Sharp Global Bounds for the Hessian on Pseudo-Hermitian Manifolds

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Summary. We consider an abstract CR manifold equipped with a strictly positive definite Levi form, which defines a pseudo-Hermitian metric on the manifold. On such a manifold it is possible to define a natural sums of squares sub-Laplacian operator. We use Bochner identities to obtain Cordes–Friedrichs type inequalities on such manifolds where the L^2 norm of the Hessian tensor of a function is controlled by the L^2 norm of the sub-Laplacian of the function with a sharp constant for the inequality. By perturbation we proceed to develop a Cordes–Nirenberg type theory for non-divergence form equations on CR manifolds. Some applications are given to the regularity of p-Laplacians on CR manifolds.

Key words: CR manifolds, Friedrichs inequalities, sub-Laplacian, Bochner identities, p-Laplacian, Alexandrov–Bakelman–Pucci estimate, Cordes–Nirenberg estimates.

Dedicated to the memory of our friend and colleague Carlos Segovia.

1 Introduction

In partial differential equation (PDE) theory, harmonic analysis enters in a fundamental way through the basic estimate valid for $f \in C_0^\infty(\mathbb{R}^n)$, which states,

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^p(\mathbb{R}^n)} \leq c(n,p) \left\| \Delta f \right\|_{L^p(\mathbb{R}^n)}, \text{ for } 1 < p < \infty. \tag{1}$$

This estimate is really a statement of the L^p boundedness of the Riesz transforms, and thus (1) is a consequence of the multiplier theorems of Marcinkiewicz and Hörmander–Mikhlin, [15]. More sophisticated variants of

(1) can be proved by relying on the square function [15] and [14]. In particular, (1) leads to *a priori* $W^{2,p}$ estimates for solutions of

$$\Delta u = f, \text{ for } f \in L^p. \tag{2}$$

Knowledge of $c(p,n)$ allows one to perform a perturbation of (2) and study

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f, \tag{3}$$

as was done by Cordes [5], where $A = (a^{ij})$ is bounded, measurable, elliptic, and close to the identity in a sense made precise by Cordes. The availability of the estimates of Alexandrov–Bakelman–Pucci and the Krylov–Safonov theory [8] allows one to obtain estimates for (3) in full generality without relying on a perturbation argument. See also [12].

Our focus here will be to study the CR analog of (3). Since at this moment there is no suitable Alexandrov–Bakelman–Pucci estimate for the CR analog of (3), we will be seeking a perturbation approach based on an analog of (1) on a CR manifold. Our main interest is the case $p = 2$ in (1). In this case a simple integration by parts suffices to prove (1) in $\mathbb{R}^n$. We easily see that for $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\sum_{i,j=1}^{n} \left\| \frac{\partial^2 f}{\partial x_i \partial x_j} \right\|_{L^2(\mathbb{R}^n)}^2 = \|\Delta f\|_{L^2(\mathbb{R}^n)}^2 . \tag{4}$$

In the case of (1) on a CR manifold, a result has been recently obtained by Domokos–Manfredi [7] in the Heisenberg group. The proof in [7] makes use of the harmonic analysis techniques in the Heisenberg group developed by Strichartz [16] which will not apply to studying such inequalities for the Hessian on a general CR manifold, although other nilpotent groups of step 2 can be treated similarly [6].

Instead, we shall proceed by integration by parts and use of the Bochner technique. A Bochner identity on a CR manifold was obtained by Greenleaf [9] and will play an important role in our computations.

