Embedded three-dimensional CR manifolds and the non-negativity of Paneitz operators

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ABSTRACT. Let $\Omega \subset \mathbb{C}^2$ be a strictly pseudoconvex domain and $M = \partial\Omega$ be a smooth, compact and connected CR manifold embedded in $\mathbb{C}^2$ with the CR structure induced from $\mathbb{C}^2$. The main result proved here is as follows. Assume the CR structure of M has zero torsion. Then if we make a small real-analytic deformation of the CR structure of M along embeddable directions, the CR structures along the deformation path continue to have non-negative Paneitz operators. We also show that any ellipsoid in $\mathbb{C}^2$ has positive Webster curvature.

1. Introduction

Throughout this paper, we will use the notation and terminology in ([**14**]) unless otherwise specified. Let (M, J, θ) be a smooth, closed and connected three-dimensional pseudo-hermitian manifold, where θ is a contact form and J is a CR structure compatible with the contact bundle $\xi = \ker\theta$. The CR structure J decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively. The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z, W \rangle_{L_\theta} = -i \langle d\theta, Z \wedge \overline{W} \rangle$. We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\langle \overline{Z}, \overline{W} \rangle_{L_\theta} = \overline{\langle Z, W \rangle_{L_\theta}}$ for all $Z, W \in T_{1,0}$. The Levi form induces a natural Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the hermitian form (when acting on sections) over M with respect to the volume form $dV = \theta \wedge d\theta$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$. For example

$$\langle \varphi, \psi \rangle = \int_M \varphi\bar{\psi}\ dV, \tag{1.1}$$

for functions φ and ψ.

Let $\{T, Z_1, Z_{\bar{1}}\}$ be a frame of $TM \otimes \mathbf{C}$, where Z_1 is any local frame of $T_{1,0}$, $Z_{\bar{1}} = \overline{Z_1} \in T_{0,1}$ and T is the characteristic vector field, that is, the unique vector field such that $\theta(T) = 1$, $d\theta(T, \cdot) = 0$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, satisfies

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} \tag{1.2}$$

for some positive function $h_{1\bar{1}}$. We can always choose Z_1 such that $h_{1\bar{1}} = 1$; hence, throughout this paper, we assume $h_{1\bar{1}} = 1$

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbf{C}$ (and extended to tensors) given in terms of a local frame $Z_1 \in T_{1,0}$ by

$$\nabla Z_1 = \theta_1{}^1 \otimes Z_1, \quad \nabla Z_{\bar{1}} = \theta_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \quad \nabla T = 0,$$

where $\theta_1{}^1$ is the 1-form uniquely determined by the following equations:

$$d\theta^1 = \theta^1 \wedge \theta_1{}^1 + \theta \wedge \tau^1$$

(1.3) $$\tau^1 \equiv 0 \mod \theta^{\bar{1}}$$

$$0 = \theta_1{}^1 + \theta_{\bar{1}}{}^{\bar{1}},$$

where $\theta_1{}^1$ and τ^1 are called the connection form and the pseudohermitian torsion, respectively. Set $\tau^1 = A^1{}_{\bar{1}}\theta^{\bar{1}}$. The structural equation for the pseudohermitian connection is given by,

(1.4) $$d\theta_1{}^1 = Rh_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} + A_1{}^{\bar{1}}{}_{,\bar{1}}\theta^1 \wedge \theta - A_{\bar{1}}{}^1{}_{,1}\theta^{\bar{1}} \wedge \theta.$$

where R is the Tanaka-Webster curvature, see [18].

We will denote components of covariant derivatives with indices preceded by a comma; thus we write $A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta$. The indices $\{0, 1, \bar{1}\}$ indicate derivatives with respect to $\{T, Z_1, Z_{\bar{1}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $\varphi_1 = Z_1\varphi$, $\varphi_{1\bar{1}} = Z_{\bar{1}}Z_1\varphi - \theta_1^1(Z_{\bar{1}})Z_1\varphi$, $\varphi_0 = T\varphi$ for a (smooth) function.

Next we introduce several natural differential operators occuring in this paper. For a detailed description, we refer the reader to the article [14]. For a smooth function φ, the Cauchy-Riemann operator ∂_b can be defined locally by

$$\partial_b\varphi = \varphi_1\theta^1,$$

and we write $\bar{\partial}_b$ for the conjugate of ∂_b. A function φ is called CR holomorphic if $\bar{\partial}_b\varphi = 0$. The divergence operator δ_b takes $(1, 0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_1{}^1$, and similarly, $\bar{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1}}{}^{\bar{1}}$.

If $\sigma = \sigma_1\theta^1$ is compactly supported, Stokes' theorem applied to the 2-form $\theta \wedge \sigma$ implies the divergence formula:

$$\int_M \delta_b\sigma\theta \wedge d\theta = 0.$$

It follows that the formal adjoint of ∂_b on functions with respect to the Levi form and the volume element $\theta \wedge d\theta$ is $\partial_b^* = -\delta_b$. The Kohn Laplacian on functions is given by the expression,

$$\Box_b = 2\bar{\partial}_b^*\bar{\partial}_b.$$

Define

(1.5) $$P_3\varphi = (\varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1)\theta^1$$

(see [14]) which is an operator whose vanishing characterizes CR-pluriharmonic functions.

We also define $\overline{P}_3\varphi = (\varphi_1{}^1{}_{\bar{1}} - iA_{\bar{1}\bar{1}}\varphi^{\bar{1}})\theta^{\bar{1}}$, the conjugate of P_3.

DEFINITION 1.1. The CR Paneitz operator P_4 is defined by

(1.6) $$P_4\varphi = \delta_b(P_3\varphi).$$

More explicitly, define Q by $Q\varphi = 2i(A^{11}\varphi_1)_{,1}$, then

$$\begin{aligned}
P_4\varphi &= \frac{1}{4}(\Box_b\overline{\Box}_b - 2Q)\varphi \\
&= \frac{1}{4}(\Box_b\overline{\Box}_b\varphi - 4i(A^{11}\varphi_1)_1) \\
&= \frac{1}{8}((\overline{\Box}_b\Box_b + \Box_b\overline{\Box}_b)\varphi + 8Im(A^{11}\varphi_1)_1).
\end{aligned}$$

By the commutation relation $[\Box_b, \overline{\Box}_b] = 4iImQ$, we see that $4P_4 = \Box_b\overline{\Box}_b - 2Q = \overline{\Box}_b\Box_b - 2\overline{Q}$. It follows that P_4 is a real and symmetric operator (see [4] for details).

DEFINITION 1.2. We say the Paneitz operator P_4 is nonnegative if and only if

$$\int_M (P_4\varphi)\bar{\varphi} \geq 0,$$

for all smooth functions φ. We use the notation $P_4 \geq 0$ to denote non-negative Paneitz operators.

Note that the nonnegativity of P_4 is a CR invariant in the sense that it is independent of the choice of the contact form θ. This follows by observing that if $\widetilde{\theta} = e^{2f}\theta$ be another contact form, we have the following transformation laws for the volume form and the CR Paneitz operator respectively (see Lemma 7.4 in [12]):

$$\widetilde{\theta} \wedge d\widetilde{\theta} = e^{4f}\theta \wedge d\theta; \qquad \widetilde{P_4} = e^{-4f}P_4.$$

In the higher dimensional case, there exists an analog of P_4 which however seems not to satisfy the covariant property. In this case, Graham and Lee, in [11], had shown the nonnegativity of P_4. To be specific, non-negativity of P_4 is a condition in dimension three but it is a given in higher dimensions. Moreover the invariance property for the Paneitz discussed above does not hold in dimensions five and higher.

