$$

where $f_B = \frac{1}{|B|}\int_B f(x)\,dx$ is the mean value of f on B, $|B|$ is the Lebesgue measure of B.

For BMO functions, we introduce $\|f\|_{BMO} = \sup\limits_B \frac{1}{|B|}\int_B |f(x) - f_B|\,dx$.

Proposition 1.1: Let L be the uniformly elliptic operator satisfying (1.1), $G = G_0$, then for any $1 \le p < 2$, $1 \le q < +\infty$, there exist some positive constants $A_1(p,\lambda)$, $A_2(q,\lambda)$, such that,

$$\left(\int_{B_2(0)} |\nabla G|^p\right)^{1/p} \le A_1(p,\lambda),$$

$$\left(\int_{B_2(0)} |G|^q\right)^{1/q} \le A_2(q,\lambda).$$

Remark 1.1: For our applications, we only need to know Proposition 1.1 for the case $p = q = 1$.

Remark 1.2: Proposition 1.1 is due to Stampacchia see also Gruter-Widman [GW] and Kinderlehrer-Stampacchia [KS].

Proof of Theorem 0.1. It is clear that we only need to prove Theorem 0.1 for $x = 0$. The general case follows from a translation. We denote G_0 by G in the following:

Case One: The pole which is the origin is the center of B, namely, $B = B_R(0)$ for some $R > 0$.

Let $\tilde G(x) = G(Rx) - \inf_{|x|\le 2} G(Rx)$, $\tilde a_{ij}(x) = a_{ij}(Rx)$, $\tilde L = \partial_i(\tilde a_{ij}(\cdot)\partial_j)$. It is very easy to see that (since we are in $\mathbb{R}^2$), $\tilde G$ is the Green's function of $-\tilde L$. It then follows from Proposition 1.1 that for $1 \le p < 2$,

$$\left(\int_{B_2(0)} |\nabla \tilde G|^p\right)^{1/p} \le A_1(p,\lambda),$$

namely,

$$R\left(\frac{1}{|B_R(0)|}\int_{B_R(0)} |\nabla G|^p\right)^{1/p} \le A_1(p,\lambda).$$

Case Two: $0 \in B = B_R(x_0)$, for some $x_0 \in \mathbb{R}^2$, $R > 0$.

In this case, $B_{2R}(0) \supset B$. Therefore we have, for $1 \leq p < 2$,

$$\begin{aligned} &R\left(\frac{1}{|B|}\int_B |\nabla G|^p\right) \\ &\leq R\left(\frac{4}{|B_{2R}(0)|}\int_{B_{2R}(0)} |\nabla G|^p\right)^{1/p} \\ &\leq \frac{4^{1/p}}{2}\cdot A_1(p,\lambda). \end{aligned}$$

The last inequality follows from Case One.

Case Three: $0 \notin B = B_R(x_0)$, $R \leq |x_0| \leq 3R$, for some $x_0 \in \mathbb{R}^2$, $R > 0$.

In this case, we have, for $1 \leq p < 2$, that

$$\begin{aligned} &R\left(\frac{1}{|B|}\int_B |\nabla G|^p\right)^{1/p} \\ &\leq R\left(\frac{(2|x_0|)^2}{R^2 B_{2|x_0|}(0)}\int_{B_{2|x_0|}(0)} |\nabla G|^p\right)^{1/p} \\ &\leq \frac{1}{2}\cdot 6^{2/p}\cdot A_1(p,\lambda). \end{aligned}$$

The last inequality again follows from Case One.

Case Four: $0 \notin B_R(x_0), |x_0| > 3R$, for some $x_0 \in \mathbb{R}^2, R > 0$.

In this case we have

$$\begin{aligned} &R\left(\frac{1}{|B|}\int_B |\nabla G|^2\right)^{1/2} \\ &\leq C_0\left(\int_{B_R(x_0)} |\nabla G|^2\right)^{1/2} \\ &\leq C_0\left(\int_{B_{\frac{|x_0|}{2}}(x_0)} |\nabla G|^2\right)^{1/2}. \end{aligned}$$

Let $\tilde{G}(x) = G(|x_0|x) - \inf_{|x|\leq 2} G(|x_0|x)$, $\tilde{a}_{ij}(x) = a_{ij}(|x_0|x)$, $\tilde{L} = \partial_i(\tilde{a}_{ij}(\cdot)\partial_j)$, $\sigma = \frac{x_0}{|x_0|}$. Note $|\sigma| = 1$.

As we have pointed out that $\tilde{G}$ is the Green's function of $-\tilde{L}$ with pole at the origin. By Cacciopoli's inequality (since $\tilde{G}$ is a solution to $L\tilde{G} = 0$ outside the pole) and Proposition 1.1, we have

$$\left(\int_{B_{\frac{1}{2}}(\sigma)} |\nabla\tilde{G}|^2\right)^{1/2} \leq C_1\left(\int_{B_{\frac{1}{2}}(\sigma)} |\tilde{G}|^2\right)^{1/2}.$$

We apply the Harnack inequality to obtain

$$\left(\int_{B_{\frac{1}{2}}(\sigma)} |\tilde{G}|^2\right)^{1/2} \leq C_1 \int_{B_{\frac{5}{8}}(\sigma)} |\tilde{G}|.$$

Further, by Proposition 1.1, we have

$$\int_{B_{\frac{5}{8}}(\sigma)} |\tilde{G}| \leq \int_{B_2(0)} |\tilde{G}| \leq C_1.$$

Scaling back, we obtain

$$R\left(\frac{1}{|B|}\int_B |\nabla G|^2\right)^{1/2} \leq C_1.$$

By Hölder's inequality we get from the estimate above that for $1 \leq p < 2$,

$$R\left(\frac{1}{|B|}\int_B |\nabla G|^p\right)^{1/p} \leq C_1.$$

Theorem 0.1 follows from Cases One - Four.

