EMBEDDABILITY FOR 3-DIMENSIONAL CAUCHY–RIEMANN MANIFOLDS AND CR YAMABE INVARIANTS

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Abstract

Let M^3 be a closed Cauchy–Riemann (CR) 3-manifold. In this article, we derive a Bochner formula for the Kohn Laplacian in which the pseudo-Hermitian torsion does not play any role. By means of this formula we show that the nonzero eigenvalues of the Kohn Laplacian have a positive lower bound, provided that the CR Paneitz operator is nonnegative and the Webster curvature is positive. This means that M^3 is embeddable when the CR Yamabe constant is positive and the CR Paneitz operator is nonnegative. Our lower bound estimate is sharp. In addition, we show that the embedding is stable in the sense of Burns and Epstein.

1. Introduction, statements, and notation

The global embedding problem in CR geometry in dimension 3 has received a lot of attention. In [5], Burns and Epstein consider perturbations of the standard CR structure on the 3-sphere S^3. They showed that the generic perturbation is nonembeddable and gave a sufficient condition for embeddability in terms of the Fourier coefficients of the perturbation function. For small perturbations, Bland [2] showed that this condition in some sense is necessary up to a contact diffeomorphism. Bland's results in [2] strongly depend on his earlier work [3] on moduli for pointed convex domains, which is closely related to Lempert's work (see [16], [17]) on constructing modular data for pointed convex domains.

In [18], Lempert asked two fundamental questions about the embeddability problem. The first one is related to the closedness property of CR structures, and the second one is related to the stability property. In this article, we introduce a CR invariant condition involving the sign of two conformally covariant operators to provide some

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affirmative answers to Lempert's questions. The first operator is the analogue of the conformal Laplacian that occurs in the computation of Webster's scalar curvature. The second, a fourth-order operator, is closely connected with pluriharmonic functions. As a consequence of our main result, after making further use of the sign conditions on these operators, it is possible to derive a positive mass theorem in this dimension (see [8]).

In [10, Definition (1.1)] and [11], Epstein introduced an invariant called *the relative index of two Szego projectors*. It quantifies the stability of the algebra of CR functions under embeddable deformations. It is used to show that every embeddable deformation of a CR structure on S^3 is stable. Our hypotheses in Theorem 3.1 and the result of Theorem 1.4 imply that the relative index is zero.

Throughout this paper, we will use the notation and terminology of [15] unless otherwise specified. Let (M, J, θ) be a closed 3-dimensional pseudo-Hermitian manifold, where θ is a contact form and where J is a CR structure compatible with the contact bundle $\xi = \ker\theta$. The CR structure J decomposes $\mathbb{C} \otimes \xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively. The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z, W\rangle_{L_\theta} = -i\langle d\theta, Z \wedge \overline{W}\rangle$. We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\langle \overline{Z}, \overline{W}\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form naturally induces a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $dV = \theta \wedge d\theta$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$; for example,

$$\langle \varphi, \psi\rangle = \int_M \varphi\bar{\psi}\, dV \tag{1.1}$$

for functions φ and ψ.

Let $\{T, Z_1, Z_{\bar{1}}\}$ be a frame of $TM \otimes \mathbf{C}$, where Z_1 is any local frame of $T_{1,0}$, $Z_{\bar{1}} = \overline{Z_1} \in T_{0,1}$, and where T is the characteristic vector field, that is, the unique vector field such that $\theta(T) = 1, d\theta(T, \cdot) = 0$. Then $\{\theta, \theta^1, \theta^{\bar{1}}\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, satisfies

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}} \tag{1.2}$$

for some positive function $h_{1\bar{1}}$. We can always choose Z_1 such that $h_{1\bar{1}} = 1$; hence, throughout this article, we will assume that $h_{1\bar{1}} = 1$

