EMBEDDABILITY FOR THREE-DIMENSIONAL CAUCHY-RIEMANN MANIFOLDS AND CR YAMABE INVARIANTS

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Abstract. Let M^3 be a closed CR 3-manifold. In this paper, we derive a Bochner formula for the Kohn Laplacian in which the pseudohermitian torsion doesn't play any role. By means of this formula we show that the nonzero eigenvalues of the Kohn Laplacian have a lower bound, provided that the CR Paneitz operator is nonnegative and the Webster curvature is positive. This means that M^3 is embeddable when the CR Yamabe constant is positive and the CR Paneitz operator is nonnegative. Our lower bound estimate is sharp. In addition, we show that the embedding is stable in the sense of Burns and Epstein. Lastly we show that the CR Paneitz operator for embeddable CR structures given by polynomial deformations and close to the standard CR structure on S^3 is positive on the subspace of spherical harmonics $\oplus_{p\geq 1} H_{p,0} \oplus H_{0,p}$.

1. Introduction,Statements and Notation

The embedding problem in CR geometry has received a lot of attention. In [5], Burns and Epstein consider perturbations of the standard CR structure on the 3-sphere S^3. They showed that the generic perturbation is nonembeddable and gave a sufficient condition for embeddability, which we call condition (BE) in this paper. For small perturbations, Bland in his paper [2] showed that this condition in some sense is necessary up to a contact diffeomorphism. Bland's results strongly depend on his earlier work on moduli for pointed convex domains, which is related to one of Lempert's work (see [16]) on constructing modular data for pointed convex domains.

AMS Classification number(2010). 32V30, 32V20.

Key words and phrases. Embedding, CR manifold, Paneitz operator, CR Yamabe.

In his paper [15], L. Lempert asked two fundamental questions about the embeddability problem. The first one is related to the closedness property of CR structures and the second one is to the stability property. In this paper we introduce a CR invariant condition which is not perturbative to address Lempert's questions. In his papers ([9]), Epstein introduced an invariant, called the relative index of two Szego projectors. It quantifies the stability of the algebra of CR functions under embeddable deformations. It is used to show that every embeddable deformation of a CR structure on S^3 is stable. Our hypotheses in Theorem 1.5 and the result of Theorem 1.3 imply the relative index is zero.

Throughout this paper, we will use the notations and terminology in ([13]) unless otherwise specified. Let (M, J, θ) be a closed three-dimensional pseudo-hermitian manifold, where θ is a contact form and J is a CR structure compatible with the contact bundle $\xi = \ker\theta$. The CR structure J decomposes $\mathbb{C}\otimes\xi$ into the direct sum of $T_{1,0}$ and $T_{0,1}$ which are eigenspaces of J with respect to i and $-i$, respectively. The Levi form $\langle\ ,\ \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z, W\rangle_{L_\theta} = -i\left\langle d\theta, Z\wedge\overline{W}\right\rangle$. We can extend $\langle\ ,\ \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\left\langle\overline{Z},\overline{W}\right\rangle_{L_\theta} = \overline{\langle Z, W\rangle}_{L_\theta}$ for all $Z, W \in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\ ,\ \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the hermitian form (when acting on sections) over M with respect to the volume form $dV = \theta\wedge d\theta$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle\ ,\ \rangle$. For example

$$\langle\varphi,\psi\rangle = \int_M \varphi\bar{\psi}\ dV, \tag{1.1}$$

for functions φ and ψ.

Let $\{T, Z_1, Z_{\bar{1}}\}$ be a frame of $TM\otimes\mathbf{C}$, where Z_1 is any local frame of $T_{1,0}$, $Z_{\bar{1}} = \overline{Z_1} \in T_{0,1}$ and T is the characteristic vector field, that is, the unique vector field such that $\theta(T) = 1,\ d\theta(T,\cdot) = 0$. Then $\left\{\theta, \theta^1, \theta^{\bar{1}}\right\}$, the coframe dual to $\{T, Z_1, Z_{\bar{1}}\}$, satisfies

$$d\theta = ih_{1\bar{1}}\theta^1\wedge\theta^{\bar{1}} \tag{1.2}$$

for some positive function $h_{1\bar{1}}$. We can always choose Z_1 such that $h_{1\bar{1}} = 1$; hence, throughout this paper, we assume $h_{1\bar{1}} = 1$

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbf{C}$ (and extended to tensors) given in terms of a local frame $Z_1 \in T_{1,0}$ by

$$\nabla Z_1 = \theta_1{}^1 \otimes Z_1, \quad \nabla Z_{\bar{1}} = \theta_{\bar{1}}{}^{\bar{1}} \otimes Z_{\bar{1}}, \quad \nabla T = 0,$$

where $\theta_1{}^1$ is the 1-form uniquely determined by the following equations:

$$\begin{aligned} d\theta^1 &= \theta^1 \wedge \theta_1{}^1 + \theta \wedge \tau^1 \\ \tau^1 &\equiv 0 \mod \theta^{\bar{1}} \\ 0 &= \theta_1{}^1 + \theta_{\bar{1}}{}^{\bar{1}}, \end{aligned} \tag{1.3}$$

where $\theta_1{}^1$ and τ^1 are called the connection form and the pseudohermitian torsion, respectively. Put $\tau^1 = A^1{}_{\bar{1}}\theta^{\bar{1}}$. The structure equation for the pseudohermitian connection is

$$d\theta_1{}^1 = R\theta^1 \wedge \theta^{\bar{1}} + 2iIm(A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta),$$

where R is the Tanaka-Webster curvature.

We will denote components of covariant derivatives with indices preceded by a comma; thus we write $A^{\bar{1}}{}_{1,\bar{1}}\theta^1 \wedge \theta$. The indices $\{0, 1, \bar{1}\}$ indicate derivatives with respect to $\{T, Z_1, Z_{\bar{1}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $\varphi_1 = Z_1\varphi$, $\varphi_{1\bar{1}} = Z_{\bar{1}}Z_1\varphi - \theta_1^1(Z_{\bar{1}})Z_1\varphi$, $\varphi_0 = T\varphi$ for a (smooth) function.

Next we consider several natural differential operators occuring in this paper. For a detailed description, we refer the reader to the article [13]. For a smooth function φ, the Cauchy-Riemann operator ∂_b can be defined locally by

$$\partial_b\varphi = \varphi_1\theta^1,$$

and we write $\bar{\partial}_b$ for the conjugate of ∂_b. A function φ is called CR holomorphic if $\bar{\partial}_b\varphi = 0$. the divergence operator δ_b takes $(1,0)$-forms to functions by $\delta_b(\sigma_1\theta^1) = \sigma_{1,}{}^1$, and similarly, $\bar{\delta}_b(\sigma_{\bar{1}}\theta^{\bar{1}}) = \sigma_{\bar{1},}{}^{\bar{1}}$.

If $\sigma = \sigma_1\theta^1$ is compactly supported, Stokes' theorem applied to the 2-form $\theta \wedge \sigma$ implies the divergence formula:

$$\int_M \delta_b\sigma\theta \wedge d\theta = 0.$$

It follows that the formal adjoint of ∂_b on functions with respect to the Levi form and the volume element $\theta \wedge d\theta$ is $\partial_b^* = -\delta_b$. The Kohn Laplacian on functions determined by θ is

$$\Box_b = 2\bar{\partial}_b^*\bar{\partial}_b,$$

Define $P\varphi = (\varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1)\theta^1$ (see [13]) which is an operator that characterizes CR-pluriharmonic functions, and $\overline{P}\varphi = (\varphi_1{}^1{}_{\bar{1}} - iA_{\bar{1}\bar{1}}\varphi^{\bar{1}})\theta^{\bar{1}}$, the conjugate of P. The CR Paneitz operator P_0 is defined by

$$P_0\varphi = \delta_b(P\varphi).$$

More explicitly, define Q by $Q\varphi = 2i(A^{11}\varphi_1)_{,1}$, then

$$\begin{aligned} P_0\varphi &= \frac{1}{4}(\Box_b\overline{\Box}_b - 2Q)\varphi \\ &= \frac{1}{4}(\Box_b\overline{\Box}_b\varphi - 4i(A^{11}\varphi_1)_1) \\ &= \frac{1}{8}\big((\overline{\Box}_b\Box_b + \Box_b\overline{\Box}_b)\varphi + 8Im(A^{11}\varphi_1)_1\big). \end{aligned}$$

By the commutation relation $[\Box_b, \overline{\Box}_b] = 4iImQ$, we see that $4P_0 = \Box_b\overline{\Box}_b - 2Q = \overline{\Box}_b\Box_b - 2\overline{Q}$. It follows that P_0 is a real and symmetric operator (see [7] for the details).

Definition 1.1. The Paneitz operator P_0 is nonnegative if

$$\int_M (P_0\varphi)\bar{\varphi} \geq 0,$$

for all smooth functions φ.

Note that the nonnegativity of P_0 is a CR invariant in the sense that it is independent of the choice of the contact form θ. This follows by observing that if $\widetilde{\theta} = e^{2f}\theta$ be another contact form, we have the following transformation laws for the volume form and the CR Paneitz operator respectively (see Lemma 7.4 in [11]):

$$\widetilde{\theta} \wedge d\widetilde{\theta} = e^{4f}\theta \wedge d\theta; \qquad \widetilde{P_0} = e^{-4f}P_0.$$

We also observe that when the Webster torsion $A_{11} \equiv 0$, then the Paneitz operator P_0 is given by,

$$P_0 = \frac{1}{4}\Box_b\overline{\Box}_b.$$

Thus the vanishing of torsion implies that $P_0 \geq 0$. This is because when M is embeddable, the two operators $\Box_b$ and $\overline{\Box}_b$ commute, and hence are simultaneously diagonalizable on each eigenspace of $\Box_b$ of a nonzero eigenvalue(see [7]). We also recall that the vanishing of torsion is equivalent to $L_T J = 0$ where L is the Lie derivative.

In the higher dimensional case, there exists an analog of P_0 which satisfies the covariant property. In this case, Graham and Lee, in [10], had shown the nonnegativity of P_0.

Definition 1.2. Suppose that $\widetilde{\theta} = e^{2f}\theta$. The CR Yamabe constant is defined by $\inf_{\widetilde{\theta}}\{\int_M \widetilde{R}\,\widetilde{\theta} \wedge d\widetilde{\theta} : \int \widetilde{\theta} \wedge d\widetilde{\theta} = 1\}$.

The CR Yamabe constant is a CR invariant. We are now in a position to describe our main theorems. In this paper we show

Theorem 1.3. *Let M^3 be a closed CR manifold. If $P_0 \geq 0$ and $R > 0$, then the non-zero eigenvalues λ of $\Box_b$ satisfy $\lambda \geq \min R$, hence the range of $\Box_b$ is closed. If $P_0 \geq 0$ and the CR Yamabe constant > 0, then M^3 can be embedded into $\mathbb{C}^n$, for some n.*

Remark 1.4. The fact $\Box_b$ has closed range is equivalent to global embedding is a result of Kohn ([12])

In section 3 we prove the stability theorem:

THEOREM 1.5. *Under (3.1), (3.4), (3.5), the embedding is stable which means that if $|t|$ is small enough then the CR embedding Ψ_t is close to Ψ_0.*

REMARK 1.6. If M is embeddable then the Paneitz operator P_0 has closed range (see [6]). In particular we point out that one consequence of the Paneitz operator being closed for embeddable structures is that the Paneitz operator has finitely many negative eigenvalues for an embeddable CR structure.