We will restrict ourselves exclusively to the three dimensional case in our paper. We next observe that when the Webster torsion $A_{11} \equiv 0$, then the Paneitz operator P_4 is given by,

$$(1.7) \qquad\qquad\qquad P_4 = \frac{1}{4}\Box_b\overline{\Box}_b.$$

It follows that the vanishing of torsion implies that $P_4 \geq 0$. This is because when the torsion vanishes identically, the two operators $\Box_b$ and $\overline{\Box}_b$ commute, and hence are simultaneously diagonalizable on each eigenspace of $\Box_b$ of a nonzero eigenvalue(see [4]). We also recall that the vanishing of torsion is equivalent to $L_T J = 0$ where L is the Lie derivative, see [18]. We summarize a part of the facts above as a proposition, which will prove useful later.

PROPOSITION 1.3. *Let the Webster torsion tensor identically vanish, i.e. $A^{11} \equiv 0$. Then,*

$$(1.8) \qquad\qquad \ker P_4 = \ker P_3 = CR\text{-pluriharmonic functions.}$$

Moreover one has,

$$(1.9) \qquad\qquad\qquad P_4 \geq 0.$$

It remains an interesting problem to determine the precise geometrical condition under which the kernel of the Paneitz operator is exactly the pluri-harmonic functions or even a direct sum of a finite dimensional subspace with the pluri-harmonic functions.

DEFINITION 1.4. Suppose that $\widetilde{\theta} = e^{2f}\theta$. The CR Yamabe constant is defined by

$\inf_{\widetilde{\theta}} \{\int_M \widetilde{R}\, \widetilde{\theta} \wedge d\widetilde{\theta} : \int \widetilde{\theta} \wedge d\widetilde{\theta} = 1\}$.

The CR Yamabe constant is a CR invariant.

We now come to the primary results of our paper. To motivate the results, it is helpful to recall the main result in our earlier paper [6].

THEOREM 1.5. Let M^3 be a closed CR manifold.
(a) If $P_4 \geq 0$ and $R > 0$, then the non-zero eigenvalues λ of $\Box_b$ satisfy

$$\lambda \geq \min R.$$

It follows the range of $\Box_b$ is closed. Coupled with the result of Kohn stated above, under the conditions $P_4 \geq 0$ and $R > 0$, M globally embeds into some $\mathbb{C}^n$.
(b) A consequence of part (a) is that: If $P_4 \geq 0$ and the CR Yamabe constant > 0, then M^3 can be globally embedded into $\mathbb{C}^n$, for some n.

Our aim is to investigate a converse to the theorem stated above. More specifically we want to know if for embedded structures, the CR Paneitz operator is non-negative. We recall the following example due to Grauert, Andreotti-Siu [1] and Rossi [17] and referred to as Rossi's example in the literature [7].

EXAMPLE 1.6. On the standard sphere (S^3, J_0), we consider the deformation J_t given by the vector field $Z_{\bar{1}} + tZ_1$, with $t \in \mathbb{R}$ and $|t| \neq 0, 1$. This structure fails to embed globally since it is known that the CR functions for this structure are even. We note $u_t = z_1$ (which is an odd function) is a continuous family of eigenfunctions for P_4^t with eigenvalue $\lambda(t) = \frac{-3t^2}{(1-t^2)^2}$. This means that we are unable to find a CR function φ_t for the CR structure (S^3, J_t) which is as close to $u_0 = z_1$ as we please. The key observation is that the Paneitz operator is negative and the structure fails to embed.

The example thus suggests that indeed it is possible that for embedded structures the CR Paneitz operator may indeed be non-negative. The main result in Section 2 of our paper is a result that ensures non-negativity of the Paneitz operator, for CR structures embedded in $\mathbb{C}^2$ along a deformation path that is real-analytic. More precisely, we are given a triple (M, J_0, θ), the background CR structure. This CR structure is given to be embedded in $\mathbb{C}^2$. Now we deform the almost complex structure J_0 via a real-analytic path J_t, keeping of course the contact form θ fixed. That is each CR structure along the path of deformation J_t is smooth for fixed t, but the dependence is real-analytic in the variable t. In the sequel when we perform deformations, the Paneitz operator associated to the deformed structures J_t will be denoted by P_4^t. The Paneitz operator for the reference structure J_0 will be denoted by P_4 instead of P_4^0.

THEOREM 1.7. Let (M, J_0, θ) be a CR structure that is embedded in $\mathbb{C}^2$. Let J_t be a deformation from J_0 along an embeddable direction with real-analytic dependence on the deformation parameter t. Assume, each structure J_t for fixed t is

smooth and embedded in $\mathbb{C}^2$. Let P_4^t denote the Paneitz operator associated to the structure (M, J_t, θ). Assume further that the Paneitz operator at $t = 0$, $P_4 (= P_4^0)$ is non-negative and

$$\ker P_4 = CR\text{-pluriharmonic functions.}$$

Then for some $\delta > 0$ and $|t| < \delta$ we have:

$$P_4^t \geq 0.$$

COROLLARY 1.8. *Under the hypothesis of zero torsion, $A^{11} \equiv 0$, the hypothesis of Theorem 1.7 are met by virtue of Proposition 1.3. Thus if the structure J_0 has zero torsion, it follows $P_4^t \geq 0$ for $|t| < \delta$.*

REMARK 1.9. Theorem 1.7 is a consequence of a local deformation theorem proved in Section 2. It is based in part on the stability of CR functions and a theorem of Lempert [15]. It is also important to note that in light of Rossi's example, the hypothesis that J_t is an embedded structure along the deformation path, cannot be removed.

REMARK 1.10. In the theorem above, we need to start deforming from a manifold which is embedded and whose CR Paneitz operator is non-negative. Examples of such manifolds are many. The sphere S^3 is such a manifold. The CR structure remains invariant under a circle action and as remarked above, this forces the CR structure to have vanishing torsion and so as observed above, the CR Paneitz operator for the sphere is non-negative.

The sphere is simply-connected. We can consider now the manifold for $(z, w) \in \mathbb{C}^2$ given by

$$|z|^2 + \frac{1}{|z|^2} + |w|^2 = 100.$$

It is evident that the CR structure is invariant under a circle action. It is also evident that the manifold is not simply connected. Thus this example provides an example of a starting structure that is not simply connected and has a non-negative Paneitz operator.