The pseudo-Hermitian connection of (J, θ) (see [20], [21]) is the connection ∇ on $TM \otimes \mathbf{C}$ (and extended to tensors) given in terms of a local frame $Z_1 \in T_{1,0}$ by

$$\nabla Z_1 = \theta_1{}^1 \otimes Z_1, \qquad \nabla Z_{\bar{1}} = \theta_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \qquad \nabla T = 0,$$

where $\theta_1{}^1$ is the 1-form uniquely determined by the following equations:

$$
\begin{aligned}
d\theta^1 &= \theta^1 \wedge \theta_1{}^1 + \theta \wedge \tau^1, \\
\tau^1 &\equiv 0 \mod \theta^{\bar{1}}, \\
0 &= \theta_1{}^1 + \theta_{\bar{1}}{}^{\bar{1}},
\end{aligned} \tag{1.3}
$$

where $\theta_1{}^1$ and τ^1 are called the *connection form* and the *pseudo-Hermitian torsion*, respectively. Put $\tau^1 = A^1{}_{\bar{1}}\theta^{\bar{1}}$. The structure equation for the pseudo-Hermitian connection is

$$
d\theta_1{}^1 = R\theta^1 \wedge \theta^{\bar{1}} + 2i\,Im(A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta),
$$

where R is the Tanaka–Webster curvature.

We denote components of covariant derivatives with indices preceded by a comma; thus we write $A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta$. The indices $\{0, 1, \bar{1}\}$ indicate derivatives with respect to $\{T, Z_1, Z_{\bar{1}}\}$. For derivatives of a scalar function, we will often omit the comma; for instance, we write $\varphi_1 = Z_1\varphi, \varphi_{1\bar{1}} = Z_{\bar{1}}Z_1\varphi - \theta_1^1(Z_{\bar{1}})Z_1\varphi, \varphi_0 = T\varphi$ for a (smooth) function.

Next we recall several natural differential operators occurring in this paper (for a detailed description, see [15]). For a smooth function φ, the Cauchy–Riemann operator ∂_b can be defined locally by

$$
\partial_b\varphi = \varphi_1\theta^1,
$$

and we write $\bar{\partial}_b$ for the conjugate of ∂_b. A function φ is called *CR holomorphic* if $\bar{\partial}_b\varphi = 0$. The divergence operator δ_b takes $(1,0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_1{}^1$, and similarly, $\bar{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1}}{}^{\bar{1}}$.

If $\sigma = \sigma_1\theta^1$ is compactly supported, then Stokes's theorem applied to the 2-form $\theta \wedge \sigma$ implies the divergence formula

$$
\int_M \delta_b\sigma\theta \wedge d\theta = 0.
$$

It follows that the formal adjoint of ∂_b on functions with respect to the Levi form and the volume element $\theta \wedge d\theta$ is $\partial_b^* = -\delta_b$. The Kohn Laplacian on functions determined by θ is

$$
\Box_b = 2\bar{\partial}_b^*\bar{\partial}_b.
$$

Define $P\varphi = (\varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1)\theta^1$ (see [15]), which is an operator that characterizes CR pluriharmonic functions, and define $\overline{P}\varphi = (\varphi_1{}^1{}_{\bar{1}} - iA_{\bar{1}\bar{1}}\varphi^{\bar{1}})\theta^{\bar{1}}$, the conjugate of P. The CR Paneitz operator P_0 is defined by

$$
P_0\varphi = \delta_b(P\varphi).
$$

More explicitly, define Q by $Q\varphi = 2i(A^{11}\varphi_1)_{,1}$; then

$$\begin{aligned} P_0\varphi &= \frac{1}{4}(\Box_b\overline{\Box}_b - 2Q)\varphi \\ &= \frac{1}{4}\big(\Box_b\overline{\Box}_b\varphi - 4i(A^{11}\varphi_1)_1\big) \\ &= \frac{1}{8}\big((\overline{\Box}_b\Box_b + \Box_b\overline{\Box}_b)\varphi + 8Im(A^{11}\varphi_1)_1\big). \end{aligned}$$

By the commutation relation $[\Box_b, \overline{\Box}_b] = 4i\,ImQ$, we see that $4P_0 = \Box_b\overline{\Box}_b - 2Q = \overline{\Box}_b\Box_b - 2\overline{Q}$. It follows that P_0 is a real and symmetric operator (see [9] for the details).

Definition 1.1
The Paneitz operator P_0 is said to be nonnegative if (denoted by $P_0 \geq 0$)

$$\int_M (P_0\varphi)\bar{\varphi}\theta \wedge d\theta \geq 0$$

for all smooth functions φ.