Corollary 1.1: For the Green's function $G_x(y)$ defined above, we have

$$\|G_x(\cdot)\|_{BMO} \leq C_1, \ \forall\, x \in \mathbb{R}^2.$$

Proof. By the Poincaré-Sobolev inequality we have, for any $x \in \mathbb{R}^2, 1 \leq p < 2$,

$$\left(\frac{1}{|B|}\int_B |G_x - \frac{1}{|B|}\int_B G_x|^p\right)^{1/p} \leq C_p R\left(\frac{1}{|B|}\int_B |\nabla G|^p\right)^{1/p}$$

for all balls $B \subset \mathbb{R}^2$ with radius $R > 0$.

The conclusion of our Corollary follows immediately from Theorem 1.1 and the above inequality.

2.

LEMMA 2.1. *Let $(a_{ij}(x))$ satisfy (0.1) and L, G_x be as before. Let $\varphi \in \mathcal{H}^1(\mathbb{R}^2)$, the Hardy space, $\psi(x) = \int_{\mathbb{R}^2} G_x(y)\varphi(y)\,dy$, then*

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} \leq C_1 \|\varphi\|_{\mathcal{H}^1}.$$

Proof. We point out that in the identity

$$\psi(x) = \int_{\mathbb{R}^2} G_x(y)\varphi(y)\,dy,$$

the integral is to be interpreted suitably on a dense class of functions in the Hardy space. See Stein's book [St] (page 225) for a suitable dense class which is composed of Schwartz functions. By using the result of C. Fefferman [Fe] on the duality between Hardy spaces and BMO and Corollary 1.1, we immediately get the conclusion of this Lemma.

Before we proceed further we give a technical extension of the main result of [Ch] which is the basis of an alternative proof pf Lemma 2.3. This is stated as Lemma 2.2.

Lemma 2.2. *Let $f \in H^1(\mathbb{R}^n)$, $g(x) = (g_i(x))_{i=1}^n$ with $\nabla \cdot g = 0$ and $g \in L^2(\mathbb{R}^n, \mathbb{R}^n)$. Further we assume for some $1 < p < +\infty$,*

$$\sup_{x,r} r \left(\frac{1}{r^n} \int_{B_r(x)} |\nabla h|^p \right)^{1/p} \equiv A < +\infty. \tag{2.1}$$

Then

$$\left| \int_{\mathbb{R}^n} fg \cdot \nabla h \, dx \right| \leq c(n) A \|\nabla f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

and

$$\left| \int_{\mathbb{R}^n} \nabla f \cdot g \, h \, dx \right| \leq c(n) A \|\nabla f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

Remark 2.1: The case $p \geq 2$ is proved in [Ch]. Moreover the weighted analog of the result above can also be proved as in [Ch] where the weight $w(x)$ has now to be chosen in the Muckenhoupt class A_p if $p < 2$ and in A_2 for $p \geq 2$. The two inequalities above are equivalent as an integration by parts shows.

Proof. The proof is exactly the same as in [Ch], except that an L^p version of Lemma 2.1 is required to prove Lemma 2.7. Lemma 2.1 is replaced in the proof of Lemma 2.7 by the following well-known inequality which is the inequality for the Littlewood-Paley $g-$*function*, see Theorem 1 on page 82 in [St]. For $F(x,y)$ the harmonic extension to $\mathbb{R}_+^{n+1}$ of $F(x)$ we have,

$$\left(\int_{\mathbb{R}^n} \left(\int_0^\infty y |\nabla F(x,y)|^2 \, dy \right)^{p/2} dx \right)^{1/p} \leq C(p) \|f\|_p. \tag{2.2}$$

Lemma 2.1 is the case $p = 2$ of the inequality above, in which case it is a consequence of Green's theorem, see [Ch].

Using this inequality in lieu of Lemma 2.1 at every application one may prove Lemma 2.7 in [Ch] under the hypothesis (2.1). To prove Lemma 2.6 in [Ch] under the hypothesis (2.1) we apply the John-Nirenberg inequality [JN] to show, with the notation of [Ch],

$$\sup_{x,t} \left(\frac{1}{t^n} \int_{B(x,t)} |h - h_{av}|^2 \, du \right)^{1/2} \leq \sup_{x,t} \frac{1}{t^n} \int_{B(x,t)} |h - h_{av}| du.$$

Now notice that the Poincaré-Sobolev inequality immediately shows that the right side of the inequality above is bounded by a uniform constant given that (2.1) holds.

The situation in dimension 2 (which is the situation of interest in this paper) is even simpler as we can dispense with the John-Nirenberg inequality and right away see by the Poincaré-Sobolev inequality that for any $p > 1$,

$$\sup_{x,t} \left(\frac{1}{t^2} \int_{B(x,t)} |h - h_{av}|^2 \, du \right)^{1/2} \leq \sup_{x,t} t \left(\frac{1}{t^2} \int_{B(x,t)} |\nabla h|^p \, du \right)^{1/p}.$$

The right side is less than a constant in view of (2.1). Given that Lemma 2.6 and Lemma 2.7 in [Ch] hold under (2.1), the rest of the proof in [Ch] remains unchanged. Since (2.2) is valid with weights in the Muckenhoupt class A_p, we also get the corresponding analogs of Lemma 2.2 with A_p weights.