REMARK 1.11. A result in [5], Prop. 4.1 states that for embedded structures M:

$$(1.10) \qquad c \int_M |f|^2 \leq \int_M |P_4 f|^2$$

which is valid $\forall\ f \in (\ker P_4^t)^\perp$, with $c > 0$ and independent of f. That is P_4 has closed range for embedded structures. However it is not obvious that when one performs a deformation along embedded directions, the constant c in the inequality above stays uniformly positive. If one were to obtain a uniform positive lower bound for c along the deformation path, one would be able to improve the conclusion of Theorem 1.7 to a global result valid for all t in any compact interval containing $t = 0$

This brings us to the remaining part of the converse in Theorem 1.5. That is, do embedded structures have positive Yamabe constant or positive Webster curvature. This is unlikely globally but certainly true if the CR structures are small perturbations of the standard CR structure of S^3. This is just by continuity. In fact by continuity if one performs a small perturbation from any CR structure

whose Webster curvature is positive, the deformed CR structure does have positive Webster curvature. However, for one large class of important hypersurfaces in $\mathbb{C}^2$, the ellipsoids, we do show that the Webster curvature is positive no matter how much deformed the ellipsoid is. The principal result in Section 3 is:

THEOREM 1.12. *The Webster curvature for all ellipsoids is positive.*

We have been informed by Song-Ying Li that he too was aware of the theorem stated above.

Now we specialize the situation to S^3 and consider small deformations of the standard CR structure of the sphere. In particular our goal is to consider the deformed structure on S^3 given by,

$$Z_{\bar{1}}^t = Z_{\bar{1}}^{\phi_t} = F(Z_{\bar{1}} + t\phi Z_1),$$

where $F = (1 - t^2|\phi|^2)^{-1/2}$, $Z_{\bar{1}} = \bar{z}_2 \frac{\partial}{\partial Z_1} - \bar{z}_1 \frac{\partial}{\partial Z_2}$ and $t \in (-\epsilon, \epsilon)$. The factor F is introduced to normalize the Levi form so that $h_{1\bar{1}} \equiv 1$. The CR Paneitz operator for the deformed structure will be denoted by P_0^t. We now consider the 3-sphere $S^3 \subset \mathbb{C}^2 \ni (z_1, z_2)$ and denote by

$$P_{p,q} = \text{span}\{z_1^a z_2^b \bar{z}_1^c \bar{z}_2^d | a + b = p, \ c + d = q\}$$

and the spherical harmonics

$$H_{p,q} = \{f \in P_{p,q}| - \Delta_{s^3} f = (p+q)(p+q+2)f\}.$$

For a given $\phi \in C^\infty(S^3)$ one has the Fourier representation

$$\phi \sim \sum \phi_{pq}$$

where ϕ_{pq} is the projection of ϕ onto $H_{p,q}$.

DEFINITION 1.13. We say ϕ satisfies condition (BE) if and only if

$$\phi_{pq} \equiv 0 \text{ for } p < q + 4, \ q = 0, 1, \cdots.$$

REMARK 1.14. Since for $p > q$

$$P_{p,q} = H_{p,q} \oplus \cdots \oplus H_{p-q,0}.$$

It follows that if $\phi \in P_{p,q}$, then ϕ satisfies (BE) if and only if $p \geq q+4$. Furthermore, the example of Rossi corresponds to $\phi = 1$ and thus fails condition (BE).

Burns and Epstein proved in [3] that for $t \in (-\epsilon, \epsilon)$ and ϕ satisfying (BE) the CR structure embeds into some $\mathbb{C}^n$. Conversely Bland [2] showed that embeddability of a CR structure close to the standard structure on S^3 implies condition (BE). To summarize we have

Theorem [**Burns-Epstein-Bland**]. A CR structure close to the standard structure on S^3 is embeddable if and only if ϕ satisfies condition (BE).

One of the results proved in Section 2 our paper, which is obtained by combining the results in our earlier paper [6], Theorem 1.5 with the results obtained in Section 2 of this paper and the theorem of Burns-Epstein-Bland cited above is:

THEOREM 1.15. *Let us consider the three sphere S^3 and a CR structure J_t obtained as a small perturbation of the standard CR structure on S^3 and whose CR vector field is given by $Z_{\bar{1}}^t$ above. Then the following are equivalent.*

(1) *The CR structure embeds in $\mathbb{C}^2$.*
(2) *$\Box_b^t$, the Kohn Laplacian for the deformed structure has closed range.*
(3) *The deformation function $\phi(\cdot)$ used to define the CR vector field $Z_{\bar{1}}^t$, satisfies the Burns-Epstein condition (BE).*
(4) *The CR Paneitz operator P_4^t for the deformed structure is non-negative and the Yamabe constant for the deformed structure is positive.*

As pointed out earlier, the Yamabe constant is positive for the deformed structure and follows simply by continuity and the fact we are only making a small deformation of the standard structure on S^3. The Yamabe constant is of course positive for the standard CR structure on S^3.

Acknowledgment

2. Small deformations of a CR structure

In the sequel we will always assume $D \subset \mathbb{C}^2$ is a strictly pseudoconvex, bounded domain with $(M, J_0) = \partial D$, in particular M is compact. Suppose that J_t be a deformation from J_0 defined by a family of smooth functions in the coordinate variable of the manifold denoted by $\cdot$ and real analytic in the deformation parameter variable t. The deformation functions on M will be denoted by $\psi(\cdot, t)$. That is, the vector field $\overline{Z}_1^t = \overline{Z}_1 + \psi(\cdot, t)Z_1$ defines a CR holomorphic vector field with respect to J_t. We also fix notation and denote the CR Paneitz operator wrt to the background CR structure J_0 as P_4 instead of P_4^0.

We now define the notion of stability for the Paneitz operator.

DEFINITION 2.1. We say the Paneitz operator $P_4^{t_0}$ associated to the CR structure J_{t_0} is stable, if given $\epsilon > 0$, there exists $\delta > 0$ such that for all t such that $|t - t_0| < \delta$, and given any $f \in \ker P_4^{t_0}$, there exists $g \in \ker P_4^t$ such that

$$||f - g||_{C^0(M)} < \epsilon.$$

There is a similar notion for the stability of CR functions. Stability of CR functions was established in a paper by Lempert [15].

The proof of the next proposition was communicated to us by C. Epstein [9]. For our purposes we need the projection operators constructed in Prop. (8.18) in [8], except for the zero eigenspace, to be continuous even at $t = 0$. This is the content of the following proposition. To state the lemma we need a few facts. We consider a family L_t of operators on M, that is holomorphic in $t \in \mathbb{C}$ for $|t| < \delta$. For real t we assume that the operators L_t are Hermitian with respect to $L^2(M)$ defined using a fixed measure independent of t which for our purposes is $\theta \wedge d\theta$. Our operators L_t are densely defined on $C^\infty(M)$ and the examples we need them for are Kohn's Laplacian $\Box_b^t$ and P_0^t. We assume moreover that

(1) Each L_t has closed range.
(2) Each L_t has pure point, discrete eigenvalues with finite dimensional eigenspaces.

(3) In particular it follows from the above two assumptions that for each L_t we do not have non-zero eigenvalues with zero as limit point.

(4) We assume the spectrum of L_t is bounded below.

Since we will apply the proposition to families of Paneitz operators P_0^t associated to embedded families of CR structures (M, J_t) and associated Kohn Laplacians $\Box_b^t$, we note that the closed range hypothesis for embedded structures is satisfied for P_0^t by a result in [5] and for $\Box_b^t$ by a result in [13]. Now further assume there exists $r > 0$, such that

$$((-r, 0) \cup (0, r)) \cap \text{spectrum } L_0 = \emptyset.$$

Non-zero eigenvalues of L_t that lie in $(-r, r)$ will be called small, using the terminology of [8].