We now turn our attention to section 4, where we study the converse to the theorem 1.3. We shall also compare our result with those of Burns-Epstein [5] and Bland [2]. Let (M, J, θ) be a CR structure. Let ϕ be a complex valued smooth function on M, such that $\|\phi\|_\infty < 1$. For θ fixed consider a deformation of the CR structure given by

$$Z_{\bar{1}}^{\phi} = Z_{\bar{1}} + \phi Z_1.$$

Our first order of business in section 4 is to compute in generality the connection forms, torsion forms and Webster-Tanaka curvature for the deformed structure. Now we specialize the situation to S^3 and consider small deformations of the standand CR structure of the sphere. In particular our goal is to consider the deformed structure on S^3 given by,

$$Z_{\bar{1}}^{t} = Z_{\bar{1}}^{\phi_t} = F(Z_{\bar{1}} + t\phi Z_1),$$

where $F = (1 - t^2|\phi|^2)^{-1/2}$, $Z_{\bar{1}} = \bar{z}_2 \frac{\partial}{\partial Z_1} - \bar{z}_1 \frac{\partial}{\partial Z_2}$ and $t \in (-\epsilon, \epsilon)$. The factor F is introduced to normalize the Levi form so that $h_{1\bar{1}} \equiv 1$. For this structure we compute the deformed Paneitz operator P_0^t. The main goal in Section 4 is to study the variations of P_0^t. We now consider the 3-sphere $S^3 \subset \mathbb{C}^2 \ni (z_1, z_2)$ and denote by

$$P_{p,q} = \operatorname{span}\{z_1^a z_2^b \bar{z}_1^c \bar{z}_2^d | a + b = p,\ c + d = q\}$$

and the spherical harmonics

$$H_{p,q} = \{f \in P_{p,q} | -\Delta_{s^3} f = (p+q)(p+q+2)f\}.$$

For a given $\phi \in C^\infty(S^3)$ one has the Fourier representation

$$\phi \sim \sum \phi_{pq}$$

where ϕ_{pq} is the projection of ϕ onto $H_{p,q}$.

DEFINITION 1.7. We say ϕ satisfies condition (BE) if and only if

$$\phi_{pq} \equiv 0 \text{ for } \; p < q+4, \; q = 0, 1, \cdots .$$

REMARK 1.8. Since for $p > q$

$$P_{p,q} = H_{p,q} \oplus \cdots \oplus H_{p-q,0}.$$

It follows that if $\phi \in P_{p,q}$, then ϕ satisfies (BE) if and only if $p \geq q+4$.

Burns and Epstein proved in [5] that for $t \in (-\epsilon, \epsilon)$ and ϕ satisfying (BE) the CR structure embeds into some $\mathbb{C}^n$. Conversely Bland [2] showed that embeddability of a CR structure close to the standard structure on S^3 implies condition (BE). To summarize we have

Theorem [**Burns-Epstein-Bland**]. A CR structure close to the standard structure on S^3 is embeddable if and only if ϕ satisfies condition (BE).

We define the space

$$\mathbf{H} = C^\infty(S^3) \cap \left(\oplus_{p\geq 1} H_{p,0} \oplus H_{0,p}\right).$$

The main result proved in section 4 is

PROPOSITION 1.9. *Let $\phi \in P_{p_1,q_1}$. For a CR structure given by deformation by ϕ and close to the standard structure on S^3, i.e., $t \in (-\epsilon, \epsilon)$, the associated CR Paneitz operator is positive on $\mathbf{H}$, provided ϕ satisfies (BE).*

We now outline the strategy of proof of Proposition 1.9. Since CR pluriharmonic functions are annihilated by the Paneitz operator, we will study for $f \in \mathbf{H}$, the quadratic form

$$I_t(f) = < P_0^t f, f > . \tag{1.4}$$

We show in Theorem 4.6 that for $f \in \mathbf{H}$ and $\phi \in C^\infty(S^3)$,

$$\dot{I}_t(f)|_{t=0} = \frac{d}{dt} I_t(f)|_{t=0} \equiv 0. \tag{1.5}$$

Thus the first variation all vanish. The (BE) condition does not appear in the first variation formula, but appears in the second variation formula. To compute the second variation, that is to compute $\ddot{P}_0^t|_{t=0}$, we perform a Morse decomposition of the functional $\ddot{I}_t$. It splits into a stable part, called D^2 which we handle via Proposition 4.7 and an unstable part which is handled by proposition 4.8. The (BE) condition enters naturally into the unstable part by means of an expressions E,

$$E = 4\phi + i\phi_0, \quad \phi_0 = T\phi.$$

In fact if ϕ satisfies (BE) then there is no unstable part.

Writing $f \in \mathbf{H}$ as

$$f = \sum_{k \geq 1} f^k + \sum_{k \geq 1} g^k, \quad f^k \in H_{k,0}, \quad g^k \in H_{0,k},$$

we get by throwing away the stable part:

Proposition 1.10. *For any $\phi \in C^\infty(S^3)$, $f \in \mathbf{H}$,*

$$\begin{aligned} \ddot{I}_t(f)|_{t=0} &= \frac{d^2}{dt^2} I_t(f)|_{t=0} \\ &\geq 2 \sum_{k,l} \int_{S^3} (k|\phi|^2 - E\bar{\phi}) f_1^k \overline{f_1^l} + 2 \sum_{k,l} \int_{S^3} (k|\phi|^2 - E\bar{\phi}) g_{\bar{1}}^k \overline{g_{\bar{1}}^l}. \end{aligned}$$

We now invoke the Hopf fibration to perform the integration in the right side of the theorem above. Viewing S^3 as a S^1 fibration over CP^1 reduces the computation to doing Fourier series on $(-\pi, \pi)$. This is the content of Proposition 4.10. This proposition works for general ϕ if $k = l$, but we have been unable to do the integration when $k \neq l$ unless $\phi \in P_{p_1,q_1}$. We get

Proposition 1.11. *For any* $\phi \in P_{p_1,q_1}$, $f \in \mathbf{H}$,

$$\ddot{I}_t(f)|_{t=0} \geq 2 \sum_k \int_{S^3} (k + p_1 - q_1 - 4)|\phi|^2 (|f_1^k|^2 + |g_1^k|^2).$$

We emphasize that the conclusion of Theorem 1.11 holds for even those $\phi \in P_{p_1,q_1}$ for which $p_1 < q_1 + 4$, i.e., for those values of p_1, q_1 that fail to satisfy condition (BE).

If however ϕ satisfies (BE), then it is evident for $k \geq 1$, $k + p_1 - q_1 - 4 \geq 1$ and thus from the above theorem $\ddot{I}_t(f)|_{t=0} > 0$, for $f \in \mathbf{H}$. Combining this fact with (1.5) and since $t \in (-\epsilon, \epsilon)$ we see readily that if ϕ satisfies (BE), then for $t \in (-\epsilon, \epsilon)$, $f \in \mathbf{H}$, we have

$$I_t(f) =< P_0^t f, f > > 0. \tag{1.6}$$

We also point out some other results in section 4 of independent interest. One such result is Corollary 4.3, which states if $\phi \in P_{p_1,q_1}$, with $p_1 = q_1 + 4$, then the deformed structure on S^3 also has zero torsion and conversely if ϕ is a homogeneous polynomial in P_{p_1,q_1}, then for precisely those for which $p_1 = q_1 + 4$, the new torsion will also vanish.

Lastly we comment that our second variation formula shows that for $\phi \in P_{p_1,q_1}$ and $f \in H_{p,0}$ or $f \in H_{0,p}$, the possible negative direction of $\ddot{I}_t(f)|_{t=0}$ can only lie in the space $f \in H_{p,0}$ or $f \in H_{0,p}$ for $p < q_1 + 4 - p_1$. A more careful computation of the second variation of Paneitz operator for Rossi's example $\phi \equiv 1$, which we have chosen not to display, shows that the negative directions are given exactly by the functions $f = z_1, z_2, \bar{z}_1, \bar{z}_2$.

Acknowledgment. The first author's research was supported in part by NSF grant DMS-0855541, the second author's research was supported in part by CIZE Foundation and in part by NSC 96-2115-M-008-017-MY3, and the third author's research was supported in part by NSF grant DMS-0758601.

2. The Embedding Criterion

In this section, we will derive the Bochner formula for the Kohn Laplacian. We need some commutation relations, for which we refer the reader to Lee's paper [13]. This formula contains no term related to pseudohermitian torsion. In this sense it seems to be more natural than the one for the sublaplacian. We have the following Bochner formula:

PROPOSITION 2.1. *For any complex-valued function φ ,we have*

$$
\begin{aligned}
-\frac{1}{2}\Box_b|\bar\partial_b\varphi|^2 = &(\varphi_{\bar1\bar1}\bar\varphi_{11}+\varphi_{\bar11}\bar\varphi_{1\bar1}) \\
&-\frac{1}{2}<\bar\partial_b\varphi,\bar\partial_b\Box_b\varphi>-<\bar\partial_b\Box_b\varphi,\bar\partial_b\varphi> \\
&-<\bar P\varphi,\bar\partial_b\varphi>+R|\bar\partial_b\varphi|^2
\end{aligned}
\tag{2.1}
$$

Proof. We calculate

$$
\begin{aligned}
-\frac{1}{2}\Box_b|\bar\partial_b\varphi|^2 &= -\frac{1}{2}\Box_b<\varphi_{\bar1}\theta^{\bar1},\varphi_{\bar1}\theta^{\bar1}> \\
&= (\varphi_{\bar1}\bar\varphi_1)_{\bar11} \\
&= (\varphi_{\bar1\bar1}\bar\varphi_1+\varphi_{\bar1}\bar\varphi_{1\bar1})_1 \\
&= \varphi_{\bar1\bar11}\bar\varphi_1+\varphi_{\bar1\bar1}\bar\varphi_{11}+\varphi_{\bar11}\bar\varphi_{1\bar1}+\varphi_{\bar1}\bar\varphi_{1\bar11}, \\
&= \varphi_{\bar1\bar1}\bar\varphi_{11}+\varphi_{\bar11}\bar\varphi_{1\bar1}-\frac{1}{2}<\bar\partial_b\varphi,\bar\partial_b\Box_b\varphi>+\varphi_{\bar1\bar11}\bar\varphi_1,
\end{aligned}
\tag{2.2}
$$

here, for the last equality, we use the identity

$$
-\frac{1}{2}<\bar\partial_b\varphi,\bar\partial_b\Box_b\varphi>=\varphi_{\bar1}\bar\varphi_{1\bar11}
$$

Therefore, the Bochner formula is completed if we show that

$$
\varphi_{\bar1\bar11}\bar\varphi_1 = -<\bar\partial_b\Box_b\varphi,\bar\partial_b\varphi>-<\bar P\varphi,\bar\partial_b\varphi>+R|\bar\partial_b\varphi|^2.
\tag{2.3}
$$

By the commutation relations, we have.

$$
\begin{aligned}
\varphi_{\bar 1\bar 1 1}\bar\varphi_1 &= (\varphi_{\bar 1 1\bar 1} - i\varphi_{\bar 1 0} + R\varphi_{\bar 1})\bar\varphi_1 \\
&= (\varphi_{1\bar 1\bar 1} - i\varphi_{0\bar 1} - i\varphi_{\bar 1 0} + R\varphi_{\bar 1})\bar\varphi_1 \\
&= (\bar P_1\varphi + iA_{\bar 1\bar 1}\varphi_1 - i\varphi_{0\bar 1} - i\varphi_{\bar 1 0} + R\varphi_{\bar 1})\bar\varphi_1 \\
&= (\bar P_1\varphi)\bar\varphi_1 + R\varphi_{\bar 1}\bar\varphi_1 + (iA_{\bar 1\bar 1}\varphi - i\varphi_{0\bar 1} - i\varphi_{\bar 1 0})\bar\varphi_1 \\
&=< \bar P\varphi, \bar\partial_b\varphi > + R|\bar\partial_b\varphi|^2 + (iA_{\bar 1\bar 1}\varphi_1 - i\varphi_{0\bar 1} - i\varphi_{\bar 1 0})\bar\varphi_1, \\
&=< \bar P\varphi, \bar\partial_b\varphi > + R|\bar\partial_b\varphi|^2 + 2(iA_{\bar 1\bar 1}\varphi_1 - i\varphi_{0\bar 1})\bar\varphi_1,
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
&-\frac{1}{2} < \bar\partial_b\Box_b\varphi, \bar\partial_b\varphi > \\
&=< \varphi_{\bar 1 1\bar 1}\theta^{\bar 1}, \varphi_{\bar 1}\theta^{\bar 1} > \\
&= \varphi_{\bar 1 1\bar 1}\bar\varphi_1 \\
&= (\varphi_{1\bar 1\bar 1} - i\varphi_{0\bar 1})\bar\varphi_1 \\
&= (\bar P_1\varphi + iA_{\bar 1\bar 1}\varphi_1 - i\varphi_{0\bar 1})\bar\varphi_1 \\
&=< \bar P\varphi, \bar\partial_b\varphi > + (iA_{\bar 1\bar 1}\varphi_1 - i\varphi_{0\bar 1})\bar\varphi_1
\end{aligned} \tag{2.5}
$$

Combining (4.55) and (4.61) ,we obtain (4.54). This completes the Proposition. □

We now prove Theorem 1.3.