LEMMA 2.3. *Let $(a_{ij}(x))$ satisfy (0.1) and L, G_x be as before. Assume $u, v \in H^1(\mathbb{R}^2)$ with compact supports, and let*

$$\psi(x) = \int_{\mathbb{R}^2} G_x(y)\{u_{x_1}v_{x_2} - u_{x_2}v_{x_1}\}\, dx_1\, dx_2,$$

then

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\psi\|_{L^2(\mathbb{R}^2)} \le C_1\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}. \tag{2.3}$$

Proof. There are two ways to see this result. The first way is to use the result of R. Coifman, P.L. Lions, Y. Meyer and S. Semmes[CLMS], $u_{x_1}v_{x_2} - v_{x_1}u_{x_2}$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$; and

$$\|u_{x_1}v_{x_2} - v_{x_1}u_{x_2}\|_{\mathcal{H}^1} \le C_0\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}.$$

Thus applying Lemma 2.1,

$$\|\psi\|_{L^\infty(\mathbb{R}^2)} \le C_1\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}. \tag{2.4}$$

Alternatively one applies Lemma 2.2, remembering that by Theorem 1.1. $G_x(y)$ satisfies (1) for some $p < 2$, and $u_{x_1}v_{x_2} - u_{x_2}v_{x_1}$ is of the form $\nabla f \cdot g$, with $g = (v_{x_2}, -v_{x_1})$ and $f \equiv u$. Thus $\nabla \cdot g \equiv 0$. (2.3) again follows from lemma 2.2. We now prove the estimate on $\nabla\psi$ in (2.3).

It is easy to see that

$$-L\psi = u_{x_1}v_{x_2} - u_{x_1}v_{x_2},$$

where u and v have compact support. Let η be a smooth cut-off function such that $\eta \equiv 1$ on B_R, $\eta \equiv 0$ outside B_{2R}, $|\nabla\eta| \le 4$, where B_R is a ball centered at the origin which contains the support of u and v.

Employing the test function $\psi\eta^2$ in an integration by parts we get,

$$\int_{\mathbb{R}^2}\eta^2\langle A\nabla\psi, \nabla\psi\rangle\, dx = \int_{\mathbb{R}^2}(u_{x_1}v_{x_2} - u_{x_2}v_{x_1})\psi\eta^2\, dx$$
$$- 2\int_{\mathbb{R}^2}\langle A\nabla\psi, \nabla\eta\rangle\psi\eta\, dx.$$

By Schwartz's inequality and straight-forward manipulation it follows from above,

$$\int_{\mathbb{R}^2} \eta^2 |\nabla\psi|^2 \, dx \le C_1 \int_{\mathbb{R}^2} |u_{x_1} v_{x_2} - u_{x_2} v_{x_2}| \, |\psi| \, dx + C_1 \int_{\mathbb{R}^2} |\nabla\eta|^2 |\psi|^2 \, dx$$
$$\le C_1 \|\psi\|_{L^\infty(\mathbb{R}^2)} \left(\|\nabla u\|_{L^2(\mathbb{R}^2)} \|\nabla v\|_{L^2(\mathbb{R}^2)} + \|\psi\|_{L^\infty(\mathbb{R}^2)} \right).$$

Applying (2.4) to the right side above and keeping in mind that $\eta \equiv 1$ on B_R we get

$$\int_{B_R} |\nabla\psi|^2 \, dx \le C_1 \|\nabla u\|^2_{L^2(\mathbb{R}^2)} \|\nabla v\|^2_{L^2(\mathbb{R}^2)}.$$

Letting $R \to +\infty$ it follows that

$$\|\nabla\psi\|_{L^2(\mathbb{R}^2)} \le C_1 \|\nabla u\|_{L^2(\mathbb{R}^2)} \|\nabla v\|_{L^2(\mathbb{R}^2)}.$$

This completes the proof of (2.3).

Corollary 2.1: Let $(a_{ij}(x))$ satisfy (0.1) and L be as before. Assume $u, v \in H^1(B_1(0))$ and $\phi \in W_0^{1,1}(B_1(0))$ be the unique solution to

$$\begin{cases} -L\phi = u_{x_1} v_{x_2} - u_{x_2} v_{x_1} & \text{in } B_1(0) \\ \phi = 0 & \text{on } \partial B_1(0) \end{cases} \tag{2.5}$$

where $B_1(0) \subset \mathbb{R}^2$ is the open unit ball. Then $\phi \in C^0(\overline{B_1(0)}) \cap H_0^1(B_1(0))$ and

$$\|\phi\|_{L^\infty(B_1(0))} + \|\nabla\phi\|_{L^2(B_1(0))} \le C_1 \|\nabla u\|_{L^2(B_1(0))} \|\nabla v\|_{L^2(B_1(0))}. \tag{2.6}$$

Proof. Let $\tilde{u} = u - \frac{1}{|B_1(0)|} \int_{B_1(0)} u, \tilde{u} = v - \frac{1}{|B_1(0)|} \int_{B_1(0)} v$. Where $|B_1(0)|$ denotes the Lebesgue measure of $B_1(0)$, it is well known that

$$\|\tilde{u}\|_{L^2(B_1(0))} \le C_0 \|\nabla\tilde{u}\|_{L^2(B_1(0))},$$
$$\|\tilde{v}\|_{L^2(B_1(0))} \le C_0 \|\nabla\tilde{v}\|_{L^2(B_1(0))}.$$

It is also well known that there exists an extension operator $P : H^1(B_1(0)) \to H_c^1(\mathbb{R}^2)$ (here $H_c^1(\mathbb{R}^2)$ denotes the set of functions in $H^1(\mathbb{R}^2)$ which have compact supports), such that, for any $w \in H^1(B_1(0))$,

$$\|Pw\|_{H^1(\mathbb{R}^2)} \le C_0 \|w\|_{H^1(B_1(0))},$$

$$Pw = w \qquad \text{a.e. in } B_1(0),$$

$$supp\, Pw \text{ is compact.}$$

Let $U = P\tilde{u}$, $V = P\tilde{v}$. It is easy to see that

$$\begin{cases} \|\nabla U\|_{L^2(\mathbb{R}^2)} \le C_0 \|\nabla u\|_{L^2(B_1(0))}, \\ \|\nabla V\|_{L^2(\mathbb{R}^2)} \le C_0 \|\nabla v\|_{L^2(B_1(0))}. \end{cases} \tag{2.7}$$