PROPOSITION 2.2. *Let L_z be a holomorphic family as above. Then the small eigenvalues of L_t are finitely many and depend real-analytically on t for $t \in (-\delta, \delta)$. The projection $\mathbb{P}_i^t$ into the eigenspace for the small eigenvalue $\lambda_i(t)$ of L_t depends real-analytically on $t \in (-\delta, \delta)$. Moreover if $\mathbb{P}^t$ denotes the projection into the small eigenvalues, then the rank of $\mathbb{P}^t$ is constant in t.*

PROOF. In Proposition (8.18) [8], the analytic dependence of the small eigenvalues is already established. What remains to be proven is the second part of our lemma. Recall the definition of P_i^t eqn. (8.23) in [8] which is,

(2.1) $$P_i^t = \lambda_i(t)\Pi_{j \neq i}(\lambda_i(t) - \lambda_j(t))\mathbb{P}_i^t.$$

For $u, v \in L^2(M)$, define the function $g(t)$

$$g(t) = \frac{< P_i^t u, v >}{\lambda_i(t)\Pi_{j \neq i}(\lambda_i(t) - \lambda_j(t))}.$$

Then $g(t)$ is holomorphic in a punctured nbhd. of $t = 0$. The function $g(t)$ can have only poles of finite order as singularities at $t = 0$ and on the real axis via (2.1), for $|t| < \varepsilon$ the function $g(t)$ is bounded. Thus the singularity at $t = 0$ is removable and then arguing now as the rest of Proposition (8.18) in [8] we conclude that the projection operators are real-analytic and converges to a finite rank projection operator at t=0. Since

$$rank \ \mathbb{P}^t = trace \ \mathbb{P}^t,$$

we obtain the integer valued function $rank \ \mathbb{P}^t$ is continuous and hence constant. $\square$

PROPOSITION 2.3. *Suppose (M, J_0) is embedded in $\mathbb{C}^n$. Let J_t be a deformation from J_0 along an embeddable direction, with t varying real-analytically and $|t| < \delta$. Let $P_4 \geq 0$ and assume further that the CR Paneitz operator P_4 for the structure J_0 is stable. Then P_4^t cannot have small eigenvalues. In particular there does not exist any continuous family of eigenfunctions u_t corresponding to non-zero eigenvalues of P_4^t branching out from a function u_0 in the kernel of P_4. One therefore concludes $P_4^t \geq 0$.*

PROOF. We argue by contradiction. Assume P_4^t has small eigenvalues. Then by Prop.2.2, the eigenvalues vary continuously in t and the projection operators to these non-zero eigenvalues $\mathbb{P}_i^t$ are also continuous. From the continuous dependence of $\mathbb{P}_i^t$ and $\lambda_i(t)$ on t we conclude that any eigenfunction u_t for a non-zero small eigenvalue can be written as $u_t = u_0 + f_t$, where u_0 is in the kernel of P_4, and $||f_t||_2 = o(1)$. We normalize $||u_0||_2 = 1$. From our stability assumption, there exists a function g_t in ker P_4^t such that $||u_0 - g_t|| < \epsilon$. Now $< u_t, g_t >= 0$ as they

are eigenfunctions for distinct eigenvalues of P_4^t. Thus for each $t \neq 0$ small enough we have

(2.2)
$$
\begin{aligned}
0 &=< u_t, g_t > \\
&=< u_t, u_0 > + < u_t, g_t - u_0 > \\
&= 1+ < f_t, u_0 > + < u_t, g_t - u_0 >= 1 + o(1) \neq 0,
\end{aligned}
$$

which is a contradiction. Thus there are no small eigenvalues of P_4^t. The operator $P_4 \geq 0$ by assumption. Thus it follows that $P_4^t \geq 0$ for $|t| < \delta$. □

We are now in a position to supply the proof of Theorem 1.7. It is a consequence of the next proposition.

PROPOSITION 2.4. *Assume the kernel of P_4 consists of exactly the CR pluriharmonic functions for the structure (M, J_0). Assume the CR structures (M, J_t) are all embedded in $\mathbb{C}^2$. Then the Paneitz operator P_4 associated to the structure J_0 is stable.*

PROOF. By assumption any function $f \in \ker P_4$ is a CR pluriharmonic function. Locally then f is the real part of a CR holomorphic function F. We may now locally extend F into Ω where $M = \partial \Omega$. We continue to denote the extension by the symbol F. We now denote points in $\mathbb{C}^2$ by (z, w). Next note in Ω that $(Re\ F)_z$ is a holomorphic function defined globally in a nbhd of M in Ω. This is because $Re\ F = f$ is globally defined on M. Since M is connected, by Hartog's theorem we can even assume that $(Re\ F)_z$ is defined in all of Ω. Let us denote the restriction to M of $(Re\ F)_z$ by Ξ. Now Ξ is a CR function. We apply the stability theorem of Lempert [15] to obtain a function Ξ_t which is a CR function for the structure J_t and such that

$$||\Xi - \Xi_t||_\infty < \epsilon.$$

Being a CR function Ξ_t lies in the kernel of P_4^t. Next we consider the extension of Ξ_t to the interior as a holomorphic function. This globally exists by Hartog's theorem again. We continue to denote this extension by Ξ_t. Next we integrate Ξ_t in the z variable, that is we consider the indefinite integral

$$F_t(z, w) = \int \Xi_t\ dz.$$

There may be an ambiguity in the definition of F_t, because of imaginary periods but the Real part of F_t is well-defined. Set $f_t = Re\ F_t$. Then f_t is pluriharmonic and its restriction to M is CR-pluriharmonic. Similarly we also consider

$$H(z, w) = \int \Xi\ dz$$

Note that $H(z, w)$ may differ from F because of imaginary periods. But their real parts do coincide.

We now easily see using the the stability estimate above,

$$||f - f_t||_{L^\infty(M)} < \epsilon.$$

We have proved stability.

If M were simply connected then the proof of the proposition is quite easy, since then f being CR pluriharmonic can be taken to be the real part of a CR function G which is defined globally on M. One may then apply the result of Lempert on

stability of CR functions to G. The stability of the pluriharmonic functions follows by consideration of the real part. □

The proof of Theorem 1.7 follows because under the hypothesis of the theorem that the kernel of P_4 is exactly the CR pluri-harmonic functions, we obtain via Proposition 2.4 that the Paneitz operator P_4 is stable. Thus the hypothesis of Proposition 2.3 is satisfied and we may conclude that $P_4^t \geq 0$ for $|t| < \delta$.

Deformation functions $\psi(\cdot, t) = t\phi(\cdot)$ where $\phi \in C^\infty(S^3)$ have been studied in an important paper by Burns-Epstein[3]. The standard CR structure on S^3 has vanishing torsion. Thus combining the results in [3] and our Theorem 1.7 we also have:

COROLLARY 2.5. *Suppose that (S^3, J_0) is the sphere S^3 equipped with the standard CR structure and $\psi(\cdot, t) = t\phi(\cdot)$ is a deformation function where $\phi(\cdot)$ satisfies the Burns-Epstein condition. If we define the deformation J_t of the CR structures by $\psi(\cdot, t)$ then $P_4^t \geq 0$ for t small enough.*

The previous Corollary when combined with the results in [2], [3] and [6], easily yields Theorem 1.15 of the introduction.