We now turn to our setup. We consider a smooth orientable manifold M^{2n+1}. Let $\mathcal{V}$ be a vector sub-bundle of the complexified tangent bundle $\mathbb{C}TM$. We say that $\mathcal{V}$ is a CR bundle if

$$\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}, \; [\mathcal{V}, \mathcal{V}] \subset \mathcal{V}, \text{ and } \dim_{\mathbb{C}} \mathcal{V} = n. \tag{5}$$

A manifold equipped with a sub-bundle satisfying (5) will be called a CR manifold. See the book by Trèves [18]. Consider the sub-bundle

$$H = \operatorname{Re}\left(\mathcal{V} \oplus \overline{\mathcal{V}}\right). \tag{6}$$

H is a real $2n$-dimensional vector sub-bundle of the tangent bundle TM. We assume that the real line bundle $H^\perp \subset T^*M$, where T^*M is the cotangent bundle, has a smooth non-vanishing global section. This is a choice of a non-vanishing 1-form θ on M and (M, θ) is said to define a pseudo-Hermitian structure. M is then called a pseudo-Hermitian manifold. Associated to θ we have the Levi form L_θ given by

$$L_\theta(V, \overline{W}) = -i\, d\theta(V \wedge \overline{W}), \text{ for } V, W \in \mathcal{V}. \tag{7}$$

We shall assume that L_θ is definite and orient θ by requiring that L_θ be positive definite. In this case, we say that M is strongly pseudo-convex. We shall always assume that M is strongly pseudo-convex.

On a manifold M that carries a pseudo-Hermitian structure, or a pseudo-Hermitian manifold, there is a unique vector field T, transverse to H defined in (6) with the properties

$$\theta(T) = 1 \quad \text{and} \quad d\theta(T, \cdot) = 0. \tag{8}$$

T is also called the Reeb vector field. The volume element on M is given by

$$dV = \theta \wedge (d\theta)^n. \tag{9}$$

A complex valued 1-form η is said to be of type $(1, 0)$ if $\eta(\overline{W}) = 0$ for all $W \in \mathcal{V}$, and of type $(0, 1)$ if $\eta(W) = 0$ for all $W \in \mathcal{V}$.

An admissible co-frame on an open subset of M is a collection of $(1, 0)$ forms $\{\theta^1, \ldots, \theta^\alpha, \ldots, \theta^n\}$ that locally form a basis for $\mathcal{V}^*$ and such that $\theta^\alpha(T) = 0$ for $1 \le \alpha \le n$. We set $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$. We then have that $\{\theta, \theta^\alpha, \theta^{\overline{\alpha}}\}$ locally form a basis of the complex co-vectors, and the dual basis is the complex vector fields $\{T, Z_\alpha, \overline{Z_\alpha}\}$. For $f \in C^2(M)$ we set

$$Tf = f_0, \qquad Z_\alpha f = f_\alpha, \qquad \overline{Z_\alpha} f = f_{\overline{\alpha}}. \tag{10}$$

We note that in what follows all our functions f will be real valued.

If follows from (5), (7), and (8) that we can express

$$d\theta = i\, h_{\alpha\overline{\beta}}\, \theta^\alpha \wedge \theta^{\overline{\beta}}. \tag{11}$$

The Hermitian matrix $(h_{\alpha\overline{\beta}})$ is called the Levi matrix.

On pseudo-Hermitian manifolds Webster [19] has defined a connection, with connection forms ω_α^β and torsion forms $\tau_\beta = A_{\beta\alpha}\theta^\alpha$, with structure relations

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau_\beta, \qquad \omega_{\alpha\overline{\beta}} + \omega_{\bar{\beta}\alpha} = dh_{\alpha\overline{\beta}} \tag{12}$$

and

$$A_{\alpha\beta} = A_{\beta\alpha}. \tag{13}$$

Webster defines a curvature form

$$\prod{}_\alpha^\beta = d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta,$$

where we have used the Einstein summation convention. Furthermore, in [19] it is shown that

$$\prod\nolimits_{\alpha}^{\beta} = R_{\alpha\bar{\beta}\rho\bar{\sigma}}\theta^{\rho} \wedge \theta^{\bar{\sigma}} + \text{ other terms.}$$

Contracting two indices using the Levi matrix $(h_{\alpha\bar{\beta}})$, we get

$$R_{\alpha\bar{\beta}} = h^{\rho\bar{\sigma}}\ R_{\alpha\bar{\beta}\rho\bar{\sigma}}. \tag{14}$$