Consider $\psi(x) = \int_{\mathbb{R}^2} G_x(y)\{U_{y_1}V_{y_2} - U_{y_2}V_{y_1}\}\, dy_1\, dy_2$. We see easily that $\phi - \psi$ satisfies

$$-L(\phi - \psi) = 0 \qquad \text{in } B_1(0).$$

It follows from the maximum principle that

$$\begin{aligned}
\|\phi\|_{L^\infty(B_1(0))} &\le \|\psi\|_{L^\infty(B_1(0))} + \|\phi - \psi\|_{L^\infty(B_1(0))} \\
&\le \|\psi\|_{L^\infty(B_1(0))} + \|\phi - \psi\|_{L^\infty(\partial B_1(0))} \\
&= \||\psi\|_{L^\infty(B_1(0))} + \|\psi\|_{L^\infty(\partial B_1(0))} \\
&\le 2\|\psi\|_{L^\infty(B_1(0))} \\
&\le C_1\|\nabla u\|_{L^2(B_1(0))}\|\nabla v\|_{L^2(B_1(0))}.
\end{aligned}$$

The last inequality follows from Lemma 2.1 and (2.7).

The estimate of $\|\nabla\phi\|_{L^2(B_1)}$ follows, as before, an integration by parts argument and the estimate for $\|\phi\|_{L^\infty(B_1(0))}$ which we have established.

Corollary 2.2: Let $(a_{ij}(x))$ satisfy (0.1) and L be as before, $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain with C^1 boundary. Let $u, v \in H^1(\Omega)$ and $\phi \in W_0^{1,1}(\Omega)$ be the unique solution of

$$\begin{cases} -L\phi = u_{x_1}v_{x_2} - v_{x_1}u_{x_2} & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.8}$$

then $\phi \in C(\overline{\Omega}) \cap H_0^1(\Omega)$ and

$$\|\phi\|_{L^\infty(\Omega)} + \|\nabla\phi\|_{L^2(\Omega)} \le C_1\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}. \tag{2.9}$$

Remark 2.2: The above constant C_1 is independent of Ω as long as Ω is a bounded simply connected domain with C^1 boundary.

Corollary 2.1 follows from the observation that the problem is invariant under conformal transformations. We only need to apply the Riemann mapping theorem to find a conformal mapping which maps Ω onto $B_1(0)$ and this reduces the situation to Corollary 2.1. We leave it to the reader to check the details.

To prove Theorem 0.2, we use the co-area formula to reduce it to Corollary 2.2. This reduction argument is due to Bethuel and Ghidaglia. We include it here for the sake of completeness and to point out that in fact the extension is not dependent on the smoothness of the coefficients $a_{ij}(x)$ which is crucial for us.

We first assume that $(a_{ij}(x))$, u, v are smooth functions. Without loss of generality, we assume $\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla v|^2 = 1$. Clearly, $\phi \in C^1(\overline{\Omega})$.

For $A > 0$, we set $W(A) = \{x \in \Omega \big| |\phi(x)| \le A\}$.

By Sard's Theorem, for almost every $A > 0$, $\partial W(A) \cap \partial\Omega = \emptyset$, $\partial W(A)$ is a one dimensional manifold.

In the following we make use of the co-area formula due to Federer and Fleming [F]. See also [ABL], [Br] and [BG] for related applications of the co-area formula.

Lemma 2.4 (Co-area formula). *Suppose $f \in C^1(\mathbb{R}^2, \mathbb{R})$ and $g \in C^0(\mathbb{R}^2, \mathbb{R}_+)$. Then for every measurable set X, we have*

$$\int_X g|\nabla f|\,dx = \int_{\mathbb{R}} \left(\int_{f^{-1}(t)\cap X} g\,d\ell \right) dt.$$

We apply the co-area formula with the choice $f = |\phi|$, $X = \Omega \cap f^{-1}([C, C+D])$, $C, D > 0$ and $g = |\nabla u| + |\nabla v|$. C and D will be specified later. It follows then that

$$\begin{aligned}\int_C^{C+D} &\left(\int_{|\phi(x)|=t} (|\nabla u| + |\nabla v|)\,d\ell \right) dt \\ &= \int_X \Big(|\nabla u| + |\nabla v| \Big) |\nabla |\phi||\,dx \\ &\le \sqrt{2} \Big(\int_\Omega |\nabla \phi|^2 \Big)^{1/2}.\end{aligned}$$

By the mean value theorem, there exists $A \in [C, C+D]$, which is a regular value of both ϕ and $-\phi$, such that

$$\int_{\partial W(A)} \big(|\nabla u| + |\nabla v|\big)\,d\ell \le \frac{2}{D} \|\nabla \phi\|_{L^2(\Omega)}. \tag{2.10}$$

The set $\partial W(A)$ is the union of a finite number of Jordan curves, say, $\partial W(A) = \bigcup_{i=1}^{N} J_i$. We denote by $H_1, \cdots, H_\ell$ the holes in Ω and ω_i the bounded component of $\mathbb{R}^2 \backslash J_i$.

We are going to establish the inequality

$$\|\phi\|_{L^\infty(\Omega)} \le A + C_1 \big(1 + \frac{1}{A^2 D^2} \int_\Omega |\nabla \phi|^2 \big)^2. \tag{2.11}$$

Case 1: $\|\phi\|_{L^\infty(\Omega)} \le A$.

In this case, (2.11) obviously holds.

Case 2: $\|\phi\|_{L^\infty(\Omega)} > A$ and $|\phi(x_0)| = \|\phi\|_{L^\infty(\Omega)}$ for some $x_0 \in \omega_{i_0}$ $(1 \le i_0 \le N)$ and J_{i_0} is contractible in Ω. Apply Corollary 2.2 to $\phi + \epsilon A (\epsilon = +1 \text{ or } -1)$, we have

$$\|\phi + \epsilon A\|_{L^\infty(\omega_{i_0})} \le C_1$$

and (2.11) follows.