3. The Webster curvature for Ellipsoids

In this section, we are going to show Theorem 1.12 of the introduction. We will need a formula for the Webster curvature for hypersurfaces embedded in $\mathbb{C}^2$ in a form suitable for our computations. Other formulae have been derived in [16], see Theorem 1.1 there.

Let $M \hookrightarrow \mathbb{C}^2$ be a hypersurface defined by a defining function $u(z_1, z_2)$:

$$M^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid u(z_1, z_2) = 0\},$$

where $du(z) \neq 0$ for all $z \in M$. Equipped with the induced CR structure from $\mathbb{C}^2$ and the contact form

$$\theta = \frac{i(\bar{\partial}u - \partial u)}{2}|_{M^3},$$

M is a pseudohermitian manifold, provided that $\theta \wedge d\theta \neq 0$. It is easy to see that the induced CR structure can be defined by the complex $(1,0)$-vector

$$(3.1) \qquad\qquad Z_1 = u_2 \frac{\partial}{\partial z_1} - u_1 \frac{\partial}{\partial z_2}.$$

We will use the notations:

$$u_j = \frac{\partial u}{\partial z_j}, \quad u_{jk} = \frac{\partial^2 u}{\partial z_j \partial z_k},$$

for all $j, k \in \{1, 2, \bar{1}, \bar{2}\}$. The characteristic vector field T is a real vector field which is uniquely defined by

$$(3.2) \qquad\qquad d\theta(T \wedge \cdot) = 0, \quad \theta(T) = 1.$$

Let $\{\theta^1, \theta^{\bar{1}}, \theta\}$ be the dual frame to $\{Z_1, Z_{\bar{1}}, T\}$. Then we have

$$(3.3) \qquad\qquad d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}},$$

for some nonzero real function $h_{1\bar{1}}$. If necessary, we could change the sign for u and assume, without loss of generality, that $h_{1\bar{1}} > 0$.

Let

(3.4)
$$J(u) = \begin{vmatrix} u & u_{\bar{1}} & u_{\bar{2}} \\ u_1 & u_{1\bar{1}} & u_{1\bar{2}} \\ u_2 & u_{2\bar{1}} & u_{2\bar{2}}. \end{vmatrix}$$

PROPOSITION 3.1. *On M^3, we have*

(3.5)
$$h_{1\bar{1}} = -J(u).$$

PROOF. We compute, on M,

(3.6)
$$\begin{aligned} d\theta &= -id(\partial u) \\ &= -id(\sum_{j=1}^{2} u_j dz_j) = -i(\sum_{j=1}^{2} du_j \wedge dz_j) \\ &= i\sum_{j,k=1}^{2} u_{jk} dz_j \wedge dz_k + i\sum_{j,k=1}^{2} u_{j\bar{k}} dz_j \wedge dz_{\bar{k}} \\ &= i\sum_{j,k=1}^{2} u_{j\bar{k}} dz_j \wedge dz_{\bar{k}}. \end{aligned}$$

Therefore

(3.7)
$$\begin{aligned} h_{1\bar{1}} &= -id\theta(Z_1 \wedge Z_{\bar{1}}) \\ &= \sum_{j,k=1}^{2} u_{j\bar{k}} dz_j \wedge dz_{\bar{k}}((u_2 \frac{\partial}{\partial z_1} - u_1 \frac{\partial}{\partial z_2}) \wedge (u_{\bar{2}} \frac{\partial}{\partial z_{\bar{1}}} - u_{\bar{1}} \frac{\partial}{\partial z_{\bar{2}}})) \\ &= u_{1\bar{1}} u_2 u_{\bar{2}} + u_{2\bar{2}} u_1 u_{\bar{1}} - u_{1\bar{2}} u_2 u_{\bar{1}} - u_{2\bar{1}} u_1 u_{\bar{2}} \\ &= -\begin{vmatrix} 0 & u_{\bar{1}} & u_{\bar{2}} \\ u_1 & u_{1\bar{1}} & u_{1\bar{2}} \\ u_2 & u_{2\bar{1}} & u_{2\bar{2}} \end{vmatrix} = -\begin{vmatrix} u & u_{\bar{1}} & u_{\bar{2}} \\ u_1 & u_{1\bar{1}} & u_{1\bar{2}} \\ u_2 & u_{2\bar{1}} & u_{2\bar{2}} \end{vmatrix} = -J(u), \quad \text{on } M. \end{aligned}$$

$\square$

Let

(3.8)
$$U = (U_{ab})_{3\times 3} = \begin{bmatrix} u & u_{\bar{1}} & u_{\bar{2}} \\ u_1 & u_{1\bar{1}} & u_{1\bar{2}} \\ u_2 & u_{2\bar{1}} & u_{2\bar{2}} \end{bmatrix}.$$

That is, $U_{ba} = \overline{U_{ab}}$, and

(3.9)
$$\begin{aligned} &U_{11} = u; \ U_{12} = u_{\bar{1}}; \ U_{13} = u_{\bar{2}}; \\ &U_{(j+1)(k+1)} = u_{j\bar{k}}, \quad 1 \le j, k \le 2. \end{aligned}$$

Note that $h_{1\bar{1}} > 0$, so the matrix U is invertible on a neighborhood of M. Let $U^{-1} = (U^{ab})$ be the inverse of U. Then it is easy to show that $U^{ba} = \overline{U^{ab}}$ and

(3.10)
$$\begin{aligned} U^{11} &= \frac{u_{1\bar{2}} u_{2\bar{1}} - u_{1\bar{1}} u_{2\bar{2}}}{h_{1\bar{1}}}; \ U^{12} = \frac{u_{2\bar{2}} u_{\bar{1}} - u_{2\bar{1}} u_{\bar{2}}}{h_{1\bar{1}}}; \ U^{13} = \frac{u_{1\bar{1}} u_{\bar{2}} - u_{1\bar{2}} u_{\bar{1}}}{h_{1\bar{1}}}; \\ U^{22} &= \frac{-u_2 u_{\bar{2}}}{h_{1\bar{1}}}; \ U^{23} = \frac{-u_1 u_{\bar{2}}}{h_{1\bar{1}}}; \ U^{33} = \frac{u_1 u_{\bar{1}}}{h_{1\bar{1}}}. \end{aligned}$$