Note that the nonnegativity of P_0 is a CR invariant in the sense that it is independent of the choice of the contact form θ. This follows by observing that if $\widetilde{\theta} = e^{2f}\theta$ is another contact form, we have the following transformation laws for the volume form and the CR Paneitz operator, respectively (see [13, Lemma 7.4]):

$$\widetilde{\theta} \wedge d\widetilde{\theta} = e^{4f}\theta \wedge d\theta, \qquad \widetilde{P_0} = e^{-4f}P_0.$$

We also observe that when the Webster torsion $A_{11} \equiv 0$, then the Paneitz operator P_0 is given by

$$P_0 = \frac{1}{4}\Box_b\overline{\Box}_b.$$

When the torsion vanishes, the two operators $\Box_b$ and $\overline{\Box}_b$ commute, and hence are simultaneously diagonalizable. It follows that $P_0 \geq 0$. We also recall that the vanishing of torsion is equivalent to $L_T J = 0$, where L is the Lie derivative. As a consequence, the CR structures with a transverse symmetry admit a contact form with vanishing torsion.

In the higher-dimensional case, there exists an analogue of P_0. In this case, Graham and Lee [12] have shown the nonnegativity of P_0.

Definition 1.2
Suppose that $\widetilde{\theta} = e^{2f}\theta$. The CR Yamabe constant is defined by

$$\inf_{\widetilde{\theta}}\left\{\int_M \widetilde{R}\widetilde{\theta}\wedge d\widetilde{\theta} : \int \widetilde{\theta}\wedge d\widetilde{\theta}=1\right\}.$$

The CR Yamabe constant is a CR invariant. We also remind the reader of the following result of Kohn.

Remark 1.3
The fact that $\Box_b$ having closed range is equivalent to global embedding is a result of Kohn [14, Theorem 5.2].

We are now in a position to state our main theorems. In this article, we show the following.

THEOREM 1.4
Let M^3 be a closed CR manifold.
(a) *If $P_0 \geq 0$ and $R > 0$, then the nonzero eigenvalues λ of $\Box_b$ satisfy*

$$\lambda \geq \min R.$$

It follows that the range of $\Box_b$ is closed. Coupled with the result of Kohn stated above, under the conditions $P_0 \geq 0$ and $R > 0$, M globally embeds into some $\mathbb{C}^n$.
(b) *A consequence of part (a) is that if $P_0 \geq 0$ and the CR Yamabe constant is positive, then M^3 can be globally embedded into $\mathbb{C}^n$ for some n.*

Lempert's embedding result (see [16, Theorem 2.1]) states that CR structures with transverse symmetry are embeddable. It follows from Lempert's assumption that there is a contact form with vanishing torsion; hence $P_0 \geq 0$. Therefore, in the special case that the CR structure has positive Yamabe invariant, the embeddability follows from our main result.

In Section 3, we also prove a stability theorem for our embedding.

Remark 1.5
If M is embeddable, then the Paneitz operator P_0 has closed range (see [6]). In particular, a consequence of the Paneitz operator being closed for embeddable structures is that the Paneitz operator has at most finitely many negative eigenvalues for an embeddable CR structure. It appears highly likely that the boundary of strictly pseudoconvex domains in $\mathbb{C}^2$ have the property that $P_0 \geq 0$. We hope to return to this question in a future publication.

2. The embedding criterion

In this section, we derive the Bochner formula for the Kohn Laplacian. We need some commutation relations, for which we refer the reader to Lee [15]. This formula contains no term related to pseudo-Hermitian torsion. In this sense it seems to be more natural than the one for the sublaplacian. We have the following Bochner formula.

PROPOSITION 2.1

For any complex-valued function φ, we have

$$
\begin{aligned}
-\frac{1}{2}\Box_b|\bar\partial_b\varphi|^2 = (\varphi_{\bar1\bar1}\bar\varphi_{11} + \varphi_{\bar11}\bar\varphi_{1\bar1})& \\
&- \frac{1}{2}\langle\bar\partial_b\varphi, \bar\partial_b\Box_b\varphi\rangle - \langle\bar\partial_b\Box_b\varphi, \bar\partial_b\varphi\rangle \\
&- \langle\bar P\varphi, \bar\partial_b\varphi\rangle + R|\bar\partial_b\varphi|^2.
\end{aligned} \tag{2.1}
$$