Case 3: $\|\phi\|_{L^\infty(\Omega)} > A$ and $|\phi(x_0)| = \|\phi\|_{L^\infty(\Omega)}$ for some $x_0 \in \omega_{i_0}$ $(1 \le i_0 \le N)$, but J_{i_0} is not contractible in Ω. This last case is more subtle.

Let $J_1, \cdots, J_k$ be all the Jordan curves which belong to ω_{i_0} and $H_1, \cdots, H_p$ be all the holes which belong to ω_{i_0} (after relabeling if necessary). Let $J_1, \cdots, J_q$ be the maximal curves with respect to inclusion in the union $J_1 \cup \cdots \cup J_k \cup \partial H_1 \cup \cdots \cup \partial H_p$ (obviously the ∂H_j cannot be maximal). The holes $H_1, \cdots, H_p$ are included in $M_{i_0} = \bigcup_{j=1}^{q} \omega_j \subset \omega_{i_0}$. By the maximality of $J_1, \cdots, J_q$, we know that $\omega_i \cap \omega_j = \phi$ if $1 \leq i < j \leq q$.

It follows from (2.10) that for any $\sigma_j \in \partial\omega_j$,

$$\sum_{j=1}^{q} \left| u(\sigma_j) - \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s)\, ds \right| \leq \frac{2}{D} \|\nabla\phi\|_{L^2(\Omega)}, \tag{2.12}$$

where $|\partial\omega_j|$ denotes the length of $\partial\omega_j$.

We first define $\bar{u}(x)$ for $x \in (\bigcup_{j=1}^{q} \omega_j) \cap \Omega$ as follows.

$$\bar{u}(x) = \begin{cases} u(x) \quad \text{if } |u(x) - \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s)\, ds \,| \leq \frac{2}{D} \|\nabla\phi\|_{L^2(\Omega)} \\ \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s) ds + \frac{2}{D} \|\nabla\phi\|_{L^2(\Omega)} \operatorname{sign}\Big(u(x) - \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s) ds \Big) \text{ elsewhere.} \end{cases}$$

Then we define $\overline{\phi}(x)$ for $x \in M_{i_0}$ by

$$\overline{\phi}(x) = \begin{cases} \phi(x) & \text{if } |\phi(x)| \leq A \\ A & \text{if } \phi(x) \geq A \\ -A & \text{if } \phi(x) \leq -A. \end{cases}$$

Now we construct $u^{\#}(x)$ for $x \in \omega_{i_0}$ as follows.

$$u^{\#}(x) = u(x) \qquad \text{if } x \in \omega_{i_0} \backslash M_{i_0}$$

$$u^{\#}(x) = \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s)\, ds \qquad \text{if } x \in \bar{\omega}_j \text{ and } x \notin \Omega,$$

and

$$u^{\#}(x) = \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s)\, ds + \frac{\bar{\phi}(x)}{\epsilon_j A} \Big(\bar{u}(x) - \frac{1}{|\partial\omega_j|} \int_{\partial\omega_j} u(s)\, ds \Big)$$

if $x \in \bar{\omega}_j \cap \Omega$. Where $\epsilon_j = \frac{\phi(x)}{A} \in \{\pm 1\}$ for $x \in \partial\omega_j$.
We make the same construction for $v^{\#}$.

It follows from a straight forward computation that

$$\begin{cases} \|\nabla u^{\#}\|^2_{L^2(\omega_{i_0})} \leq C_0 \Big(1 + \frac{1}{A^2 D^2} (\int_{M_{i_0}} |\nabla\phi|^2)^2 \Big), \\ \|\nabla v^{\#}\|^2_{L^2(\omega_{i_0})} \leq C_0 \Big(1 + \frac{1}{A^2 D^2} (\int_{M_{i_0}} |\nabla\phi|^2)^2 \Big). \end{cases} \tag{2.13}$$

Consider

$$\begin{cases} -L\psi = \frac{\partial u^{\#}}{\partial x_1} \frac{\partial v^{\#}}{\partial x_2} - \frac{\partial u^{\#}}{\partial x_2} \frac{\partial v^{\#}}{\partial x_1} & \text{in } \omega_{i_0} \\ \psi = 0 & \text{on } \partial\omega_{i_0}. \end{cases}$$

Notice that ω_{i_0} is simply connected, it follows from Corollary 2.2 that

$$\|\psi\|_{L^\infty(\omega_{i_0})} \leq C_1 \|\nabla u^\#\|_{L^2(\omega_{i_0})} \|\nabla v^\#\|_{L^2(\omega_{i_0})}.$$

Using (2.13) we obtain

$$\|\psi\|_{L^\infty(\omega_{i_0})} \leq C_1 \left(1 + \frac{1}{A^2 D^2} \int_{\omega_{i_0}} |\nabla \phi|^2\right)^2.$$

Noticing that $u^\# = u$, $v^\# = v$ in $\omega_{i_0} \backslash M_{i_0}$, it follows from the maximum principle that

$$\|\psi - \phi\|_{L^\infty(\omega_{i_0})} \leq \|\phi\|_{L^\infty(\partial(\omega_{i_0} \backslash M_{i_0}))} + \|\psi\|_{L^\infty(\omega_{i_0})}.$$

Hence

$$\|\phi\|_{L^\infty(\Omega)} = \|\phi\|_{L^\infty(\omega_{i_0})} \leq A + C_1 \left(1 + \frac{1}{A^2 D^2} \int_\Omega |\nabla \phi|^2\right)^2.$$

We have thus established (2.11) in Case 3. Clearly, Cases 1-3 are all the possibilities. Hence we have established (2.11).