Proof of Theorem 1.3 : Let φ be an eigenfunction with respect to a nonzero eigenvalue λ, that is, φ is not a CR function. Taking the integral of both sides of the Bochner formula (2.1), we have

$$
\begin{aligned}
0 = &\int \varphi_{\bar 1\bar 1}\bar\varphi_{11} + \int \varphi_{\bar 1 1}\bar\varphi_{1\bar 1} - \frac{3}{2}\lambda \int |\bar\partial_b\varphi|^2 \\
&+ \int < P_0\varphi, \varphi > + \int R|\bar\partial_b\varphi|^2.
\end{aligned} \tag{2.6}
$$

On the other hand,

$$\int \varphi_{\bar{1}1}\bar{\varphi}_{1\bar{1}} = \frac{1}{4}\int <\Box_b\varphi, \Box_b\varphi> = \frac{\lambda}{2}\int |\bar{\partial}_b\varphi|^2.$$

Taking together the above two formulae, we obtain

$$\begin{aligned} \lambda\int |\bar{\partial}_b\varphi|^2 &= \int |\varphi_{\bar{1}\bar{1}}|^2 + \int <P_0\varphi, \varphi> + \int R|\bar{\partial}_b\varphi|^2 \\ &\geq \int <P_0\varphi, \varphi> + \int R|\bar{\partial}_b\varphi|^2. \end{aligned} \tag{2.7}$$

Therefore, if P_0 is nonnegative and $R > 0$, then we immediately have that $\lambda \geq \min R$. Since the spectrum spec($\Box_b$) of the Kohn Laplacian in $(0, \infty)$ only consists of point eigenvalues (see Theorem 1.3 in [5]), it follows that the range of $\Box_b$ is closed. Applying the result of Kohn [12], we conclude M is embeddable.

To prove the second part of Theorem 1.3 note if the CR Yamabe constant > 0, then we can choose a contact form such that the Webster curvature with respect to this contact form is positive and so we conclude by the first part of Theorem 1.3.

Remark 2.2. The estimate for the nonzero eigenvalues is sharp. For example, the standard sphere S^3 as a pseudohermitian 3-manifold has the smallest nonzero eigenvalue $\lambda = 2 = R$, for details, see [7].

Remark 2.3. In general, let M^{2n+1} be a pseudohermitian manifold. The Bochner formula for the Kohn Laplacian is as follows:

$$\begin{aligned} -\frac{1}{2}\Box_b|\bar{\partial}_b\varphi|^2 = &\sum_{\alpha,\beta}(\varphi_{\bar{\alpha}\bar{\beta}}\bar{\varphi}_{\alpha\beta} + \varphi_{\bar{\alpha}\beta}\bar{\varphi}_{\alpha\bar{\beta}}) \\ &- \frac{1}{2n} <\bar{\partial}_b\varphi, \bar{\partial}_b\Box_b\varphi> - \frac{(n+1)}{2n} <\bar{\partial}_b\Box_b\varphi, \bar{\partial}_b\varphi> \\ &- \frac{1}{n} <\bar{P}\varphi, \bar{\partial}_b\varphi> + \frac{(n-1)}{n} <P\bar{\varphi}, \partial_b\bar{\varphi}> \\ &+ Ric(\nabla_b\varphi_{\mathbb{C}}, \nabla_b\varphi_{\mathbb{C}}), \end{aligned} \tag{2.8}$$

where $\nabla_b \varphi_{\mathbb{C}}$ is the corresponding complex $(1,0)$-vector of $\nabla_b \varphi$. The proof of (2.8) is the same as (2.1). Again, in case $n = 2$, using this formula we also obtain that the sharp lower bound of nonzero eigenvalues of the Kohn Laplacian $\Box_b$ is $\frac{4}{3}k_0$, provided that the Ricci curvature has the lower bound:

$$Ric(X, X) \geq k_0 |X|^2,$$

for some k_0 and for all complex $(1,0)$-vector X. Unfortunately, in the higher dimensional cases $n \geq 3$, the coefficient $\frac{(n-1)}{n}$ of the term $< P\bar{\varphi}, \partial_b \bar{\varphi} >$ is too large to get the lower bound of nonzero eigenvalues of the Kohn Laplacian $\Box_b$ immediately.

Example 2.4. In this example, we shall compute the Webster curvature of Rossi's global nonembeddability example together with a suitable contact structure and show that the associated CR Paneitz operators are not nonnegative. Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,||z_1|^2 + |z_2|^2 - 1 = 0\}$ be the boundary of the unit ball in $\mathbb{C}^2$ with the induced CR structure given by the complex vector field

$$Z_1 = \bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2},$$

and contact form

$$\theta = \frac{i(\bar{\partial} u - \partial u)}{2}|_{S^3},$$

where $u = |z_1|^2 + |z_2|^2 - 1$. Taking the admissible coframe

$$\theta^1 = z_2 dz_1 - z_1 dz_2,$$

we have $d\theta = i\theta^1 \wedge \theta^{\bar{1}}$. Rossi's example is the CR manifold S^3 together with the CR structure given by

$$L_t = Z_1 + t\bar{Z}_1,$$

for all $t \in \mathbb{R}$ and $t \neq 1, -1$. Now, for $|t| < 1$, taking the contact form

$$\theta(t) = \theta,$$

and the admissible coframe

$$\theta^1(t) = \frac{1}{\sqrt{1-t^2}}(\theta^1 - t\theta^{\bar{1}}),$$

we have $d\theta(t) = i\theta^1(t) \wedge \theta^{\bar{1}}(t)$ and the following Proposition:

Proposition 2.5. *For $|t| < 1$, with respect to the coframe $\{\theta(t), \theta^1(t), \theta^{\bar{1}}(t)\}$, the connection form and pseudohermitian torsion are as follows*

$$\theta_1{}^1(t) = \theta_1{}^1 - \frac{4t^2 i}{1-t^2}\theta = \frac{-2(1+t^2)}{1-t^2}i\theta; \tag{2.9}$$

and

$$\tau^1(t) = \frac{4ti}{1-t^2}\theta^{\bar{1}}(t),$$

where

$$\theta_1{}^1 = -\bar{z}_1 dz_1 - \bar{z}_2 dz_2 + z_1 d\bar{z}_1 + z_2 d\bar{z}_2 = -2i\theta.$$

In addition, the Webster curvature is $R(t) = \frac{2(1+t^2)}{1-t^2}$.

Proof. We just check that forms $\theta_1{}^1(t)$, $\tau^1(t)$ satisfy the equations (1.3). Finally, after a direct computation, we see that $d\theta_1{}^1(t) = \frac{2(1+t^2)}{1-t^2}\theta^1 \wedge \theta^{\bar{1}}$, so $R(t) = \frac{2(1+t^2)}{1-t^2}$. □

Similarly, for $|t| > 1$, take the contact form $\theta(t)$ and the admissible coframe $\theta^1(t)$ as follows:

$$\theta(t) = -\theta, \qquad \theta^1(t) = \frac{1}{\sqrt{t^2-1}}(\theta^1 + t\theta^{\bar{1}}).$$

Then we have

$$\tau^1(t) = \frac{4ti}{1-t^2}\theta^{\bar{1}}(t) \text{ and } R(t) = \frac{2(1+t^2)}{t^2-1}$$

From Theorem 1.3 and the above Proposition, we immediately obtain that the CR Paneitz operator of Rossi's nonembeddable manifolds are not nonnegative.

3. Stability of Embeddability

We now consider stability issues, see [5] and [14] for earlier work. We have a fixed CR structure on a compact manifold (M^3, θ, J). Let us denote by $\bar{L}$ the CR vector field on M^3. We now perturb $\bar{L}$ by a smooth family of functions $\varphi(\cdot, t) = \varphi_t(\cdot)$,where $(\cdot)$ represents a point on M, and $t \in (-\varepsilon, \varepsilon)$. We assume always,

$$D^{\alpha}_{z,s}\varphi(z,s,t)|_{t=0} = 0, \quad |\alpha| \le l_0, \quad l_0 \ge 4, \quad (z,s) \in M. \tag{3.1}$$

We define

$$\bar{L}_t = \bar{L} + \varphi(\cdot, t)L. \tag{3.2}$$

Associated to $\bar{L}_t$ we form the associated $\bar{\partial}_b^{(t)}$-Laplacian operator,

$$\Box_b^{(t)} = \bar{\partial}_b^{(t)*}\bar{\partial}_b^{(t)}. \tag{3.3}$$

We now use our main result to guarantee embedding of our CR structure in $\mathbb{C}^N$. Thus we assume that along the deformation path in t,

(3.4) the associated Paneitz operator $P_0^{(t)} \ge 0$,

(3.5) the CR Yamabe constant $\ge c > 0$.

By our main result (Theorem 1.3), using (3.4) and (3.5) it follows that

$$\lambda_1(\Box_b^{(t)}) \ge \nu > 0, \tag{3.6}$$

with ν independent of t. Thus by using the construction of Boutet de Monvel in [4] or the exposition in Chen and Shaw's book [8], we can embed for $\varepsilon > 0$ small enough the CR structures via a map Ψ_t into the same $\mathbb{C}^N$, i.e.,

$$\Psi_t : (M, \theta, J_t) \longrightarrow \mathbb{C}^N\,. \tag{3.7}$$

The question arises if the maps Ψ_t are close in say the sup-norm in t. We have the following theorem which we re-state from the introduction:

THEOREM 1.5. *Under* (3.1), (3.4), (3.5), *for any* $\delta > 0$, *there exists* $\varepsilon > 0$, *so that*

$$\sup_{t\in[-\varepsilon,\varepsilon]} \|\Psi_t - \Psi_0\|_{C^k(M)} < \delta, \quad k = k(l_0).$$

Proof. The proof of this theorem is abstract and relies on an identity in [5]. We use Proposition 5.55 in [5]. We denote the projection into the zero eigenspace of $\Box_b^{(t)}$ by $\Im^{\varphi_t}$, which is the Szego projector. By the spectral theorem and (3.6) if $|\lambda| = \nu/2$, the resolvent $(\Box_b^{(t)} - \lambda)^{-1}$ is well-defined and so

$$\Im^{\varphi_t} = \int_{|\lambda|=\nu/2} (\Box_b^{(t)} - \lambda)^{-1} d\lambda,$$

and it is immediate that $\Im^{\varphi_t}$ is a bounded operator on $L^2(M)$. As observed in [5] as a consequence of the above fact and their identity (5.58) they obtain the inequality (5.60) which we re-state,

$$\|\Im^{\varphi_t} - \Im^{\varphi_0}\|_{L^2(M)} \le C\nu A\|\varphi_t - \varphi_0\|_{L^\infty(M)}, \tag{3.8}$$

where A is the sup norm of some high enough derivative of $\varphi_t - \varphi_0$. But by our hypothesis (3.1) the right side of (3.8) is smaller than $\delta > 0$, for $\varepsilon > 0$, sufficiently small.