Proof

We calculate

$$
\begin{aligned}
-\frac{1}{2}\Box_b|\bar\partial_b\varphi|^2 &= -\frac{1}{2}\Box_b\langle\varphi_{\bar1}\theta^{\bar1}, \varphi_{\bar1}\theta^{\bar1}\rangle \\
&= (\varphi_{\bar1}\bar\varphi_1)_{\bar11} \\
&= (\varphi_{\bar1\bar1}\bar\varphi_1 + \varphi_{\bar1}\bar\varphi_{1\bar1})_1 \\
&= \varphi_{\bar1\bar11}\bar\varphi_1 + \varphi_{\bar1\bar1}\bar\varphi_{11} + \varphi_{\bar11}\bar\varphi_{1\bar1} + \varphi_{\bar1}\bar\varphi_{1\bar11}, \\
&= \varphi_{\bar1\bar1}\bar\varphi_{11} + \varphi_{\bar11}\bar\varphi_{1\bar1} - \frac{1}{2}\langle\bar\partial_b\varphi, \bar\partial_b\Box_b\varphi\rangle + \varphi_{\bar1\bar11}\bar\varphi_1,
\end{aligned} \tag{2.2}
$$

here; for the last equality, we use the identity

$$
-\frac{1}{2}\langle\bar\partial_b\varphi, \bar\partial_b\Box_b\varphi\rangle = \varphi_{\bar1}\bar\varphi_{1\bar11}.
$$

Therefore, the Bochner formula is completed if we show that

$$
\varphi_{\bar1\bar11}\bar\varphi_1 = -\langle\bar\partial_b\Box_b\varphi, \bar\partial_b\varphi\rangle - \langle\bar P\varphi, \bar\partial_b\varphi\rangle + R|\bar\partial_b\varphi|^2. \tag{2.3}
$$

By the commutation relations, we have

$$
\begin{aligned}
\varphi_{\bar1\bar11}\bar\varphi_1 &= (\varphi_{\bar11\bar1} - i\varphi_{\bar10} + R\varphi_{\bar1})\bar\varphi_1 \\
&= (\varphi_{1\bar1\bar1} - i\varphi_{0\bar1} - i\varphi_{\bar10} + R\varphi_{\bar1})\bar\varphi_1 \\
&= (\bar P_1\varphi + iA_{\bar1\bar1}\varphi_1 - i\varphi_{0\bar1} - i\varphi_{\bar10} + R\varphi_{\bar1})\bar\varphi_1 \\
&= (\bar P_1\varphi)\bar\varphi_1 + R\varphi_{\bar1}\bar\varphi_1 + (iA_{\bar1\bar1}\varphi_1 - i\varphi_{0\bar1} - i\varphi_{\bar10})\bar\varphi_1
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \bar{P}\varphi, \bar{\partial}_b\varphi\rangle + R|\bar{\partial}_b\varphi|^2 + (iA_{\bar{1}\bar{1}}\varphi_1 - i\varphi_{0\bar{1}} - i\varphi_{\bar{1}0})\bar{\varphi}_1,\\
&= \langle \bar{P}\varphi, \bar{\partial}_b\varphi\rangle + R|\bar{\partial}_b\varphi|^2 + 2(iA_{\bar{1}\bar{1}}\varphi_1 - i\varphi_{0\bar{1}})\bar{\varphi}_1, \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
-\frac{1}{2}&\langle \bar{\partial}_b\Box_b\varphi, \bar{\partial}_b\varphi\rangle\\
&= \langle \varphi_{\bar{1}1\bar{1}}\theta^{\bar{1}}, \varphi_{\bar{1}}\theta^{\bar{1}}\rangle\\
&= \varphi_{\bar{1}1\bar{1}}\bar{\varphi}_1\\
&= (\varphi_{1\bar{1}\bar{1}} - i\varphi_{0\bar{1}})\bar{\varphi}_1\\
&= (\bar{P}_1\varphi + iA_{\bar{1}\bar{1}}\varphi_1 - i\varphi_{0\bar{1}})\bar{\varphi}_1\\
&= \langle \bar{P}\varphi, \bar{\partial}_b\varphi\rangle + (iA_{\bar{1}\bar{1}}\varphi_1 - i\varphi_{0\bar{1}})\bar{\varphi}_1. \qquad (2.5)
\end{aligned}
$$

Combining (2.4) and (2.5), we obtain (2.3). This completes the proposition. $\square$

We now prove Theorem 1.3.