By choosing $C = 1$, $D = \|\nabla \phi\|_{L^2(\Omega)}$, we have, for some $1 \leq A \leq 1 + \|\nabla \phi\|_{L^2(\Omega)}$, that

$$\begin{aligned}\int_\Omega |\nabla \phi|^2 &\leq \|\phi\|_{L^\infty(\Omega)} \\ &\leq A + C_1 \left(1 + \frac{1}{A}\right)^2 \\ &\leq \|\nabla \phi\|_{L^2(\Omega)} + C_1.\end{aligned}$$

Notice that the first inequality follows from multiplying (2.8) by ϕ and integrating by parts. This gives a bound on $\|\nabla \phi\|_{L^2(\Omega)}$:

$$\|\nabla \phi\|_{L^2(\Omega)} \leq C_1.$$

The bound of $\|\phi\|_{L^\infty(\Omega)}$ follows from (2.11) and the bound for $\|\nabla \phi\|_{L^2(\Omega)}$.

We have now proved Theorem 0.2 under the assumption that $a_{ij}(x)$, u, v are smooth functions. Observe that the constant C_1 in (2.9) does not depend at all on the smoothness of $a_{ij}(x)$, u, v. Smoothness was only used so as to apply the co-area formula. Thus we are using smoothness in a qualitative way only.

This makes it possible to use a simple approximation argument to establish (2.9) in the general case.

Let $a_{ij}^{(k)}(x)$, $u^{(k)}$, $v^{(k)}(k=1,2,3,\dots)$ be a sequence of smooth functions with the following properties.

$$\gamma^{-1}|\xi|^2 \le a_{ij}^{(k)}(x)\xi_i\xi_j \le \gamma|\xi|^2,\ \forall x,\xi \in \mathbb{R}^2,\ k=1,2,3,\dots$$
$$a_{ij}^{(k)}(x) \xrightarrow{k\to+\infty} a_{ij}(x) \quad \text{in } L^p(\Omega),$$
$$a_{ij}^{(k)}(x) \xrightarrow{k\to+\infty} a_{ij}(x) \quad \text{a.e., in } \Omega,$$
$$u^{(k)} \xrightarrow{k\to+\infty} u \qquad \text{in } H^1(\Omega),$$
$$v^{(k)} \xrightarrow{k\to+\infty} v \qquad \text{in } H^1(\Omega).$$

Let $\phi^{(k)}$ be the unique solution of

$$\begin{cases} -\partial_i\left(a_{ij}^{(k)}(x)\partial_j\phi^{(k)}\right) = u_{x_1}^{(k)}v_{x_2}^{(k)} - u_{x_2}^{(k)}v_{x_1}^{(k)} & \text{in } \Omega, \\ \phi^{(k)} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since we have proved (2.9) in the smooth coefficients use, we have

$$\|\phi^{(k)}\|_{L^\infty(\Omega)} + \|\nabla\phi^{(k)}\|_{L^2(\Omega)} \le C_1\|\nabla u^{(k)}\|_{L^2(\Omega)}\|\nabla v^{(k)}\|_{L^2(\Omega)}.$$

By passing to the limit, we can assume that for some $\tilde{\phi} \in H_0^1(\Omega), \phi^{(k)} \xrightarrow{k\to+\infty} \tilde{\phi}$ weakly in $H_0^1(\Omega)$.

It follows easily that $\tilde{\phi}$ satisfies

$$\begin{cases} -L\tilde{\phi} = u_{x_1}v_{x_2} - u_{x_2}v_{x_1}, & \text{in } \Omega \\ \tilde{\phi} = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle, $\phi \equiv \tilde{\phi}$.

For any $p > 1$, $\phi^{(k)} \xrightarrow{k\to+\infty} \phi$ strongly in $L^p(\Omega)$. Therefore we deduce from

$$\|\phi^{(k)}\|_{L^p(\Omega)} \le |\Omega|^{1/p}\|\phi^{(k)}\|_{L^\infty(\Omega)} \le C_1|\Omega|^{1/p}C_1\|\nabla u^{(k)}\|_{L^2}\|\nabla v^{(k)}\|_{L^2}$$

that

$$\|\phi\|_{L^p(\Omega)} \le C_1|\Omega|^{1/p}\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

Letting $p \to +\infty$, we have

$$\|\phi\|_{L^\infty(\Omega)} \le C_1\|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}.$$

The estimate for $\|\nabla\phi\|_{L^2(\Omega)}$ follows as before.

3.

We first study (0.5) and prove Theorem 0.3.

Let $R = \frac{1}{2}\operatorname{diam}\Omega$ and $\Omega \subset B_R$. Extend f to be zero outside Ω.

Set

$$\phi(x) = \int_{\mathbb{R}^2} G_x(y)|f(y)|\,dy.$$

Clearly ϕ satisfies

$$-L\phi = |f| \quad \text{in } \mathbb{R}^2.$$

Because of the maximum principle, (0.6) follows from

$$\int_\Omega \exp(C_1|\phi(x)|/\|f\|_1)\,dx \leq C_2(diam\,\Omega)^2, \tag{3.1}$$

for some $C_1 = C_1(\lambda) > 0, C_2 = C_2(\lambda) > 0$.

Proposition3.1: ϕ is in BMO and $\|\phi\|_{BMO} \leq C_1\|f\|_1$.

Proof. For any ball $B \subset \mathbb{R}^2$,

$$\phi(x) - \frac{1}{|B|}\int_B \phi = \int_{\mathbb{R}^2} [G_x(y) - \frac{1}{|B|}\int G_z(y)\,dz\,]|f(y)|\,dy\,.$$

Thus,

$$\frac{1}{|B|}\int_B |\phi(x) - \frac{1}{|B|}\int_B \phi|\,dx \leq \frac{1}{|B|}\int_{\mathbb{R}^2}\int_B |G_x(y) - \frac{1}{|B|}\int_B G_z(y)\,dz|\,|f(y)|\,dxdy\,.$$

Notice that for fixed y, $G_x(y)$ is the Green's function of the adjoint operator of L, which is also of divergence form with the same structure constant λ.