Now recall the construction of Boutet de Monvel. Using the notation in [8], page 318, the embedding for each coordinate chart is given by a CR function h_t (we are in CR dimension 1), where

$$h_t = \Im^{\varphi_t}\big(\psi e^{-\tau\varphi_p}\big), \quad \tau \to \infty. \tag{3.9}$$

Now note $h_t - h_0$ also satisfy an equation, that is,

$$\Box_b^{(t)}(h_t - h_0) = \big(\Box_b^{(0)} - \Box_b^{(t)}\big)(h_0). \tag{3.10}$$

From (3.8), (3.9), $\|h_t - h_0\|_{L^2(M)} < \delta$.

From (3.1) and (3.10), the right side of (3.10) is small in the C^∞-norm. Since we have (3.6), it now implies by sub-elliptic regularity that for $\delta > 0$, there exists ε_0,

$$\sup_{t\in(-\varepsilon_0,\varepsilon_0)} \|h_t - h_0\|_{C^\infty(M)} \leq \delta. \tag{3.11}$$

In fact by differentiation of (3.10) in t, we may also obtain higher stability in t, provided we replace (3.1) by the stronger hypothesis that $D^\alpha_{z,s} D^\beta_t \varphi(z,s,0) = 0$ for large enough $|\alpha|, |\beta|$. This proves our theorem since on coordinate charts of M, the map Ψ_t is given by h_t. □

4. The second variation of the Paneitz operator

Our goal in this section is to investigate CR structures close to the standard structure on S^3 and prove Theorem 1.9, which is a converse to Theorem 1.3. To achieve our goal we compute the second variation of the Paneitz operator. Let (M, J, θ) be a three-dimensional pseudo-hermitian manifold. In the computation, the contact form θ is always fixed and we suppose that the CR structure J is given by the the $(0,1)$-complex vector field $Z_{\bar 1}$.

Suppose $\phi \in C^\infty(M)$ with $|\phi| < 1$. Then the complex vector field

$$Z_{\bar 1}{}^{\phi} = Z_{\bar 1} + \phi Z_1 \tag{4.1}$$

defines a strictly pseudoconvex CR structure on M.

For the purpose of computing the second variation, we need to know exactly what the connection and torsion forms are for the manifold with CR structure defined by the complex vector field (4.1). Therefore, first of all, we focus on the computation of the connection form and torsion form and then use them to obtain the second variation of the Paneitz operator.

Let θ^1 denote the $(1,0)$-form dual to Z_1. We take

$$\theta^1{}_{\phi} = F(\phi)(\theta^1 - \phi\theta^{\bar 1}) \tag{4.2}$$

as an admissible coframe, where

$$F = F(\phi) = \frac{1}{(1-|\phi|^2)^{1/2}},$$

which is a real function. For simplifying the computation, we normalize ${Z_{\bar 1}}^{\phi}$ by setting

$$ {Z_{\bar 1}}^{\phi} = F(\phi)(Z_{\bar 1} + \phi Z_1) $$

such that $\{{Z_1}^{\phi}, {Z_{\bar 1}}^{\phi}, T\}$ is dual to $\{{\theta^1}_{\phi}, {\theta^{\bar 1}}_{\phi}, \theta\}$ and ${h_{1\bar 1}}^{\phi} \equiv h_{1\bar 1}$. Now we are ready to compute the connection and torsion forms, which are denoted by ${{\theta_1}^1}_{\phi}$ and ${\tau^1}_{\phi}$, respectively. They are determined by the following structure equations:

$$ \begin{aligned} d{\theta^1}_{\phi} &= {\theta^1}_{\phi} \wedge {{\theta_1}^1}_{\phi} + \theta \wedge {\tau^1}_{\phi} \\ {\tau^1}_{\phi} &= 0, \quad \mod {\theta^{\bar 1}}_{\phi} \\ {h^{1\bar 1}}_{\phi} d{h_{1\bar 1}}^{\phi} &= {{\theta_1}^1}_{\phi} + {{\theta_{\bar 1}}^{\bar 1}}_{\phi}, \end{aligned} \tag{4.3} $$

where ${h^{1\bar 1}}_{\phi}$ is the inverse of ${h_{1\bar 1}}^{\phi}$. Denote ${\tau^1}_{\phi} = {A^1}_{\bar 1}{}^{\phi}{\theta^{\bar 1}}_{\phi}$. Then we have the following proposition

Proposition 4.1. *We have*

$$ \begin{aligned} {{\theta_1}^1}_{\phi} &= {\theta_1}^1 - F^{-1}dF - F^{-1}(B_{11}\theta^1 + B_{12}\theta^{\bar 1} + B_{13}\theta); \\ {A^1}_{\bar 1}{}^{\phi} &= {A^1}_{\bar 1} - F^2\Big(\phi_0 + \phi{\theta_1}^1(T) - \phi{\theta_{\bar 1}}^{\bar 1}(T) + \phi^2{A^{\bar 1}}_1 - |\phi|^2{A^1}_{\bar 1}\Big), \end{aligned} \tag{4.4} $$

where

$$ \begin{aligned} B_{11} &= F^2\Big(- 2F_1 - \bar\phi F{\theta_{\bar 1}}^{\bar 1}(Z_{\bar 1}) + \bar\phi F{\theta_1}^1(Z_{\bar 1}) \\ &\quad - F\bar\phi_{\bar 1} - \bar\phi F\phi_1 - 2\bar\phi F_{\bar 1} - |\phi|^2F{\theta_1}^1(Z_1) + |\phi|^2F{\theta_{\bar 1}}^{\bar 1}(Z_1)\Big); \\ B_{12} &= F^2\Big(2|\phi|^2F_{\bar 1} + \phi F{\theta_1}^1(Z_1) - \phi F{\theta_{\bar 1}}^{\bar 1}(Z_1) \\ &\quad + F\phi_1 + \phi F\bar\phi_{\bar 1} + 2\phi F_1 + |\phi|^2F{\theta_{\bar 1}}^{\bar 1}(Z_{\bar 1}) - |\phi|^2F{\theta_1}^1(Z_{\bar 1})\Big); \\ B_{13} &= F^3\Big(- \bar\phi(\phi_0 + \phi{\theta_1}^1(T) - \phi{\theta_{\bar 1}}^{\bar 1}(T)) + \bar\phi{A^1}_{\bar 1} - \phi{A^{\bar 1}}_1\Big). \end{aligned} \tag{4.5} $$

Proof. From the structure equations (1.3), we have

$$ d\theta^1 = \theta^1 \wedge {\theta_1}^1 + \theta \wedge \tau^1, $$

thus

$$
\begin{aligned}
d{\theta^1}_\phi &= d(F\theta^1 - \phi F\theta^{\bar 1})\\
&= dF\wedge\theta^1 + Fd\theta^1 - d(\phi F)\wedge\theta^{\bar 1} - \phi F d\theta^{\bar 1}\\
&= dF\wedge\theta^1 + F\theta^1\wedge{\theta_1}^1 + F\theta\wedge\tau^1\\
&- d(\phi F)\wedge\theta^{\bar 1} - \phi F\theta^{\bar 1}\wedge{\theta_{\bar 1}}^{\bar 1} - \phi F\theta\wedge\tau^{\bar 1}\\
&= (dF - F{\theta_1}^1)\wedge\theta^1 + (d(\phi F) - \phi F{\theta_{\bar 1}}^{\bar 1})\wedge\theta^{\bar 1} + (-F\tau^1 + \phi F\tau^{\bar 1})\wedge\theta.
\end{aligned}
\tag{4.6}
$$

On the other hand,

$$
\begin{aligned}
&{\theta^1}_\phi\wedge{{\theta_1}^1}_\phi + \theta\wedge{\tau^1}_\phi\\
=&(F\theta^1 - \phi F\theta^{\bar 1})\wedge{{\theta_1}^1}_\phi + \theta\wedge{\tau^1}_\phi\\
=&(-F{{\theta_1}^1}_\phi)\wedge\theta^1 + (\phi F{{\theta_1}^1}_\phi)\wedge\theta^{\bar 1} + (-{\tau^1}_\phi)\wedge\theta.
\end{aligned}
\tag{4.7}
$$

Comparing (4.6) and (4.7), we see, by Cartan's lemma, that there exists complex-valued functions $B_{ij},\ i,j = 1,2,3$ such that

$$
\begin{aligned}
-F{{\theta_1}^1}_\phi &= (dF - F{\theta_1}^1) + B_{11}\theta^1 + B_{12}\theta^{\bar 1} + B_{13}\theta;\\
\phi F{{\theta_1}^1}_\phi &= -d(\phi F) + \phi F{\theta_{\bar 1}}^{\bar 1}) + B_{21}\theta^1 + B_{22}\theta^{\bar 1} + B_{23}\theta;\\
-{\tau^1}_\phi &= -F\tau^1 + \phi F\tau^{\bar 1} + B_{31}\theta^1 + B_{32}\theta^{\bar 1} + B_{33}\theta,
\end{aligned}
\tag{4.8}
$$

and

$$
B_{ij} = B_{ji}. \tag{4.9}
$$

Therefore, from (4.8), we have

$$
\begin{aligned}
& A^1{}_{\bar{1}}{}^{\phi}(F\theta^{\bar{1}} - \bar{\phi}F\theta^1) \\
=& A^1{}_{\bar{1}}{}^{\phi}\theta^{\bar{1}} \\
=& \tau^1{}_{\phi} \\
=& F\tau^1 - \phi F\tau^{\bar{1}} - B_{31}\theta^1 - B_{32}\theta^{\bar{1}} - B_{33}\theta \\
=& FA^1{}_{\bar{1}}\theta^{\bar{1}} - \phi FA^{\bar{1}}{}_1\theta^1 - B_{31}\theta^1 - B_{32}\theta^{\bar{1}} - B_{33}\theta,
\end{aligned}
$$

hence, comparing the coeficients, we get

$$
FA^1{}_{\bar{1}}{}^{\phi} = FA^1{}_{\bar{1}} - B_{32}; \tag{4.10}
$$

$$
-F\bar{\phi}A^1{}_{\bar{1}}{}^{\phi} = -F\phi A^{\bar{1}}{}_1 - B_{31}; \tag{4.11}
$$

$$
B_{33} = 0 \tag{4.12}
$$

Taken together (4.10) and (4.11) implies

$$
A^1{}_{\bar{1}}{}^{\phi} = A^1{}_{\bar{1}} - \frac{B_{32}}{F} = \frac{\phi}{\bar{\phi}}A^{\bar{1}}{}_1 + \frac{B_{31}}{\bar{\phi}F},
$$

that is,

$$
\bar{\phi}FA^1{}_{\bar{1}} - \bar{\phi}B_{32} = \phi FA^{\bar{1}}{}_1 + B_{31}. \tag{4.13}
$$