Proof of Theorem 1.3

Let φ be an eigenfunction with respect to a nonzero eigenvalue λ; that is, φ is not a CR function. Taking the integral of both sides of the Bochner formula (2.1), we have

$$
\begin{aligned}
0 = \int \varphi_{\bar{1}\bar{1}}\bar{\varphi}_{11} + \int \varphi_{\bar{1}1}\bar{\varphi}_{1\bar{1}} - \frac{3}{2}\lambda\int |\bar{\partial}_b\varphi|^2\\
+ \int \langle P_0\varphi, \varphi\rangle + \int R|\bar{\partial}_b\varphi|^2. \qquad (2.6)
\end{aligned}
$$

On the other hand,

$$
\int \varphi_{\bar{1}1}\bar{\varphi}_{1\bar{1}} = \frac{1}{4}\int \langle \Box_b\varphi, \Box_b\varphi\rangle = \frac{\lambda}{2}\int |\bar{\partial}_b\varphi|^2.
$$

Taking together the above two formulas, we obtain

$$
\begin{aligned}
\lambda\int |\bar{\partial}_b\varphi|^2 &= \int |\varphi_{\bar{1}\bar{1}}|^2 + \int \langle P_0\varphi, \varphi\rangle + \int R|\bar{\partial}_b\varphi|^2\\
&\geq \int \langle P_0\varphi, \varphi\rangle + \int R|\bar{\partial}_b\varphi|^2. \qquad (2.7)
\end{aligned}
$$

Therefore, if P_0 is nonnegative and $R > 0$, then we immediately have that $\lambda \geq \min R$. Since the spectrum $\mathrm{spec}(\Box_b)$ of the Kohn Laplacian in $(0, \infty)$ consists only of point eigenvalues (see [5, Theorem 1.3]), it follows that the range of $\Box_b$ is closed. Applying the result of Kohn [14], we conclude that M is embeddable.

To prove the second part of Theorem 1.4 note that if the CR Yamabe constant is greater than zero, then we can choose a contact form such that the Webster curvature with respect to this contact form is positive, and so we conclude by the first part of Theorem 1.4. $\Box$

Remark 2.2
The estimate for the nonzero eigenvalues is sharp. For example, $\Box_b$ on the standard sphere S^3 as a pseudo-Hermitian 3-manifold has the smallest nonzero eigenvalue $\lambda = 2 = R$ (for details, see [9]).

Remark 2.3
In general, let M^{2n+1} be a pseudo-Hermitian manifold. The Bochner formula for the Kohn Laplacian is as follows:

$$\begin{aligned} -\frac{1}{2}\Box_b|\bar\partial_b\varphi|^2 = {} & \sum_{\alpha,\beta}(\varphi_{\bar\alpha\bar\beta}\bar\varphi_{\alpha\beta} + \varphi_{\bar\alpha\beta}\bar\varphi_{\alpha\bar\beta}) \\ & - \frac{1}{2n}\langle\bar\partial_b\varphi, \bar\partial_b\Box_b\varphi\rangle - \frac{(n+1)}{2n}\langle\bar\partial_b\Box_b\varphi, \bar\partial_b\varphi\rangle \\ & - \frac{1}{n}\langle\bar P\varphi, \bar\partial_b\varphi\rangle + \frac{(n-1)}{n}\langle P\bar\varphi, \partial_b\bar\varphi\rangle \\ & + \mathrm{Ric}(\nabla_b\varphi_{\mathbb{C}}, \nabla_b\varphi_{\mathbb{C}}), \end{aligned} \tag{2.8}$$

where $\nabla_b\varphi_{\mathbb{C}}$ is the corresponding complex $(1,0)$-vector of $\nabla_b\varphi$. The proof of (2.8) is the same as (2.1). Again, in the case when $n = 2$, using this formula we also obtain that the sharp lower bound of nonzero eigenvalues of the Kohn Laplacian $\Box_b$ is $(4/3)k_0$, provided that the Ricci curvature has the lower bound

$$\mathrm{Ric}(X, X) \geq k_0|X|^2$$

for some k_0 and for all complex $(1,0)$-vectors X. Unfortunately, in the higher-dimensional cases $n \geq 3$, the coefficient $(n-1)/n$ of the term $\langle P\bar\varphi, \partial_b\bar\varphi\rangle$ is too large to get the lower bound of nonzero eigenvalues of the Kohn Laplacian $\Box_b$ immediately.