PROPOSITION 3.2. *On M,*

$$
(3.11)\quad T = \sum_{j=1}^{2} iU^{1(j+1)}\frac{\partial}{\partial z_j} + \ complex\ conjugate
$$

$$
\theta^1 = U^{13}dz_1 - U^{12}dz_2.
$$

PROOF. We just check that T satisfies
$$
d\theta(T \wedge \cdot) = 0, \quad \theta(T) = 1.
$$
We compute

$$
\begin{aligned}
d\theta(T \wedge \cdot) &= i(\sum_{j,k=1}^{2} u_{j\bar{k}}dz_j \wedge dz_{\bar{k}})(T \wedge \cdot) \\
&= i\sum_{j,k=1}^{2} u_{j\bar{k}}(dz_j(T)dz_{\bar{k}} - dz_{\bar{k}}(T)dz_j) \\
&= -\sum_{j,k=1}^{2}(U_{(j+1)(k+1)}U^{1(j+1)}dz_{\bar{k}} + U_{(j+1)(k+1)}U^{(k+1)1}dz_j) \\
(3.12)\quad &= -\sum_{k=1}^{2}(\delta_{k3} - U_{1(k+1)}U^{11})dz_{\bar{k}} - \sum_{j=1}^{2}(\delta_{3j} - U^{11}U_{(j+1)1})dz_j \\
&= \sum_{j=1}^{2}U_{1(j+1)}U^{11}dz_{\bar{j}} + U^{11}U_{(j+1)1}dz_j \\
&= U^{11}(\sum_{j=1}^{2} u_j dz_j + u_{\bar{j}}dz_{\bar{j}}) \\
&= U^{11}du = 0, \quad \text{on } M,
\end{aligned}
$$

and

$$
\begin{aligned}
\theta(T) &= -i(\sum_{j=1}^{2} u_j dz_j)(T) \\
(3.13)\quad &= \sum_{j=1}^{2} u_j U^{1(j+1)} = \sum_{j=1}^{2} U_{(j+1)1}U^{1(j+1)} \\
&= \sum_{b=1}^{3} U_{b1}U^{1b}, \quad \text{on } M\ (U_{11} = 0, \text{ on } M) \\
&= 1
\end{aligned}
$$

Similarly, after a direct computation, we get $\theta^1(T) = 0$ and $\theta^1(Z_1) = 1$. $\square$

PROPOSITION 3.3. *With respect to the frame Z_1, the connection form $\theta_1{}^1$ and the torsion form τ^1 are expressed by*

$$
(3.14)\quad \theta_1{}^1 = (h^{1\bar{1}}Z_1 h_{1\bar{1}})\theta^1 + ((u_1 Tc_2 - u_2 Tc_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1))\theta,
$$

$$
\tau^1 = -i(c_1 Z_{\bar{1}}c_2 - c_2 Z_{\bar{1}}c_1)\theta^{\bar{1}},
$$

where $h^{1\bar{1}} = h_{1\bar{1}}^{-1}$, $c_1 = U^{13}$ and $c_2 = -U^{12}$.

PROOF. First we point out that all equalities are only true on M. Now let $\theta^1 = c_1 dz_1 + c_2 dz_2$, i.e., $c_1 = U^{13}$ and $c_2 = -U^{12}$. We have that $1 = \theta^1(Z_1) = c_1 u_2 - c_2 u_1$. Therefore

$$(3.15) \quad \begin{aligned} u_2 \theta^1 &= u_2 c_1 dz_1 + u_2 c_2 dz_2 \\ &= dz_1 + c_2 u_1 dz_1 + u_2 c_2 dz_2 \\ &= dz_1 + c_2(u_1 dz_1 + u_2 dz_2) \\ &= dz_1 + ic_2\theta, \end{aligned}$$

or

$$(3.16) \quad dz_1 = u_2\theta^1 - ic_2\theta,$$

hence,

$$(3.17) \quad \begin{aligned} 0 = d(dz_1) &= d(u_2\theta^1 - ic_2\theta) \\ &= du_2 \wedge \theta^1 + u_2 d\theta^1 - idc_2 \wedge \theta - ic_2 d\theta, \end{aligned}$$

or

$$(3.18) \quad u_2 d\theta^1 = -du_2 \wedge \theta^1 + idc_2 \wedge \theta + ic_2 d\theta.$$

Similarly, we have

$$(3.19) \quad dz_2 = -u_1\theta^1 + ic_1\theta,$$

and thus,

$$(3.20) \quad u_1 d\theta^1 = -du_1 \wedge \theta^1 + idc_1 \wedge \theta + ic_1 d\theta.$$

Taking together (3.18) and (3.20), one obtains that

$$(3.21) \quad \begin{aligned} d\theta^1 &= (c_1 u_2 - c_2 u_1) d\theta^1 \\ &= \theta^1 \wedge (c_1 du_2 - c_2 du_1) + \theta \wedge i(c_2 dc_1 - c_1 dc_2) \end{aligned}$$

On the other hand, we have

$$(3.22) \quad d\theta^1 = \theta^1 \wedge \theta_1{}^1 + \theta \wedge \tau^1.$$

From (3.21), (3.22) and by the Cartan lemma, there exists functions a, b and c such that

$$(3.23) \quad \begin{aligned} \theta_1{}^1 &= c_1 du_2 - c_2 du_1 + a\theta^1 + b\theta \\ \tau^1 &= i(c_2 dc_1 - c_1 dc_2) + b\theta^1 + c\theta. \end{aligned}$$

Since $\tau^1 = A^1{}_{\bar{1}}\theta^{\bar{1}}$, from (3.23), this means that

$$(3.24) \quad \begin{aligned} A^1{}_{\bar{1}} &= i(c_2 Z_{\bar{1}} c_1 - c_1 Z_{\bar{1}} c_2), \\ b &= -i(c_2 Z_1 c_1 - c_1 Z_1 c_2), \\ c &= -i(c_2 T c_1 - c_1 T c_2), \end{aligned}$$

hence,

$$(3.25) \quad \theta_1{}^1 = c_1 du_2 - c_2 du_1 + a\theta^1 - i(c_2 Z_1 c_1 - c_1 Z_1 c_2)\theta.$$

Finally, from the structural equation $h^{1\bar{1}} dh_{1\bar{1}} = \theta_1{}^1 + \theta_{\bar{1}}{}^{\bar{1}}$, we get

$$(3.26) \quad a = c_2 Z_1 u_1 - c_1 Z_1 u_2 + h^{1\bar{1}} Z_1 h_{1\bar{1}},$$

hence,

(3.27)
$$\theta_1{}^1 = c_1 du_2 - c_2 du_1 + (c_2 Z_1 u_1 - c_1 Z_1 u_2 + h^{1\bar{1}} Z_1 h_{1\bar{1}})\theta^1 - i(c_2 Z_1 c_1 - c_1 Z_1 c_2)\theta$$
$$= (h^{1\bar{1}} Z_1 h_{1\bar{1}})\theta^1 + ((u_1 T c_2 - u_2 T c_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1))\theta,$$

where in the last equality, we used the identities $u_1 Z_{\bar{1}} c_2 - u_2 Z_{\bar{1}} c_1 = 0$ and $c_1 u_2 - c_2 u_1 = 1$. $\square$

PROPOSITION 3.4. *The Webster curvature can be expressed as*

(3.28) $$R = -h^{1\bar{1}}(Z_{\bar{1}} Z_1 \log h_{1\bar{1}}) + i\Big((u_1 T c_2 - u_2 T c_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1)\Big).$$

PROOF. Let $E = (u_1 T c_2 - u_2 T c_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1)$. Taking the exterior differential of $\theta_1{}^1$

$$d\theta_1{}^1 = d(Z_1 \log h_{1\bar{1}}) \wedge \theta^1 + (Z_1 \log h_{1\bar{1}})d\theta^1 + dE \wedge \theta + Ed\theta$$

(3.29)
$$= (-Z_{\bar{1}} Z_1 \log h_{1\bar{1}} + \theta_1{}^1(Z_{\bar{1}})(Z_1 \log h_{1\bar{1}}) + ih_{1\bar{1}}E)\theta^1 \wedge \theta^{\bar{1}}, \text{ mod } \theta$$

$$= (-Z_{\bar{1}} Z_1 \log h_{1\bar{1}} + ih_{1\bar{1}}E)\theta^1 \wedge \theta^{\bar{1}}, \text{ mod } \theta.$$