Now, from (4.8) again, multiplying the first formula by ϕ and adding the second formula, we get

$$\begin{aligned}
0 &= \phi(dF - F\theta_1{}^1) + \phi B_{11}\theta^1 + \phi B_{12}\theta^{\bar 1} + \phi B_{13}\theta \\
&\quad - d(\phi F) + \phi F\theta_{\bar 1}{}^{\bar 1} + B_{21}\theta^1 + B_{22}\theta^{\bar 1} + B_{23}\theta \\
&= -Fd\phi - \phi F\theta_1{}^1 + \phi F\theta_{\bar 1}{}^{\bar 1} + (\phi B_{11} + B_{21})\theta^1 + (\phi B_{12} + B_{22})\theta^{\bar 1} + (\phi B_{13} + B_{23})\theta \\
&= (-F\phi_1 - \phi F\theta_1{}^1(Z_1) + \phi F\theta_{\bar 1}{}^{\bar 1}(Z_1) + \phi B_{11} + B_{21})\theta^1 \\
&\quad + (-F\phi_{\bar 1} - \phi F\theta_1{}^1(Z_{\bar 1}) + \phi F\theta_{\bar 1}{}^{\bar 1}(Z_{\bar 1}) + \phi B_{12} + B_{22})\theta^{\bar 1} \\
&\quad + (-F\phi_0 - \phi F\theta_1{}^1(T) + \phi F\theta_{\bar 1}{}^{\bar 1}(T) + \phi B_{13} + B_{23})\theta,
\end{aligned}$$

that is,

$$-F\phi_1 - \phi F\theta_1{}^1(Z_1) + \phi F\theta_{\bar 1}{}^{\bar 1}(Z_1) + \phi B_{11} + B_{21} = 0 \tag{4.14}$$

$$-F\phi_{\bar 1} - \phi F\theta_1{}^1(Z_{\bar 1}) + \phi F\theta_{\bar 1}{}^{\bar 1}(Z_{\bar 1}) + \phi B_{12} + B_{22} = 0 \tag{4.15}$$

$$-F\phi_0 - \phi F\theta_1{}^1(T) + \phi F\theta_{\bar 1}{}^{\bar 1}(T) + \phi B_{13} + B_{23} = 0. \tag{4.16}$$

Multiplying (4.13) by ϕ and subtracting (4.16) we obtain

$$\begin{aligned}
B_{23} &= \frac{F\phi_0 + \phi F\theta_1{}^1(T) - \phi F\theta_{\bar 1}{}^{\bar 1}(T) + \phi^2 FA^{\bar 1}{}_1 - |\phi|^2 FA^1{}_{\bar 1}}{(1 - |\phi|^2)} \\
&= F^3(\phi_0 + \phi\theta_1{}^1(T) - \phi\theta_{\bar 1}{}^{\bar 1}(T) + \phi^2 A^{\bar 1}{}_1 - |\phi|^2 A^1{}_{\bar 1}).
\end{aligned} \tag{4.17}$$

Since $F^2 = 1 + |\phi|^2 F^2$, substituting (4.17) into (4.13), we obtain

$$B_{13} = F^3\Big(- \bar\phi(\phi_0 + \phi\theta_1{}^1(T) - \phi\theta_{\bar 1}{}^{\bar 1}(T)) + \bar\phi A^1{}_{\bar 1} - \phi A^{\bar 1}{}_1\Big) \tag{4.18}$$

Now, substituting (4.17) into (4.10), we obtain

$$A^1{}_{\bar 1}{}^{\phi} = A^1{}_{\bar 1} - F^2\Big(\phi_0 + \phi\theta_1{}^1(T) - \phi\theta_{\bar 1}{}^{\bar 1}(T) + \phi^2 A^{\bar 1}{}_1 - |\phi|^2 A^1{}_{\bar 1}\Big) \tag{4.19}$$

Finally, to complete the proof of the proposition, we need to determine B_{11} and B_{12}. Taking the conjugate of the first formula of (4.8), we get

$$\begin{aligned} -F\theta_{\bar{1}}{}^{\bar{1}}{}_{\phi} &= (dF - F\theta_{\bar{1}}{}^{\bar{1}}) + \overline{B_{11}}\theta^{\bar{1}} + \overline{B_{12}}\theta^{1} + \overline{B_{13}}\theta \\ &= (dF - F(-\theta_{1}{}^{1} + h^{1\bar{1}}dh_{1\bar{1}})) + \overline{B_{11}}\theta^{\bar{1}} + \overline{B_{12}}\theta^{1} + \overline{B_{13}}\theta \\ &= dF + F\theta_{1}{}^{1} - Fh^{1\bar{1}}dh_{1\bar{1}} + \overline{B_{11}}\theta^{\bar{1}} + \overline{B_{12}}\theta^{1} + \overline{B_{13}}\theta. \end{aligned}$$

On the other hand,

$$\begin{aligned} -F\theta_{\bar{1}}{}^{\bar{1}}{}_{\phi} &= -F(-\theta_{1}{}^{1}{}_{\phi} + h^{1\bar{1}}dh_{1\bar{1}}) \\ &= -(dF - F\theta_{1}{}^{1}) - B_{11}\theta^{1} - B_{12}\theta^{\bar{1}} - B_{13}\theta - Fh^{1\bar{1}}dh_{1\bar{1}}, \end{aligned}$$

where the last equality is due to the first formula of (4.8). Comparing the above two formula, we have

$$2dF = (-B_{11} - \overline{B_{12}})\theta^{1} + (-B_{12} - \overline{B_{11}})\theta^{\bar{1}} + (-B_{13} - \overline{B_{13}})\theta,$$

which implies that

$$B_{12} = -\overline{B_{11}} - 2F_{\bar{1}}. \tag{4.20}$$

Substituting (4.20) into (4.14), we get

$$\phi B_{11} = \overline{B_{11}} + 2F_{\bar{1}} + F\phi_{1} + \phi F\theta_{1}{}^{1}(Z_{1}) - \phi F\theta_{\bar{1}}{}^{\bar{1}}(Z_{1}). \tag{4.21}$$

Now multiplying (4.21) by $\bar{\phi}$ and subtracting the conjugate of (4.21), we obtain

$$\begin{aligned} B_{11} = F^{2}\Big(&-2F_{1} - \bar{\phi}F\theta_{\bar{1}}{}^{\bar{1}}(Z_{\bar{1}}) + \bar{\phi}F\theta_{1}{}^{1}(Z_{\bar{1}}) \\ &- F\bar{\phi}_{\bar{1}} - \bar{\phi}F\phi_{1} - 2\bar{\phi}F_{\bar{1}} - |\phi|^{2}F\theta_{1}{}^{1}(Z_{1}) + |\phi|^{2}F\theta_{\bar{1}}{}^{\bar{1}}(Z_{1})\Big). \end{aligned} \tag{4.22}$$

Substituting this into (4.20), we get

$$\begin{aligned} B_{12} = F^{2}\Big(&2|\phi|^{2}F_{\bar{1}} + \phi F\theta_{1}{}^{1}(Z_{1}) - \phi F\theta_{\bar{1}}{}^{\bar{1}}(Z_{1}) \\ &+ F\phi_{1} + \phi F\bar{\phi}_{\bar{1}} + 2\phi F_{1} + |\phi|^{2}F\theta_{\bar{1}}{}^{\bar{1}}(Z_{\bar{1}}) - |\phi|^{2}F\theta_{1}{}^{1}(Z_{\bar{1}})\Big). \end{aligned} \tag{4.23}$$

This finishes the proof of the proposition. ∎

According to Example 2.4, we see that if S^3 is the 3-sphere with the standard CR structure and contact form then

$$A^1{}_{\bar 1} \equiv 0, \quad h_{1\bar 1} \equiv 1, \quad R \equiv 2, \quad \theta_1{}^1(T) = -2i, \quad \theta_1{}^1(Z_1) = \theta_1{}^1(Z_{\bar 1}) = 0.$$

Therefore we have the following corollary.

Corollary 4.2. *On S^3, the connection form and torsion with respect to the CR structure given by*

$$Z_{\bar 1}{}^\phi = Z_{\bar 1} + \phi Z_1$$

are

$$\begin{aligned} &\theta_1{}^1{}_\phi = \theta_1{}^1 - F^{-1}dF - F^{-1}(B_{11}\theta^1 + B_{12}\theta^{\bar 1} + B_{13}\theta); \\ &A^1{}_{\bar 1}{}^\phi = -F^2(\phi_0 - 4i\phi), \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} B_{11} &= F^2(-2F_1 - F\bar\phi_{\bar 1} - \bar\phi F\phi_1 - 2\bar\phi F_{\bar 1}); \\ B_{12} &= F^2(2|\phi|^2 F_{\bar 1} + F\phi_1 + \phi F\bar\phi_{\bar 1} + 2\phi F_1); \\ B_{13} &= -\bar\phi F^3(\phi_0 - 4i\phi). \end{aligned} \tag{4.25}$$

Corollary 4.3. *We have that on S^3*

$$A^1{}_{\bar 1}{}^\phi = 0 \Longleftrightarrow \phi_0 = 4i\phi \Longleftrightarrow \phi \in P_{p,q}, \ p = q + 4, \tag{4.26}$$

where

$$P_{p,q} = sp\{z_1^a z_2^b \bar z_1{}^c \bar z_2{}^d | \ a + b = p, \ c + d = q\}. \tag{4.27}$$

We are now ready to compute the first and second variations of the Paneitz operator. The background space is the standard 3-sphere $S^3 \subset C^2$ with the CR structure given by the complex vector field

$$Z_{\bar 1} = z_2 \frac{\partial}{\partial \bar z_1} - z_1 \frac{\partial}{\partial \bar z_2}.$$

Fix ϕ, we use $\phi_t = t\phi$ to define the deformation of the CR structures along ϕ, i.e., for each t, the CR structure is defined by the complex vector field

$$Z_{\bar{1}}^t = Z_{\bar{1}}{}^{\phi_t} = F(Z_{\bar{1}} + t\phi Z_1), \tag{4.28}$$

where $F = \frac{1}{(1-t^2|\phi|^2)^{1/2}}$. The Kohn Laplacian for the deformed structure will be denoted by $\Box_b^t$. Then the coresponding Paneitz operator satisfies

$$4P_0^t = \Box_b^t \overline{\Box}_b^t - 2Q^t, \tag{4.29}$$

where Q^t is a second order differential operator defined by $Q^t f = 2i(A^{t11} f_1)_1$ for each smooth function f, i.e.,

$$Q^t = 2i\Big(A^{t11} Z_1^t Z_1^t + (Z_1^t A^{t11}) Z_1^t - A^{t11} \theta_{\bar{1}}^{t\bar{1}}(Z_1^t) Z_1^t\Big). \tag{4.30}$$

We would like to compute the first and second variations of $4P_0$ and use "$\cdot$" to denote differentiation with respect to t. We have the following proposition.