Therefore $G_x(y)$ is also in BMO as a function of x for fixed y. Thus, for $G_x(y) = G(x,y)$,

$$\begin{aligned}\|\phi\|_{BMO} &\leq \sup_y \|G(\cdot,y)\|_{BMO}\|f\|_1 \\ &\leq C_1\|f\|_1.\end{aligned}$$

Since ϕ is in BMO, we may use the John-Nirenberg estimate to get,

$$\begin{aligned}|\{x \in B_R: |\phi(x) - \bar{\phi}| > \mu\}| &\leq C_0 e^{-\mu/\|\phi\|_{BMO}} R^2 \\ &\leq C_0 e^{-\mu/C_2\|f\|_1} R^2,\end{aligned}$$

where $\bar{\phi} = \frac{1}{|B_R|}\int_{B_R}\phi$.

It follows immediately that

$$\frac{1}{|B_R|}\int_{B_R} e^{C_1|\phi(x)-\bar{\phi}|/\|f\|_1}\,dx \le C_2. \tag{3.2}$$

Now recall that

$$\begin{aligned}\bar{\phi} &= \frac{1}{|B_R|}\int_{B_R}\phi(x)\,dx\\ &= \frac{1}{|B_R|}\int_{B_R}\int_{\mathbb{R}^2} G_x(y)|f(y)|\,dy\,dx\,.\end{aligned}$$

Thus,

$$|\bar{\phi}| \le \sup_{\{y\in B_R\}} \frac{1}{|B_R|}\int_{B_R} G_x(y)\,dx\cdot\|f\|_1. \tag{3.3}$$

If $R=1$, it follows from Proposition 1.1 that

$$\sup_{y\in B_1}\int_{B_2} G_x(y)\,dx \le C_1. \tag{3.4}$$

Thus from (3.3) we obtain, when $R=1, |\bar{\phi}|/\|f\|_1 \le C_1$. Therefore we can deduce (3.1) in the case $diam\,\Omega = 2$ from (3.2), (3.3) and (3.4). As indicated before, we have established (0.6) in the case $diam\,\Omega = 1$.

The general case follows simply from a scaling argument as follows.

Without loss of generality, $0\in\Omega$.

Let $\tilde{\Omega} = \frac{1}{R}\Omega$, then $diam\,\tilde{\Omega} = 2$ and for $x\in\tilde{\Omega}, \tilde{f}(x) = R^2 f(Rx), \tilde{a}_{ij}(x) = a_{ij}(Rx), \tilde{L} = \partial_i(\tilde{a}_{ij}(x)a_{ij}), \tilde{u}(x) = u(Rx)$. It is obvious that $\tilde{u}$ satisfies

$$\begin{cases} -\tilde{L}\tilde{u} = \tilde{f} & \text{in } \tilde{\Omega}\\ \tilde{u} = 0 & \text{on } \partial\tilde{\Omega}.\end{cases}$$

Notice that $diam\,\tilde{\Omega} = 2$, thus,

$$\int_{\tilde{\Omega}} \exp\Big(C_1|\tilde{u}(x)|/\|\tilde{f}\|_1\Big)\,dx \le C_2. \tag{3.5}$$

Notice $\|\tilde{f}\|_1 = \|f\|_1$. Changing variables in (3.5) we thus obtain (0.6).

Theorem 0.3 has been established.

In the following we state a number of corollaries which can be proved essentially the same way as in [BM], provided we have Theorem 0.3.

Corollary 3.1: Let u be a solution of (0.5) with $f\in L^1(\Omega)$. Then for every constant $k>0$,

$$e^{k|u|}\in L^1(\Omega).$$

Corollary 3.2: Suppose u is a solution of

$$\text{(3.6)} \qquad \begin{cases} -Lu = V(x)e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $V \in L^p(\Omega)$ and $e^u \in L^{p'}(\Omega)$ for some $1 < p \leq \infty$, $p' = \frac{p}{p-1}$. Then $u \in L^\infty(\Omega)$.

Corollary 3.3: Suppose $u \in L^1_{loc}(\mathbb{R}^2)$ satisfies

$$-Lu = V(x)e^u \quad \text{in} \quad \mathbb{R}^2$$

with $V \in L^p(\mathbb{R}^2)$ and $e^u \in L^{p'}(\mathbb{R}^2)$ for some $1 < p \leq \infty$, $p' = \frac{p}{p-1}$. Then $u \in L^\infty(\mathbb{R}^2)$.

Corollary 3.4: There exists some $\epsilon_0 = \epsilon_0(\lambda) > 0$, such that, for any (u_n) a sequence of solutions of

$$\text{(3.7)} \qquad -Lu_n = V_n(x)e^{u_n} \quad \text{in} \quad \Omega$$

with $u_n = 0$ on $\partial\Omega$,

$$\|V_n\|_{L^p} \leq A \quad \text{for some} \quad 1 < p < \infty,$$

and

$$\int_\Omega |V_n|e^{u_n} \leq \epsilon_0$$

we have $\|u_n\|_{L^\infty} \leq C(\lambda, A)$.

Corollary 3.5: There exists some $\epsilon_1 = \epsilon_1(\lambda) > 0$, such that, for any (u_n) a sequence of solutions of (3.7) with

$$\|V_n\|_{L^p} \leq A_1 \quad \text{for some} \quad 1 < p < \infty,$$

$$\|u_n^+\|_{L^1} \leq A_2$$

and

$$\int_\Omega |V_n|e^{u_n} \leq \epsilon_1$$

we have (u_n^+) is bounded in $L^\infty_{loc}(\Omega)$.