Example 2.4
In this example, we compute the Webster curvature of Rossi's global nonembeddability example (see [1], [19], [7, Theorem 12.4.1]) together with a suitable contact structure and show that the associated CR Paneitz operators are not nonnegative. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 - 1 = 0\}$ be the boundary of the unit ball in $\mathbb{C}^2$ with the induced CR structure given by the complex vector field

$$Z_1 = \bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2}$$

and contact form

$$\theta = \frac{i(\bar{\partial} u - \partial u)}{2}\Big|_{S^3},$$

where $u = |z_1|^2 + |z_2|^2 - 1$. Taking the admissible coframe

$$\theta^1 = z_2\, dz_1 - z_1\, dz_2,$$

we have $d\theta = i\theta^1 \wedge \theta^{\bar{1}}$. Rossi's example is the CR manifold S^3 together with the CR structure given by

$$L_t = Z_1 + t\bar{Z}_1,$$

for all $t \in \mathbb{R}$ and $t \neq 1, -1$. Now, for $|t| < 1$, taking the contact form

$$\theta(t) = \theta$$

and the admissible coframe

$$\theta^1(t) = \frac{1}{\sqrt{1-t^2}}(\theta^1 - t\theta^{\bar{1}}),$$

we have $d\theta(t) = i\theta^1(t) \wedge \theta^{\bar{1}}(t)$ and the following proposition.

PROPOSITION 2.5

For $|t| < 1$, with respect to the coframe $\{\theta(t), \theta^1(t), \theta^{\bar{1}}(t)\}$, the connection form and pseudo-Hermitian torsion are as follows:

$$\theta_1{}^1(t) = \theta_1{}^1 - \frac{4t^2 i}{1-t^2}\theta = \frac{-2(1+t^2)}{1-t^2} i\theta \tag{2.9}$$

and

$$\tau^1(t) = \frac{4ti}{1-t^2}\theta^{\bar{1}}(t),$$

where

$$\theta_1{}^1 = -\bar{z}_1\, dz_1 - \bar{z}_2\, dz_2 + z_1\, d\bar{z}_1 + z_2\, d\bar{z}_2 = -2i\theta.$$

In addition, the Webster curvature is $R(t) = \frac{2(1+t^2)}{1-t^2}$.

Proof
We just check that the forms $\theta_1{}^1(t), \tau^1(t)$ satisfy the equations (1.3). Finally, after a direct computation, we see that $d\theta_1{}^1(t) = \frac{2(1+t^2)}{1-t^2}\theta^1 \wedge \theta^{\bar{1}}$, and so $R(t) = \frac{2(1+t^2)}{1-t^2}$. $\square$

Similarly, for $|t| > 1$, take the contact form $\theta(t)$ and the admissible coframe $\theta^1(t)$ as follows:

$$\theta(t) = -\theta, \qquad \theta^1(t) = \frac{1}{\sqrt{t^2-1}}(\theta^1 + t\theta^{\bar{1}}).$$

Then we have

$$\tau^1(t) = \frac{4ti}{1-t^2}\theta^{\bar{1}}(t) \qquad \text{and} \qquad R(t) = \frac{2(1+t^2)}{t^2-1}.$$

From Theorem 1.4 and Proposition 2.5, we immediately obtain that the CR Paneitz operators of Rossi's nonembeddable manifolds are not nonnegative.

3. Stability of embeddability

We now consider stability issues (see [5], [18] for earlier work). We have a fixed CR structure on a compact manifold (M^3, θ, J). Let us denote by $\bar{L}$ the CR vector field on M^3. We now perturb $\bar{L}$ by a smooth family of functions $\varphi(\cdot, t) = \varphi_t(\cdot)$, where $(\cdot)$ represents a point on M and $t \in (-\varepsilon, \varepsilon)$. On a local chart of M we denote local coordinates by (z, s), where z is the holomorphic variable and s is the variable corresponding to the real vector field $T = \frac{\partial}{\partial s}$. We assume always that

$$D^{\alpha}_{z,s}\varphi(z,s,t)|_{t=0} = 0, \quad |\alpha| \le l_0, l_0 \ge 4, (z,s) \in M. \tag{3.1}$$