Proposition 4.4. *We have*

$$4\dot{P}_0^t|_{t=0} = -2D\overline{\Box}_b - 2\Box_b D + 4(EZ_1Z_1 + E_1Z_1), \tag{4.31}$$

and

$$\begin{aligned} 4\ddot{P}_0^t|_{t=0} &= 16|\phi|^2 P_0 + 2|\phi|^2(\Box_b\Box_b + \overline{\Box}_b\overline{\Box}_b) + 8D^2 - 8E\bar{\phi}\Delta_b + 8\nabla_b(E\bar{\phi}) \\ &\quad + 4(\Box_b|\phi|^2)\Delta_b - 8(\nabla_b|\phi|^2)\Delta_b - 4(\nabla_b|\phi|^2)\overline{\Box}_b - 4\Box_b(\nabla_b|\phi|^2), \end{aligned} \tag{4.32}$$

where

$$D = \phi Z_1 Z_1 + \bar{\phi} Z_{\bar{1}} Z_{\bar{1}} + \phi_1 Z_1 + \bar{\phi}_{\bar{1}} Z_{\bar{1}};$$

$$E = 4\phi + i\phi_0$$

Proof. The first and the second derivative with respect to t are, respectively,

$$4\dot{P}_0^t = \dot{\Box}_b^t \overline{\Box}_b^t + \Box_b^t \dot{\overline{\Box}}_b^t - 2\dot{Q}^t; \tag{4.33}$$

and

$$4\ddot{P}_0^t = \ddot{\Box}_b^t \overline{\Box}_b^t + \Box_b^t \ddot{\overline{\Box}}_b^t + 2\dot{\Box}_b^t \dot{\overline{\Box}}_b^t - 2\ddot{Q}^t. \tag{4.34}$$

From (4.24) and (4.25), we have

$$\begin{aligned} \theta_1^{t1}(Z_{\bar{1}}^t) &= \theta_1^{t1}(F(Z_{\bar{1}} + t\phi Z_1)) \\ &= -(F_{\bar{1}} + B_{12} + t\phi F_1 + t\phi B_{11}), \end{aligned} \tag{4.35}$$

where

$$\begin{aligned} B_{11} &= F^2(-2F_1 - tF\bar{\phi}_{\bar{1}} - t^2\bar{\phi}F\phi_1 - 2t\bar{\phi}F_{\bar{1}}) \\ &= -t(F^3\bar{\phi}_{\bar{1}}) - t^2(F^3\bar{\phi}\phi_1 + F^5(Z_1|\phi|^2)) + O(t^3), \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} B_{12} &= F^2(2t^2|\phi|^2F_{\bar{1}} + tF\phi_1 + t^2F\phi\bar{\phi}_{\bar{1}} + 2t\phi F_1) \\ &= t(F^3\phi_1) + t^2(F^3\phi\bar{\phi}_{\bar{1}}) + O(t^3). \end{aligned} \tag{4.37}$$

Now we compute the Kohn Laplacian and its variations. We have

$$\begin{aligned} \overline{\Box}_b^t &= -2Z_{\bar{1}}^t Z_1^t + 2\theta_1^{t1}(Z_{\bar{1}}^t)Z_1^t \\ &= -2Z_{\bar{1}}^t Z_1^t - 2(F_{\bar{1}} + B_{12} + t\phi F_1 + t\phi B_{11})Z_1^t; \end{aligned} \tag{4.38}$$

thus

$$\begin{aligned} \dot{\overline{\Box}}_b^t = {} & -2(\dot{Z}_{\bar{1}}^t Z_1^t + Z_{\bar{1}}^t \dot{Z}_1^t) \\ & - 2(\dot{F}_{\bar{1}} + \dot{B}_{12} + \phi F_1 + t\phi\dot{F}_1 + \phi B_{11} + t\phi\dot{B}_{11})Z_1^t \\ & - 2(F_{\bar{1}} + B_{12} + t\phi F_1 + t\phi B_{11})\dot{Z}_1^t, \end{aligned} \tag{4.39}$$

where for any complex vector field $Z \in TS^3$, we have

$$ZF = \frac{1}{2}t^2F^3(Z|\phi|^2)$$
$$\dot{F} = t|\phi|^2F^3$$
$$Z\dot{F} = tF^3(Z|\phi|^2) + \frac{3}{2}t^3|\phi|^2F^5(Z|\phi|^2),$$

and

$$\dot{Z}^t_{\bar{1}} = \dot{F}(Z_{\bar{1}} + t\phi Z_1) + F\phi Z_1.$$

Next, we would like to expand $-2Q^t$ with respect to t at $t = 0$. Denote $4\phi + i\phi_0$ as E. From (4.24), we see that

$$A^{t1}{}_{\bar{1}} = F^2(4it\phi - t\phi_0) = itF^2E,$$

hence, from (4.30),

$$\begin{aligned} -2Q^t &= -4i\Big(A^{t11}Z^t_1Z^t_1 + (Z^t_1A^{t11})Z^t_1 - A^{t11}\theta^{t\bar{1}}_{\bar{1}}(Z^t_1)Z^t_1\Big) \\ &= -4i(A^{t11}Z^t_1Z^t_1 + (Z^t_1A^{t11})Z^t_1) - 4iA^{t11}(F_1 + \overline{B_{12}} + t\bar{\phi}F_{\bar{1}} + t\bar{\phi}\overline{B_{11}})Z^t_1 \\ &= 4tF^2EZ^t_1Z^t_1 + (Z^t_1(4tF^2E))Z^t_1 + 4tF^2E(F_1 + \overline{B_{12}} + t\bar{\phi}F_{\bar{1}} + t\bar{\phi}\overline{B_{11}})Z^t_1, \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} 4tF^2EZ^t_1Z^t_1 &= 4tF^2EF(Z_1 + t\bar{\phi}Z_{\bar{1}})\big(F(Z_1 + t\bar{\phi}Z_{\bar{1}})\big) \\ &= 4F^4E(tZ_1Z_1 - t^2\bar{\phi}\Delta_b + t^2\bar{\phi}_1Z_{\bar{1}}) + O(t^3); \\ (Z^t_1(4tF^2E))Z^t_1 &= 4tF^2\Big((Z_1 + t\bar{\phi}Z_{\bar{1}})(F^2E)\Big)(Z_1 + t\bar{\phi}Z_{\bar{1}}) \\ &= 4t(F^4E_1)Z_1 + 4t^2F^4(\bar{\phi}E_{\bar{1}}Z_1 + \bar{\phi}E_1Z_{\bar{1}}) + O(t^3); \end{aligned} \tag{4.41}$$

and

$$\begin{aligned} &F_1 + \overline{B_{12}} + t\bar{\phi}F_{\bar{1}} + t\bar{\phi}\overline{B_{11}} \\ =&\frac{1}{2}t^2F^3(Z_1|\phi|^2) + tF^3\bar{\phi}_{\bar{1}} + t^2(F^3\bar{\phi}\phi_1) + O(t^3) - t^2F^3\phi_1\bar{\phi} \\ =&\frac{1}{2}t^2F^3(Z_1|\phi|^2) + tF^3\bar{\phi}_{\bar{1}} + O(t^3). \end{aligned} \tag{4.42}$$

Substituting (4.41) and (4.42) into (4.40), we get

$$
\begin{aligned}
-2Q^t &= 4tF^4(EZ_1Z_1 + E_1Z_1) + 4t^2F^4E(-\bar{\phi}\Delta_b + \bar{\phi}_1 Z_{\bar{1}}) \\
&\quad + 4t^2F^4(\bar{\phi}E_{\bar{1}}Z_1 + \bar{\phi}E_1Z_{\bar{1}}) + 4t^2F^6E\bar{\phi}_{\bar{1}}Z_1 + O(t^3).
\end{aligned} \tag{4.43}
$$

Therefore, from (4.33) together with (4.36), (4.37), (4.38), (4.39) and (4.43), we get

$$
\begin{aligned}
\overline{\Box}_b^t|_{t=0} &= \overline{\Box}_b; \\
\dot{\overline{\Box}}_b^t|_{t=0} &= -2(\phi Z_1Z_1 + \bar{\phi}Z_{\bar{1}}Z_{\bar{1}} + \bar{\phi}_{\bar{1}}Z_{\bar{1}} + \phi_1 Z_1); \\
-2\dot{Q}^t|_{t=0} &= 4(EZ_1Z_1 + E_1Z_1),
\end{aligned}
$$

hence the first variation of $4P_0$:

$$
4\dot{P}_0^t|_{t=0} = -2D\overline{\Box}_b - 2\Box_b D + 4(EZ_1Z_1 + E_1Z_1), \tag{4.44}
$$

where

$$
D = \phi Z_1Z_1 + \bar{\phi}Z_{\bar{1}}Z_{\bar{1}} + \phi_1 Z_1 + \bar{\phi}_{\bar{1}}Z_{\bar{1}}.
$$

Finally, for the second variation of Paneitz operator, we also need to compute the second variation of the Kohn Laplacian. From (4.39), taking the derivative with respect to t, we get

$$
\begin{aligned}
\ddot{\overline{\Box}}_b^t &= -2(\ddot{Z}_{\bar{1}}^t Z_1^t + Z_{\bar{1}}^t \ddot{Z}_1^t + 2\dot{Z}_{\bar{1}}^t \dot{Z}_1^t) \\
&\quad - 2(\ddot{F}_{\bar{1}} + \ddot{B}_{12} + 2\phi\dot{F}_1 + t\phi\ddot{F}_1 + 2\phi\dot{B}_{11} + t\phi\ddot{B}_{11})Z_1^t \\
&\quad - 4(\dot{F}_{\bar{1}} + \dot{B}_{12} + \phi F_1 + t\phi\dot{F}_1 + \phi B_{11} + t\phi\dot{B}_{11})\dot{Z}_1^t \\
&\quad - 2(F_{\bar{1}} + B_{12} + t\phi F_1 + t\phi B_{11})\ddot{Z}_1^t,
\end{aligned} \tag{4.45}
$$

where, for all $Z \in TS^3$,

$$\ddot{Z}^t_{\bar{1}} = \ddot{F}(Z_{\bar{1}} + t\phi Z_1) + 2\dot{F}\phi Z_1;$$

$$\ddot{F} = |\phi|^2 F^3 + 3t^2|\phi|^4 F^5;$$

$$Z\ddot{F} = F^3(Z|\phi|^2) + \frac{15}{2}t^2|\phi|^2 F^5(Z|\phi|^2) + \frac{15}{2}t^4|\phi|^4 F^7(Z|\phi|^2),$$

which implies that

$$\ddot{\overline{\Box}}^t_b|_{t=0} = 2|\phi|^2(\overline{\Box}_b + \Box_b) - 4(Z_1|\phi|^2)Z_{\bar{1}} - 4(Z_{\bar{1}}|\phi|^2)Z_1. \tag{4.46}$$

Therefore, from (4.34) together with (4.36), (4.37), (4.38), (4.39), (4.43) and (4.46), we get

$$\begin{aligned} 4\ddot{P}^t_0|_{t=0} = {} & 2|\phi|^2(\overline{\Box}_b + \Box_b)\overline{\Box}_b + \Box_b(2|\phi|^2(\overline{\Box}_b + \Box_b)) \\ & - 4((Z_{\bar{1}}|\phi|^2)Z_1 + (Z_1|\phi|^2)Z_{\bar{1}})\overline{\Box}_b - 4\Box_b((Z_{\bar{1}}|\phi|^2)Z_1 + (Z_1|\phi|^2)Z_{\bar{1}}) \\ & + 8D^2 + 8E(-\bar{\phi}\Delta_b) + 8\bar{\phi}(E_{\bar{1}}Z_1 + E_1 Z_{\bar{1}}) + 8E(\bar{\phi}_{\bar{1}}Z_1 + \bar{\phi}_1 Z_{\bar{1}}), \end{aligned}$$

where

$$\begin{aligned} & \Box_b(2|\phi|^2(\overline{\Box}_b + \Box_b)) \\ = {} & 4\Box_b(|\phi|^2\Delta_b) \\ = {} & 4[(\Box_b|\phi|^2)\Delta_b + |\phi|^2\Box_b\Delta_b - 2(|\phi|^2_{\bar{1}}Z_1\Delta_b + |\phi|^2_1 Z_{\bar{1}}\Delta_b)] \\ = {} & 2|\phi|^2\Box_b(\overline{\Box}_b + \Box_b) + 4(\Box_b|\phi|^2)\Delta_b - 8(\nabla_b|\phi|^2)\Delta_b, \end{aligned}$$

hence

$$\begin{aligned} 4\ddot{P}^t_0|_{t=0} = {} & 16|\phi|^2 P_0 + 2|\phi|^2(\Box_b\Box_b + \overline{\Box}_b\overline{\Box}_b) + 8D^2 - 8E\bar{\phi}\Delta_b + 8\nabla_b(E\bar{\phi}) \\ & + 4(\Box_b|\phi|^2)\Delta_b - 8(\nabla_b|\phi|^2)\Delta_b - 4(\nabla_b|\phi|^2)\overline{\Box}_b - 4\Box_b(\nabla_b|\phi|^2). \end{aligned} \tag{4.47}$$