Corollary 3.6: Assume (u_n) is a sequence of solutions of (3.7) with $u_n = 0$ on $\partial\Omega$, satisfying, for some $1 < p < \infty$,

$$\|e^{u_n}\|_{L^{p'}} \leq A$$

and one of the following conditions:

either

$$|V_n(x)| \leq W(x) \ \forall\, n \quad \text{with} \quad W \in L^p(\Omega)$$

or

$$V_n \to V \quad \text{in} \quad L^p(\Omega).$$

Then $\|u_n\|_{L^\infty} \leq C$.

4.

In this section we study a sequence $\{u_n\}$ of solutions of

$$-\frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}u_n\big) = V_n(x)e^{u_n} \qquad \text{in } \Omega \tag{4.1}$$

where Ω is a bounded domain in $\mathbb{R}^2, (a_{ij}(x)) \in C^{\gamma}_{loc}(\Omega)$ for some $0 < \gamma < 1$ satisfies (0.1).

We define the blow up set by

$S = \{x \in \Omega\colon$ there exists a sequence x_n in Ω such that $x_n \to x$ and $u_n(x_n) \to +\infty\}$.

THEOREM 4.1. *Assume that (u_n) is a sequence of solutions of (4.1), $(a_{ij}(x))$ satisfies the conditions mentioned above. If for some $1 < p \le +\infty$,*

$$V_n \ge 0 \qquad in\ \Omega \tag{4.2}$$

$$\|V_n\|_{L^p} \le C_1 \tag{4.3}$$

and

$$\|e^{u_n}\|_{L^{p'}} \le C_2, \tag{4.4}$$

where $p' = \frac{p}{p-1}$ if $1 < p < +\infty$, $p' = 1$ if $p = +\infty$.

Then there exists a subsequence u_{n_k} satisfying the following alternative: Either

(i) (u_{n_k}) is bounded in $L^{\infty}_{loc}(\Omega)$, or
(ii) $u_{n_k}(x) \to -\infty$ uniformly on compact subsets of Ω, or
(iii) the blow up set S (relative to (u_{n_k})) is finite, nonempty and $u_{n_k}(x) \to = -\infty$ uniformly on compact subsets of $\Omega\backslash S$. In addition $V_{n_k}e^{u_{n_k}}$ converges in the sense of measures on Ω to $\Sigma_i \alpha_{a_i}\delta_{a_i}$ with $\alpha_i \ge 2\pi\alpha_{a_i}/p'$ and $S = \bigcup_i\{a_i\}$.

Where $\alpha_{a_i} = 2/det(a_{ij}(a_i))^{-1/2}$ for $a_i \in \Omega$.

The proof of Theorem 4.1 will be essentially the same as the proof of Theorem 3 in [BM] once we have Lemma 4.1. We will only indicate some changes.

LEMMA 4.1. *Under the hypothesis of Theorem 4.1, for any $x_0 \in \Omega$, $G_{x_0}(x) = \overline{G}_{x_0} + h_{x_0}(x)$, where $\overline{G}_{x_0}(x)$ is the Green's function of the constant coefficient operator $-\frac{\partial}{\partial x_i}(a_{ij}(x_0)\frac{\partial}{\partial x_j})$ and h_{x_0} is Hölder continuous.*

Proof of Lemma 4.1. Clearly, $\overline{G}_{x_0} = C_{x_0}\log\frac{1}{|E_{x_0}\cdot(x-x_0)|}$, where $E_{x_0} = A_{x_0}^{-1/2}$, $A_{x_0} = (a_{ij}(x_0)), C_{x_0} = \det(E_{x_0})$.

It is also clear that h_{x_0} satisfies

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}h_{x_0}\right) = \frac{\partial}{\partial x_i}\left((a_{ij}(x) - a_{ij}(x_0))\frac{\partial \overline{G}_{x_0}}{\partial x_j}\right)$$
$$\equiv \frac{\partial}{\partial x_i}(f_i).$$

It follows from the Hölder continuity of $(a_{ij}(x))$, that $f_i \in L^p_{loc}$ for some $p > 2$. It then follows from standard elliptic theory that h_{x_0} is locally Hölder continuous in Ω.

It is very easy to see that $\int_\Omega e^{\alpha_{x_0}\bar{G}_{x_0}(x)}\,dx = +\infty$ and, due to the Hölder continuity of h_{x_0}, $\int_\Omega e^{\alpha_{x_0}G_{x_0}(x)}\,dx = +\infty$.

Proof of Theorem 4.1. *Since $\{V_n e^{u_n}\}$ is bounded in $L^1(\Omega)$ we may extract a subsequence (still denoted as $V_n e^{u_n}$) such that $V_n e^{u_n}$ converges in the sense of measures on Ω to some nonnegative bounded measure μ, namely*

$$\int V_n e^{u_n}\psi \longrightarrow \int \psi\,d\mu \tag{4.5}$$

for every $\psi \in C_c(\Omega)$.

Definition 4.1: We say $x_0 \in \Omega$ is a regular point if there is a function $\psi \in C_c(\Omega)$, $0 \le \psi \le 1$, with $\psi = 1$ in some neighborhood of x_0, such that,

$$\int \psi\,d\mu < 2\pi\alpha_{x_0}/p'.$$

We denote by Σ the set of nonregular points in Ω. Clearly $x_0 \in \Sigma$ iff $\mu(\{x_0\}) \ge 2\pi\alpha_{x_0}/p'$ and

$$card(\Sigma) \le C_1C_2 \cdot \inf_{x_0\in\Omega} p'/2\pi\alpha_{x_0} \le C_1C_2p'\lambda/4\pi.$$

Step 1: $S = \Sigma$.

Step 2: $S = \emptyset$ implies (i) or (ii) holds.

Step 3: $S \ne \emptyset$ implies (iii) holds.

All the three steps can be proved essentially the same way as in [BM], with our definition of Σ. We leave the details to the reader.

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(Received February 26, 1992;
in revised form July 23, 1992)