Comparing (3.29) with the structure equation $d\theta_1{}^1 = h_{1\bar{1}} R \theta^1 \wedge \theta^{\bar{1}}$, mod θ, we immediately get formula (3.28) for the Webster curvature. $\square$

Finally combining (3.14) and (3.28), we get another representation for the connection form

(3.30) $$\theta_1{}^1 = (Z_1 \log h_{1\bar{1}})\theta^1 - i\left(R + h^{1\bar{1}}(Z_{\bar{1}} Z_1 \log h_{1\bar{1}})\right)\theta.$$

REMARK 3.5. There is another expression for the Webster curvature, which was proved by S.-Y. Li and H.-S. Luk in [16]. It is

(3.31) $$R = -h^{1\bar{1}} \sum_{j,k=1}^{2} \frac{\partial^2 \log(-J(u))}{\partial z_j \partial z_{\bar{k}}} w^j w^{\bar{k}} + 2\frac{\det H(u)}{h_{1\bar{1}}}.$$

Next an ellipsoid is given by

(3.32) $$A_1 x_1{}^2 + B_1 y_1{}^2 + A_2 x_2{}^2 + B_2 y_2{}^2 - 1 = 0,$$

where $A_1, A_2, B_1, B_2 > 0$. Set $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then our defining function becomes

(3.33) $$u = b_1 |z_1|^2 + b_2 |z_2|^2 + a_1 z_1{}^2 + a_1 \bar{z}_1{}^2 + a_2 z_2{}^2 + a_2 \bar{z}_2{}^2 - 1 = 0,$$

where $a_j = \frac{1}{4}(A_j - B_j)$, $b_j = \frac{1}{2}(A_j + B_j) > 0$, $j = 1, 2$. We want to make use of the formula for Webster curvature (3.28)

$$R = -h^{1\bar{1}} Z_{\bar{1}} Z_1 (\log h_{1\bar{1}}) + i\big[(u_1 T c_2 - u_2 T c_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1)\big],$$

where we recall $Z_1 = u_2 \frac{\partial}{\partial z_1} - u_1 \frac{\partial}{\partial z_2}$ and $\theta = \frac{1}{2i}(\partial u - \bar{\partial} u)$. The functions c_1, c_2 satisfy the identities:

(3.34) $$Z_{\bar{1}} u_1 = h_{1\bar{1}} c_1, \text{ and } Z_{\bar{1}} u_2 = h_{1\bar{1}} c_2.$$

So,

(3.35)
$$Z_1 Z_{\bar{1}} u_1 = Z_1(h_{1\bar{1}})c_1 + h_{1\bar{1}} Z_1 c_1$$
$$Z_1 Z_{\bar{1}} u_2 = Z_1(h_{1\bar{1}})c_2 + h_{1\bar{1}} Z_1 c_2.$$

Multiplying the first equation in (3.35) by c_2 and the second by c_1 and subtracting, we get

$$(3.36) \qquad h_{1\bar{1}}(c_2 Z_1 c_1 - c_1 Z_1 c_2) = c_2 Z_1 Z_{\bar{1}} u_1 - c_1 Z_1 Z_{\bar{1}} u_2.$$

So,

(3.37)
$$\begin{aligned}
c_2 Z_1 c_1 - c_1 Z_1 c_2 &= \frac{c_2 Z_1 Z_{\bar{1}} u_1 - c_1 Z_1 Z_{\bar{1}} u_2}{h_{1\bar{1}}} \\
&= \frac{c_2([Z_1, Z_{\bar{1}}]u_1) - c_1([Z_1, Z_{\bar{1}}]u_2)}{h_{1\bar{1}}} + \frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}} \\
&= \frac{-i h_{1\bar{1}}(c_2 T u_1 - c_1 T u_2)}{+} \frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}} \\
&= i(c_1 T u_2 - c_2 T u_1) + \frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}}.
\end{aligned}$$

Next note $\theta^1(Z_1) = c_1 u_2 - u_1 c_2 = 1$. Thus, $T(c_1 u_2 - u_1 c_2) = 0$. So we have,

$$(3.38) \qquad u_2 T c_1 - u_1 T c_2 = c_2 T u_1 - c_1 T u_2.$$

From (3.37) and (3.38) we get,

(3.39)
$$\begin{aligned}
&(u_1 T c_2 - u_2 T c_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1) \\
&= (c_1 T u_2 - c_2 T u_1) + i(-i)(c_1 T u_2 - c_2 T u_1) - i\left(\frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}}\right) \\
&= 2(c_1 T u_2 - c_2 T u_1) - i\left(\frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}}\right).
\end{aligned}$$

Substituting (3.39) into the Webster curvature formula (3.28), we get

(3.40)
$$\begin{aligned}
R &= -h^{1\bar{1}} Z_{\bar{1}} Z_1 (\log h_{1\bar{1}}) + i\big[(u_1 T c_2 - u_2 T c_1) + i(c_1 Z_1 c_2 - c_2 Z_1 c_1)\big] \\
&= -h^{1\bar{1}} Z_{\bar{1}} Z_1 (\log h_{1\bar{1}}) + 2i(c_1 T u_2 - c_2 T u_1) + \left(\frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}}\right).
\end{aligned}$$

We next compute an expression for the Levi form. We have,

(3.41)
$$\begin{aligned}
J(u) &= \begin{vmatrix} u & u_{\bar{1}} & u_{\bar{2}} \\ u_1 & u_{1\bar{1}} & u_{1\bar{2}} \\ u_2 & u_{\bar{1}2} & u_{2\bar{2}} \end{vmatrix} = \begin{vmatrix} u & u_{\bar{1}} & u_{\bar{2}} \\ u_1 & b_1 & 0 \\ u_2 & 0 & b_2 \end{vmatrix} \\
&= \begin{vmatrix} u & 2a_1 \bar{z}_1 + b_1 z_1 & 2a_2 \bar{z}_2 + b_2 z_2 \\ 2a_1 z_1 + b_1 \bar{z}_1 & b_1 & 0 \\ 2a_2 z_2 + b_2 \bar{z}_2 & 0 & b_2 \end{vmatrix} \\
&= b_1 b_2 u - b_1 u_2 u_{\bar{2}} - b_2 u_1 u_{\bar{1}}.
\end{aligned}$$

Thus when $u = 0$, one has

$$(3.42) \qquad h_{1\bar{1}} = b_1 u_2 u_{\bar{2}} + b_2 u_1 u_{\bar{1}}.$$

Next we compute the first term in the Webster curvature formula (3.40):

$$
(3.43) \quad -h^{1\bar{1}} Z_{\bar{1}} Z_1 (\log h_{1\bar{1}}) = -h^{1\bar{1}} Z_{\bar{1}} \left(\frac{Z_1 h_{1\bar{1}}}{h_{1\bar{1}}} \right)
$$

$$
= \frac{|Z_1 h_{1\bar{1}}|^2}{(h_{1\bar{1}})^3} - \frac{Z_{\bar{1}} Z_1 h_{1\bar{1}}}{(h_{1\bar{1}})^2}, \text{ using } h^{1\bar{1}} = \frac{1}{h_{1\bar{1}}}.
$$