This completes the proposition. □

For the reader's convenience we list the following useful facts [7] that are necessary for the subsequent computations. We recall $H_{p,q}$ denotes the space of bi-graded spherical harmonics of type (p,q) on S^3. Then for $f \in H_{p,q}$, and the operators associated to the standard structure on S^3,

$$\Box_b f = 2(p+1)qf, \quad \overline{\Box}_b f = 2(q+1)pf,$$
$$P_0 f = pq(p+1)(q+1)f, \quad \Delta_b f = -(f_{1\bar{1}} + f_{\bar{1}1}) = (2pq+p+q)f. \tag{4.48}$$

Proposition 4.5. *Let $\phi \in C^\infty(S^3)$. Let $g_{\bar{1}} = 0$ and $f \in H_{p,0}$ or $H_{0,p}$. Then for any $p \geq 0$*

$$< \dot{P}_0^t|_{t=0} f, g > \equiv 0. \tag{4.49}$$

Proof. We only display the proof for $f \in H_{p,0}$. The proof for $f \in H_{0,p}$ is similar. We repeatedly integrate by parts and use $g_{\bar{1}} = 0$ to finish the proof. From the first variation (4.31), we have

$$4 < \dot{P}_0^t|_{t=0} f, g > = -2 < D\overline{\Box}_b f, g > -2 < Df, \Box_b g >$$
$$+ 4 < Ef_{11}, g > +4 < E_1 f_1, g > . \tag{4.50}$$

Integration by parts in the fourth term, and using $g_{\bar{1}} = 0$ in the integration by parts and in the second term yields

$$4 < \dot{P}_0^t|_{t=0} f, f > = -2 < D\overline{\Box}_b f, g >,$$
$$= -4p(q+1) < Df, g >= -4p(q+1) < \phi f_{11} + \phi_1 f_1, g >$$
$$= 0, \tag{4.51}$$

where the last equality is due to the integration by parts and using $g_{\bar{1}} = 0$. □

Proposition 4.6. *Let $\phi \in C^\infty(S^3)$. Let $f \in \mathbf{H} = C^\infty(S^3) \cap \oplus_{p \geq 1} H_{p,0} \oplus H_{0,p}$. Then*

$$< \dot{P}_0^t|_{t=0} f, f > \equiv 0.$$

Proof. The Proposition follows from Proposition 4.5 and the fact that the operator $\dot{P}_0^t|_{t=0}$ is real. □

Proposition 4.7. *Let* $\phi \in C^\infty(S^3)$*. Then*

$$< D^2 f, f > \geq 0, \quad \textit{for all } f \in Ker P_0, \tag{4.52}$$

where D *is the operator defined in formula (4.44).*

Proof. Since $f \in Ker P_0$, its Fourier representation has the form

$$f = \sum_{p,q=0}^{\infty} f_{pq}, \quad \text{with } p = 0 \text{ or } q = 0.$$

Thus we divide it into a CR holomorphic part and a anti-CR holomorphic part, that is, f has an expression $f = u + v$, where u and v is a CR function and anti-CR function, respectively. This means $u_{\bar{1}} = 0$ and v_1=0. Now we compute

$$< D^2 f, f > = < D^2 u, u > + < D^2 v, v > + < D^2 u, v > + < D^2 v, u > . \tag{4.53}$$

Since $u_{\bar{1}} = 0$, we get $Du = \phi u_{11} + \phi_1 u_1$. Thus,

$$\begin{aligned} < D^2 u, u > &= < \phi(\phi u_{11} + \phi_1 u_1)_{11}, u > + < \bar{\phi}(\phi u_{11} + \phi_1 u_1)_{\bar{1}\bar{1}}, u > \\ &\quad + < \phi_1(\phi u_{11} + \phi_1 u_1)_1, u > + < \bar{\phi}_{\bar{1}}(\phi u_{11} + \phi_1 u_1)_{\bar{1}}, u > . \end{aligned} \tag{4.54}$$

Integrate by parts the last two terms to get

$$\begin{aligned} &< \phi_1(\phi u_{11} + \phi_1 u_1)_1, u > + < \bar{\phi}_{\bar{1}}(\phi u_{11} + \phi_1 u_1)_{\bar{1}}, u > \\ &= - < \phi(\phi u_{11} + \phi_1 u_1)_{11}, u > - < \phi(\phi u_{11} + \phi_1 u_1)_1, u_{\bar{1}} > \\ &\quad - < \bar{\phi}(\phi u_{11} + \phi_1 u_1)_{\bar{1}\bar{1}}, u > - < \bar{\phi}(\phi u_{11} + \phi_1 u_1)_{\bar{1}}, u_1 > . \end{aligned} \tag{4.55}$$

Taking together (4.54) and (4.55), and using $u_{\bar{1}} = 0$ and integrating by parts again, we see

$$\begin{aligned} < D^2 u, u > &= - \int_{S^3} \bar{\phi}(\phi u_{11} + \phi_1 u_1)_{\bar{1}} \bar{u}_{\bar{1}} \\ &= \int_{S^3} |\phi|^2 |u_{11}|^2 + |\phi_1|^2 |u_1|^2 + \int_{S^3} (\bar{\phi}\phi_1 u_1 \bar{u}_{\bar{1}\bar{1}} + \bar{\phi}_{\bar{1}} \phi u_{11} \bar{u}_{\bar{1}}) \\ &= \int_{S^3} |\phi u_{11} + \phi_1 u_1|^2 \;\geq 0. \end{aligned} \tag{4.56}$$

Similarly, using $v_1 = 0$, we get

$$< D^2u, v >= \int_{S^3} (\phi u_{11} + \phi_1 u_1)(\phi \bar{v}_{11} + \phi_1 \bar{v}_1). \tag{4.57}$$

Using the conjugate and the fact D^2 is real, we see

$$< D^2v, v >= \overline{< D^2\bar{v}, \bar{v} >} = \int_{S^3} |\phi \bar{v}_{11} + \phi_1 \bar{v}_1|^2, \tag{4.58}$$

and

$$< D^2v, u >= \int_{S^3} \overline{(\phi u_{11} + \phi_1 u_1)(\phi \bar{v}_{11} + \phi_1 \bar{v}_1)}. \tag{4.59}$$

Substituting (4.56), (4.57), (4.58) and (4.59) into (4.53), we get

$$< D^2f, f >= \int_{S^3} |(\phi u_{11} + \phi_1 u_1) + (\bar{\phi} v_{\bar{1}\bar{1}} + \bar{\phi}_{\bar{1}} v_{\bar{1}})|^2 \geq 0.$$

This finishes the proof. □

Using (4.47) write

$$4 < \ddot{P}_0^t|_{t=0} f, f >= 8 < D^2 f, f > + < Rf, f > . \tag{4.60}$$

Proposition 4.8. *Let $\phi \in C^\infty(S^3)$. Let $f \in H_{p,0}$ or $f \in H_{0,p}$, and $g_{\bar{1}} = 0$. Then for all $p \geq 0$, we have*

$$(a) < Rf, g >= 0, \quad \textit{if } f \in H_{0,p},$$

and

$$(b) < Rf, g >= 8 \int_{S^3} (p|\phi|^2 - E\bar{\phi}) f_1 \bar{g}_{\bar{1}} \quad \textit{if } f \in H_{p,0}.$$

Proof. We first compute each term in the formula of the second variation of the Paneitz operator (see (4.32)). For all $f \in H_{p,0}$, and $g_{\bar{1}} = 0$.

By (4.48),

$$< 16|\phi|^2 P_0 f, g >= 0. \tag{4.61}$$

$$< 2|\phi|^2(\Box_b\Box_b + \overline{\Box}_b\overline{\Box}_b)f, g >= 8p^2 \int_{S^3} |\phi|^2 f\bar{g}. \tag{4.62}$$

$$< -8E\bar{\phi}\Delta_b f, g >= -8p \int_{S^3} E\bar{\phi} f\bar{g}. \tag{4.63}$$

Integrating by parts gives

$$\begin{aligned} < 8\nabla_b(E\bar{\phi})f, g > &= 8 < (E\bar{\phi})_{\bar{1}} f_1 + (E\bar{\phi})_1 f_{\bar{1}}, g >= 8 < (E\bar{\phi})_{\bar{1}} f_1, g > \\ &= -8 < E\bar{\phi} f_{1\bar{1}}, g > -8 < E\bar{\phi} f_1, g_1 > \\ &= 8p \int_{S^3} E\bar{\phi} f\bar{g} - 8 \int_{S^3} E\bar{\phi} f_1 \bar{g}_{\bar{1}}. \end{aligned} \tag{4.64}$$

$$\begin{aligned} < 4(\Box_b|\phi|^2)\Delta_b f, g > &= 4p < (\Box_b|\phi|^2)f, g > \\ &= 4p < \Box_b(|\phi|^2 f) - |\phi|^2\Box_b f + 2(|\phi|^2_{\bar{1}} f_1 + |\phi|^2_1 f_{\bar{1}}), g > \\ &= 8p < (\nabla_b|\phi|^2)f, g > \\ &=< 8(\nabla_b|\phi|^2)\Delta_b f, g > . \end{aligned} \tag{4.65}$$

$$\begin{aligned} < -4(\nabla_b|\phi|^2)\overline{\Box}_b f, g > &= -8p < (\nabla_b|\phi|^2)f, g >= -8p < |\phi|^2_{\bar{1}} f_1, g > \\ &= 8p < |\phi|^2 f_{1\bar{1}}, g > +8p < |\phi|^2 f_1, g_1 > \\ &= 8p \int_{S^3} |\phi|^2 f_1 \bar{g}_{\bar{1}} - 8p^2 \int_{S^3} |\phi|^2 f\bar{g}. \end{aligned} \tag{4.66}$$

$$< -4\Box_b(\nabla_b|\phi|^2)f, g >=< -4(\nabla_b|\phi|^2)f, \Box_b g >= 0 \tag{4.67}$$

We collect similar terms from (4.61) to (4.67) and observe that both the coefficients of terms

$$\int_{S^3} |\phi|^2 f\bar{g} \quad \text{and} \quad \int_{S^3} E\bar{\phi} f\bar{g}$$

are zero. The coefficient of the term $\int_{S^3} |\phi|^2 f_1 \bar{g}_{\bar{1}}$ is $8p$ and The coefficient of the term $\int_{S^3} E\bar{\phi} f_1 \bar{g}$ is -8. This completes the proof of (b).