We define

$$\bar{L}_t = \bar{L} + \varphi(\cdot, t)L. \tag{3.2}$$

Associated to $\bar{L}_t$ we form the associated $\bar{\partial}_b^{(t)}$-Laplacian operator

$$\square_b^{(t)} = \bar{\partial}_b^{(t)*}\bar{\partial}_b^{(t)}. \tag{3.3}$$

We now use our main result to guarantee embedding of our CR structure in $\mathbb{C}^N$. Thus we assume that, along the deformation path in t,

$$\text{the associated Paneitz operator } P_0^{(t)} \ge 0, \tag{3.4}$$

$$\text{the CR Yamabe constant } \ge c > 0. \tag{3.5}$$

By our main result (Theorem 1.4), using (3.4) and (3.5) it follows that

$$\lambda_1(\Box_b^{(t)}) \geq \nu > 0, \tag{3.6}$$

with ν independent of t. Thus by using the construction of Boutet de Monvel in [4] or the exposition of Chen and Shaw in [7], we can embed for $\varepsilon > 0$ small enough the CR structures via a map Ψ_t into the same $\mathbb{C}^N$, that is,

$$\Psi_t : (M, \theta, J_t) \longrightarrow \mathbb{C}^N. \tag{3.7}$$

The question arises if the maps Ψ_t are close in, say, the sup-norm in t. We have the following stability theorem.

THEOREM 3.1
Under (3.1), (3.4), and (3.5), for any $\delta > 0$ there exists $\varepsilon > 0$, so that

$$\sup_{t\in[-\varepsilon,\varepsilon]} \|\Psi_t - \Psi_0\|_{C^k(M)} < \delta, \quad k = k(l_0).$$

Proof
The proof of this theorem is abstract and relies on an identity in [5]. We use [5, Proposition 5.55]. We denote the projection into the zero eigenspace of $\Box_b^{(t)}$ by $\Im^{\varphi_t}$, which is the Szego projector. By the spectral theorem and (3.6), if $|\lambda| = \nu/2$, then the resolvent $(\Box_b^{(t)} - \lambda)^{-1}$ is well defined, and so

$$\Im^{\varphi_t} = \int_{|\lambda|=\nu/2} (\Box_b^{(t)} - \lambda)^{-1}\, d\lambda,$$

and it is immediate that $\Im^{\varphi_t}$ is a bounded operator on $L^2(M)$. As observed in [5], as a consequence of the above fact and their identity (5.58), Burns and Epstein obtain the inequality (5.60) which we restate:

$$\|\Im^{\varphi_t} - \Im^{\varphi_0}\|_{L^2(M)} \leq C\nu A \|\varphi_t - \varphi_0\|_{L^\infty(M)}, \tag{3.8}$$

where A is the sup-norm of some high enough derivative of $\varphi_t - \varphi_0$. But by our hypothesis (3.1) the right-hand side of (3.8) is smaller than $\delta > 0$, for $\varepsilon > 0$ sufficiently small.

Now recall the construction of Boutet de Monvel. Using the notation in [7, p. 318], the embedding for each coordinate chart is given by a CR function h_t (we are in CR dimension 1), where

$$h_t = \Im^{\varphi_t}(\psi e^{-\tau\varphi_p}), \quad \tau \to \infty. \tag{3.9}$$

Now note that $h_t - h_0$ also satisfy an equation, that is,

$$\square_b^{(t)}(h_t - h_0) = (\square_b^{(0)} - \square_b^{(t)})(h_0). \tag{3.10}$$

From (3.8) and (3.9), $\|h_t - h_0\|_{L^2(M)} < \delta$. From (3.1) and (3.10), the right-hand side of (3.10) is small in the C^k-norm. Since we have (3.6), it now implies by subelliptic regularity that, for $\delta > 0$, there exists ε_0,

$$\sup_{t\in(-\varepsilon_0,\varepsilon_0)} \|h_t - h_0\|_{C^k(M)} \leq \delta. \tag{3.11}$$

In fact by differentiation of (3.10) in t, we may also obtain higher stability in t, provided that we replace (3.1) by the stronger hypothesis that $D_{z,s}^{\alpha} D_t^{\beta} \varphi(z,s,0) = 0$ for large enough $|\alpha|, |\beta|$. This proves our theorem since on coordinate charts of M, the map Ψ_t is given by h_t. $\square$

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