A straightforward computation using the defining function yields,

$$
(3.44) \quad \begin{aligned} Z_{\bar{1}} u_2 &= -b_2 u_{\bar{1}}, \ Z_{\bar{1}} u_1 = b_1 u_{\bar{2}} \\ Z_{\bar{1}} u_{\bar{1}} &= 2a_1 u_{\bar{2}}, \ Z_{\bar{1}} u_{\bar{2}} = -2a_2 u_{\bar{1}}. \end{aligned}
$$

So,

$$
(3.45) \quad \begin{aligned} Z_{\bar{1}} Z_1 h_{1\bar{1}} &= 2a_1 b_2 Z_{\bar{1}}(u_2 u_{\bar{1}}) - 2a_2 b_1 Z_{\bar{1}}(u_1 u_{\bar{2}}) \\ &= 2a_1 b_2 [(Z_{\bar{1}} u_2) u_{\bar{1}} + u_2 (Z_{\bar{1}} u_{\bar{1}})] - 2a_2 b_1 [(Z_{\bar{1}} u_1) u_{\bar{2}} + u_1 (Z_{\bar{1}} u_{\bar{2}})] \\ &= 4a_1{}^2 b_2 |u_2|^2 + 4a_2{}^2 b_1 |u_1|^2 - 2a_1 b_2{}^2 (u_{\bar{1}})^2 - 2a_2 b_1{}^2 (u_{\bar{2}})^2. \end{aligned}
$$

Next we claim that

$$
(3.46) \quad 2i(c_1 T u_2 - c_2 T u_1) = \frac{2}{h_{1\bar{1}}^2} \left(b_1 b_2^2 |u_1|^2 + b_2 b_1^2 |u_2|^2 - 2a_1 b_2^2 |u_{\bar{1}}|^2 - 2a_2 b_1^2 |u_{\bar{1}}|^2 \right);
$$

and

$$
(3.47) \quad \frac{c_2 Z_{\bar{1}} Z_1 u_1 - c_1 Z_{\bar{1}} Z_1 u_2}{h_{1\bar{1}}} = \frac{2a_1 b_2^2 |u_{\bar{1}}|^2 + 2a_2 b_1^2 |u_{\bar{1}}|^2}{(h_{1\bar{1}})^2}
$$

These follow because,

$$
(3.48) \quad Z_{\bar{1}} Z_1 u_1 = Z_{\bar{1}}(2a_1 u_2) = -2a_1 b_2 u_{\bar{1}}.
$$

So,

$$
(3.49) \quad c_2 Z_{\bar{1}} Z_1 u_1 = \frac{-b_2 u_{\bar{1}}}{h_{1\bar{1}}}(-2a_1 b_2 u_{\bar{1}}) = \frac{2a_1 b_2^2 (u_{\bar{1}})^2}{h_{1\bar{1}}}.
$$

Similarly,

$$
(3.50) \quad c_1 Z_{\bar{1}} Z_1 u_2 = \frac{b_1 u_{\bar{2}}}{h_{1\bar{1}}}(-2a_2 b_1 u_{\bar{2}}) = \frac{-2a_2 b_1^2 (u_{\bar{2}})^2}{h_{1\bar{1}}}.
$$

Taking the two expressions above together, we get the second claim (3.47). To prove the first claim (3.46), we use

$$
(3.51) \quad T = \frac{i}{h_{1\bar{1}}} \left(b_2 u_{\bar{1}} \frac{\partial}{\partial z_1} + b_1 u_{\bar{2}} \frac{\partial}{\partial z_2} \right) + \text{ complex conjugate.}
$$

So

$$
(3.52) \quad \begin{aligned} T u_1 &= \frac{2i a_1 b_2 u_{\bar{1}}}{h_{1\bar{1}}} - \frac{i b_1 b_2 u_1}{h_{1\bar{1}}} \\ T u_2 &= \frac{2i a_2 b_1 u_{\bar{2}}}{h_{1\bar{1}}} - \frac{i b_1 b_2 u_2}{h_{1\bar{1}}}, \end{aligned}
$$

In conjunction with $c_1 = \frac{b_1 u_{\bar{2}}}{h_{1\bar{1}}}$, $c_2 = \frac{-b_2 u_{\bar{1}}}{h_{1\bar{1}}}$, we get the first claim (3.46). Now we substitute (3.45) into (3.43) and substitute (3.43), (3.46) and (3.47) into formula

(3.40), to obtain

$$(3.53) \quad \begin{aligned} R &= \frac{|Z_1 h_{1\bar{1}}|^2}{(h_{1\bar{1}})^3} + \frac{2b_1(b_2^2 - 2a_2^2)|u_1|^2 + 2b_2(b_1^2 - 2a_1^2)|u_2|^2}{(h_{1\bar{1}})^2} \\ &> 0, \end{aligned}$$

The last inequality is a consequence of $b_i^2 - 2a_i^2 > 0$, $i = 1, 2$. This follows because,

$$(3.54) \quad \begin{aligned} b_i^2 &= \frac{1}{4}(A_i + B_i)^2 \\ 2a_i^2 &= \frac{2}{16}(A_i - B_i)^2, \end{aligned}$$

hence $b_i^2 - 2a_i^2 = \frac{1}{8}(A_i^2 + B_i^2 + 6A_iB_i) > 0$, (note that $A_i, B_i > 0$).

4. Further Remarks

It remains an interesting problem to determine the precise geometrical condition when the notion of being in the kernel of P_4 coincides with CR-pluriharmonicity for a general CR structure. One problem of immediate interest is to determine if for embedded structures, the CR-pluriharmonic functions coincide with the functions in the kernel of the Paneitz operator. It is unclear if such an equivalence is true even for CR structures close to the standard structure on S^3. Under the assumption that the CR pluriharmonic functions coincide with the kernel of the Paneitz operator, C. R. Graham, K. Hirachi and J. M. Lee proved the theorem stated below. Thus our question has further geometric consequences beyond a possible link with embedding of CR structures.

THEOREM 4.1. *Let $\Omega \subset \mathbb{C}^2$ be a strictly pseudoconvex domain with a defining function u. Suppose $M = \partial\Omega$. Then the following are equivalent:*

(1) *$Q = 0$.*
(2) *u satisfies Fefferman's Monge-Ampere equation $-J(u) \equiv 1$ along M up to multiplication by a CR pluriharmonic function.*
(3) *$\theta \wedge d\theta$ is the invariant volume element up to multiplication by a CR pluriharmonic function.*

We recall that $J(u)$ is Fefferman's Monge-Ampere equation, which is defined by

$$J(u) = \det \begin{pmatrix} u_{1\bar{1}} & u_{1\bar{2}} & u_1 \\ u_{2\bar{1}} & u_{2\bar{2}} & u_2 \\ u_{\bar{1}} & u_{\bar{2}} & u \end{pmatrix}$$

In section 3, we showed that the Webster curvature for ellipsoids are positive. It is interesting to know if the Webster curvature is also positive for a strictly convex domain? If so, then from our earlier result in [**6**], there is an uniform positive lower bound for the first nonzero eigenvalues λ_t of the Kohn Laplacian $\Box_b^t$ for the family of strictly convex domains Ω_t, which is smoothly dependent on t.

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