Similarly, for $f \in H_{0,p}$, integrating by parts and using $f_1 = 0$ and $g_{\bar{1}} = 0$, we get

$$
\begin{aligned}
&< 16|\phi|^2 P_0 f, g > = 0;\\
&< 2|\phi|^2(\Box_b\Box_b + \overline{\Box}_b\overline{\Box}_b) f, g > = 8p^2 \int_{S^3} |\phi|^2 f\bar{g};\\
&< -8E\bar{\phi}\Delta_b f, g > = -8p \int_{S^3} E\bar{\phi} f\bar{g};\\
&< 8\nabla_b(E\bar{\phi}) f, g > = 8p \int_{S^3} E\bar{\phi} f\bar{g};\\
&< 4(\Box_b|\phi|^2)\Delta_b f, g > =< 8(\nabla_b|\phi|^2)\Delta_b f, g > -8p^2 \int_{S^3} |\phi|^2 f\bar{g};\\
&< -4(\nabla_b|\phi|^2)\overline{\Box}_b f, g > = 0;\\
&< -4\Box_b(\nabla_b|\phi|^2) f, g > =< -4(\nabla_b|\phi|^2) f, \Box_b g >= 0,
\end{aligned}
\tag{4.68}
$$

Taking all together these terms, we get $< Rf, g >= 0$. This completes the proof of (a). □

The proof of Proposition 1.10: From (4.60),

$$< \ddot{P}_0^t f, f >= 2 < D^2 f, f > + \frac{1}{4} < Rf, f > .$$

Using Proposition 4.7,

$$< \ddot{P}_0^t f, f >\geq \frac{1}{4} < Rf, f > .$$

Now,

$$f = \sum_{k\geq 1} f^k + \sum_{k\geq 1} g^k, \quad f^k \in H_{k,0}, \quad g^k \in H_{0,k}.$$

Using the relations in Proposition 4.8, one computes

$$\frac{1}{4} < Rf, f >= 2\sum_{k,l} \int_{S^3} (k|\phi|^2 - E\bar{\phi}) f_1^k \overline{f_1^l} + 2\sum_{k,l} \int_{S^3} (k|\phi|^2 - E\bar{\phi}) g_{\bar{1}}^k \overline{g_{\bar{1}}^l}.$$

This ends the proof. □

We are going to recall the Hopf fibration of $S^3 \subset C^2$. We consider the space BC^2 by blowing up the orign from C^2. Let (z_1, z_2) be the linear coordinates on C^2 and (ζ, w) denote blow up coordinates on BC^2. These coordinates are related by

$$\zeta = z_1, \quad w = \frac{z_2}{z_1}.$$

BC^2 is the tautological line bundle over CP^1. We see that w is an affine coordinate on $CP^1 = S^2$, which is the blow up of the origin, and ζ is the fiber coordinate. Now S^3 is defined by $r(z_1, z_2) = |z_1|^2 + |z_2|^2 = 1$, so we have

$$\begin{aligned} 1 = & r = |z_1|^2 + |z_2|^2 \\ & = |\zeta|^2(1 + |w|^2) \\ & = |\zeta|^2 e^{H(w)}, \end{aligned} \tag{4.69}$$

where $H(w) = \ln(1 + |w|^2)$, defines a circle bundle over $CP^1 = S^2$. This is a fibration of S^3. Let θ be the standard contact form on S^3. Then we have

$$\theta \wedge d\theta = i \frac{\partial^2 H(w)}{\partial w \partial \bar{w}} d\psi \wedge dw \wedge d\bar{w}, \tag{4.70}$$

where $\zeta = |\zeta| e^{i\psi}$ and ψ is the fiber coordinate. Also, we see that T is the generator of the circular action $(z_1, z_2) \to (e^{i\psi} z_1, e^{i\psi} z_2)$ with period 2π, that is,

$$T = \frac{\partial}{\partial \psi}.$$

Proposition 4.9. *For $f \in H_{p,0}$, $f_1 = e^{i(p-2)\psi} H(w, \bar{w})$ and $g \in H_{0,p}$ $g_{\bar{1}} = e^{-i(p-2)\psi} G(w, \bar{w})$, so in particular both $|f_1|$ and $|g_{\bar{1}}|$ does not depend on the fiber coordinate ψ.*
For $\phi \in C^\infty(S^3)$, we have the following Fourier expansion of ϕ with respect to ψ

$$\phi = \sum_{p,q=0}^{\infty} \phi_{pq} = \sum_{m \in Z} e^{im\psi} \phi_m(w, \bar{w}),$$

where $m = p - q$ and ϕ_m is a function only defined on S^2.

Proof. If $f \in H_{p,0}$, say, $f = \sum_{a+b=p} c_{ab} z_1^a z_2^b$ then

$$
\begin{aligned}
f_1 = Z_1 f &= (\bar{z}_2 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2}) f \\
&= \sum_{a+b=p} c_{ab}(a z_1^{a-1} z_2^b \bar{z}_2 - b z_1^a z_2^{b-1} \bar{z}_1) \\
&= \sum_{a+b=p} c_{ab}(a\zeta^{a+b-1}\bar{\zeta} w^b \bar{w} - b\zeta^{a+b-1}\bar{\zeta} w^{b-1}) \\
&= \zeta^{a+b-1}\bar{\zeta}\left(\sum_{a+b=p} c_{ab}(a w^b \bar{w} - b w^{b-1})\right).
\end{aligned} \tag{4.71}
$$

Since on S^3, $1 = |\zeta|^2 e^{H(w)}$, we have

$$
\begin{aligned}
|f_1| =& |\zeta|^{a+b} \left| \sum_{a+b=p} c_{ab}(a w^b \bar{w} - b w^{b-1}) \right| \\
&= e^{-\frac{a+b}{2} H(w)} \left| \sum_{a+b=p} c_{ab}(a w^b \bar{w} - b w^{b-1}) \right|,
\end{aligned} \tag{4.72}
$$

which does not depend on ψ. On the other hand, for $\phi \in C^\infty(S^3)$, it has the Fourier representation $\phi = \sum_{p,q=0}^{\infty} \phi_{pq}$. We would like to express it by the coordinates (ζ, w). We denote $\phi_{pq} = \sum_{a+b=p,c+d=q} c_{abcd} z_1^a z_2^b \bar{z}_1^c \bar{z}_2^d$. Then

$$
\begin{aligned}
\phi_{pq} &= \sum_{a+b=p,c+d=q} c_{abcd} \zeta^{a+b} w^b \bar{\zeta}^{c+d} \bar{w}^d \\
&= \sum_{a+b=p,c+d=q} c_{abcd} e^{i(p-q)\psi} |\zeta|^{p+q} w^b \bar{w}^d \\
&= \sum_{a+b=p,c+d=q} c_{abcd} e^{i(p-q)\psi} e^{\frac{-(p+q)H(w)}{2}} w^b \bar{w}^d \\
&= e^{im\psi} \phi_m(w, \bar{w}),
\end{aligned} \tag{4.73}
$$

where $m = p - q$ and $\phi_m = \left(\sum_{a+b=p,c+d=q} c_{abcd} e^{\frac{-(p+q)H(w)}{2}} w^b \bar{w}^d\right)$. □

Proposition 4.10. *(a) Let $\phi \in C^\infty(S^3)$, then for ϕ satisfying (BE)*

$$
\int_{S^3} (k|\phi|^2 - E\bar{\phi})|f_1^k|^2 \geq \int_{S^3} |\phi|^2 |f_1^k|^2, \quad k \geq 1;
$$

$$\int_{S^3}(k|\phi|^2 - E\bar{\phi})|g_{\bar{1}}^k|^2 \geq \int_{S^3}|\phi|^2|g_{\bar{1}}^k|^2, \quad k \geq 1.$$

(b) For any $\phi \in P_{p_1,q_1}$,

$$\int_{S^3}(k|\phi|^2 - E\bar{\phi})f_1^k\overline{f_1^l} = \begin{cases} 0 & ,k \neq l \\ \int_{S^3}(k+p_1-q_1-4)|\phi|^2|f_1^k|^2 & ,k = l. \end{cases}$$

$$\int_{S^3}(k|\phi|^2 - E\bar{\phi})g_{\bar{1}}^k\overline{g_{\bar{1}}^l} = \begin{cases} 0 & ,k \neq l \\ \int_{S^3}(k+p_1-q_1-4)|\phi|^2|g_{\bar{1}}^k|^2 & ,k = l. \end{cases}$$

Proof. We only display the proof for the first statement of part (a). From Proposition 4.9, $|f_1^k|$ is independent of the fiber variable ψ. Changing variables using (4.70), we have using the Fourier expansion of ϕ in Proposition 4.9 that

$$\phi = \sum \phi_{pq} = \sum_{m\in Z} e^{im\psi}\phi_m(w,\bar{w}), \quad m = p - q.$$

Now

$$E = 4\phi + i\phi_0.$$

Thus

$$E\bar{\phi} = (\sum_{m\in Z} e^{im\psi}(4-m)\phi_m(w,\bar{w}))\bar{\phi}.$$

So by Plancherel's theorem,

$$\begin{aligned} \int_{S^3}(k|\phi|^2 - E\bar{\phi})|f_1^k|^2 &= \int_{S^2}\left(\int_{S^1}(k|\phi|^2 - E\bar{\phi})\right)|f_1^k|^2 \\ &= \int_{S^2}\left(\sum_m (k+m-4)|\phi_m|^2\right)|f_1^k|^2. \end{aligned} \tag{4.74}$$

The (BE) condition implies $m - 4 = p - q - 4 \geq 0$, and since $k \geq 1$, the term above is bounded below by

$$\int_{S^2}\left(\sum_m |\phi_m|^2\right)|f_1^k|^2.$$

Using Plancherel's theorem again we obtain our result.

We only consider the case for f_1^k in part (b), the case for $g_{\bar{1}}^k$ is similar. Observe that if $\phi \in P_{p_1,q_1}$, then $|\phi|$ is independent of the fiber variable ψ, and since,

$$E\bar{\phi} = (4\phi + i\phi_0) = 4|\phi|^2 + i\phi_0\bar{\phi}$$

$$= 4|\phi|^2 + (q_1 - p_1)|\phi|^2 = (4 + q_1 - p_1)|\phi|^2.$$

$E\bar{\phi}$ is also independent of the fiber variable ψ, and only depends on $w, \bar{w}$.Thus if $k \neq l$, the integrand may be written as

$$e^{\pm i(k-l)\psi} G_{\pm}(w, \bar{w}),$$

from which it immediately follows that if $k \neq l$, the integral vanishes. When $k = l$, from the computation (4.74) in part (a), ,

$$\int_{S^3} (k|\phi|^2 - E\bar{\phi})|f_1^k|^2 = (k + p_1 - q_1 - 4) \sum_{m=p_1-q_1} \int_{S^2} |\phi_m|^2 |f_1^k|^2$$

$$= (k + p_1 - q_1 - 4) \int_{S^3} |\phi|^2 |f_1^k|^2.$$

We have our conclusion. □

The proof of Proposition 1.11: We put together Proposition 1.10 and the computation of the integrals in the right side of Proposition 1.10 which are done in Proposition 4.10. The Proposition follows. □

We emphasize that in Proposition 1.11 we do not hypothesize that ϕ satisfies (BE). The following corollaries are therefore immediate consequences of Proposition 1.11.

COROLLARY 4.11. *Let $\phi \in P_{p_1,q_1}$. Then $< \ddot{P}_0^t|_{t=0} f, f >$ is positive for all $f \in H_{p,0}$ or $f \in H_{0,p}$, $p \geq 1$ if ϕ satisfies condition (BE),i.e., $p_1 \geq 4 + q_1$.*

COROLLARY 4.12. *Let $\phi \in P_{p_1,q_1}$ and W denote the subspace of $\oplus H_{p,0} \oplus H_{0,p}$ on which $\ddot{P}_0^t|_{t=0} < 0$. Then $W \subset \oplus H_{p,0} \oplus H_{0,p}$, for $p < q_1 + 4 - p_1$.*

REMARK 4.13. The reader may verify by making an explicit calculation for $\phi_t = t$, using the above formula, that is in Rossi's example, we have

$$< \ddot{P}_0^t|_{t=0} f, f > < 0, \quad \text{for} \quad f = z_1,\ z_2,\ \bar{z}_1,\ \bar{z}_2.$$

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