①

Solve $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = e^{x+2y}$, $u(x,0)=0$

by the method of characteristics.

The characteristic eqns. are $\frac{dx}{ds} = 1 = \frac{dy}{ds}$.

and the characteristic curves are $(x(s), y(s))$.

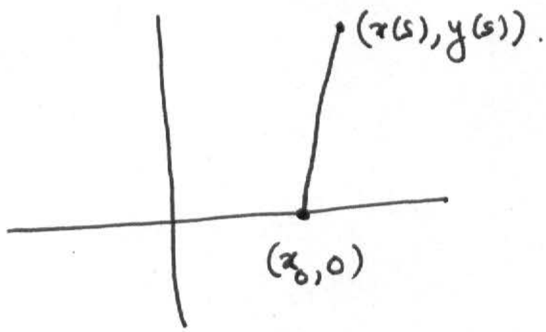
So along the characteristic curves we have

$$\frac{d}{ds} u(x(s), y(s)) \underset{\text{chain rule}}{=} \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

$$= u_x + u_y \quad \left(\text{because } \frac{dx}{ds} = \frac{dy}{ds} = 1\right). \quad ①$$

Next solve the characteristic curve eqn. to get

$$x(s) = s + c_1, \quad y(s) = s + c_2 \quad \left(\frac{dx}{ds} = \frac{dy}{ds} = 1\right)$$



Let us choose c_1, c_2 so that at $s=0$ we are at $(x_0, 0)$ along the characteristic curve drawn above. We select $c_1 = x_0$, $c_2 = 0$ and get

$$\boxed{x(s) = s + x_0, \quad y(s) = s.} \quad \text{—— (2)}$$

This is the parametrization we use of the curve in the picture.

Use (1) and (2). in our PDE; our PDE transforms to,

$$u_x + u_y + u = \frac{du}{ds} + u = e^{x+2y} = e^{s+x_0}\, e^{2s}$$

So we get the ODE (this is the point of the characteristic method PDE $\longrightarrow$ ODE)

$$\frac{du}{ds} + u = e^{3s+x_0}.$$

Using the integrating factor method we solve,

$$u(s) = u(x(s), y(s)) = \frac{e^{3s+x_0}}{4} + Be^{-s}.$$

Next $u(x(0), y(0)) = u(0) = u(x_0, 0) = \frac{e^{x_0}}{4} + B = 0$

$B = -\frac{e^{x_0}}{4}$. To get

$$u(x(s), y(s)) = \frac{e^{3s+x_0}}{4} - \frac{e^{x_0}}{4} e^{-s}. \qquad (3)$$

Lastly we want the answer as $u(x,y)$. From (2)

$x_0 = x(s) - s$, $y(s) = s$, so $x_0 = x - y$, $s = y$

Use this fact in (3) to get,

$$u(x,y) = \frac{e^{3y+x-y}}{4} - \frac{e^{x-y-y}}{4} = \frac{e^x}{4}\left(e^{2y} - e^{-2y}\right)$$

$$= \frac{e^x}{2}\left(\frac{e^{2y} - e^{-2y}}{2}\right) = \frac{1}{2} e^x \sinh(2y)$$

$\boxed{u(x,y) = \frac{1}{2} e^x \sinh(2y)}$ is the solution.