The Webster–Ricci tensor $\mathrm{Ric}(V,V)$ for $V \in \mathcal{V}$ is then defined as

$$\mathrm{Ric}(V,V) = R_{\alpha\bar{\beta}}x^{\alpha}\overline{x^{\beta}},\ \text{for } V = \sigma_{\alpha}x^{\alpha}Z_{\alpha}. \tag{15}$$

The torsion tensor is defined for $V \in \mathcal{V}$ as follows:

$$\mathrm{Tor}(V,V) = i\left(A_{\bar{\alpha}\bar{\beta}}\overline{x^{\alpha}}\bar{x}_{\beta} - A_{\alpha\beta}x^{\alpha}x^{\beta}\right). \tag{16}$$

In [19], Prop. (2.2), Webster proves that the torsion vanishes if $\mathcal{L}_T$ preserves H, where $\mathcal{L}_T$ is the Lie derivative. In particular, if M is a hypersurface in $\mathbb{C}^{n+1}$ given by the defining function ρ,

$$\mathrm{Im} z_{n+1} = \rho(z,\bar{z}), \qquad z = (z_1, z_2, \ldots, z_n), \tag{17}$$

then Webster's hypothesis is fulfilled, and the torsion tensor vanishes on M. Thus, for the standard CR structure on the sphere S^{2n+1} and on the Heisenberg group, the torsion vanishes.

Our main focus will be the sub-Laplacian Δ_b. We define the horizontal gradient ∇_b and Δ_b as follows:

$$\nabla_b f = \sum_{\alpha} f_{\overline{\alpha}} Z_{\alpha}, \tag{18}$$

$$\Delta_b f = \sum_{\alpha} f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha}. \tag{19}$$

When $n = 1$ we will need to frame our results in terms of the CR Paneitz operator. Define the Kohn Laplacian $\Box_b$ by

$$\Box_b = \Delta_b + i\,T. \tag{20}$$

Then the CR Paneitz operator P_0 is defined by

$$P_0 f = \left(\overline{\Box}_b\Box_b + \Box_b\overline{\Box}_b\right) f - 2\left(Q + \overline{Q}\right) f, \tag{21}$$

where

$$Qf = 2i\,(A^{11}f_1)_1.$$

See [10] and [4] for further details.

2 The main theorem

Theorem 1 *Let M^{2n+1} be a strictly pseudo-convex pseudo-Hermitian manifold. When M is non-compact, assume that $f \in C_0^\infty(M)$. When M is compact with $\partial M = \emptyset$, we may assume $f \in C^\infty(M)$. When f is real valued and $n \geq 2$, we have*

$$\sum_{\alpha,\beta} \int_M |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \int_M \Big(Ric + \frac{n}{2}\, Tor\Big)(\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{2n} \int_M |\Delta_b f|^2. \quad (1)$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$\int_M |f_{11}|^2 + |f_{1\bar{1}}|^2 + \int_M \left(Ric - \frac{3}{2}\, Tor\right)(\nabla_b f, \nabla_b f) \leq \frac{3}{2} \int_M |\Delta_b f|^2. \quad (2)$$

Here by $\sum_{\alpha,\beta} |f_{\alpha\beta}|^2$ we mean the Hilbert–Schmidt norm square of the tensor $f_{\alpha\beta}$ and similarly for $\sum_{\alpha,\beta} |f_{\alpha\bar{\beta}}|^2$.

Proof We begin by noting the Bochner identity established by Greenleaf, Lemma 3 in [9]:

$$\frac{1}{2}\Delta_b \left(|\nabla_b f|^2\right) = \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \mathrm{Re}\left(\nabla_b f, \nabla_b(\Delta_b f)\right) \quad (3)$$
$$+ \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b, \nabla_b) + i \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}\right),$$

where for $V, W \in \mathcal{V}$ we use the notation $(V, W) = L_\theta(V, \overline{W})$ and $|V| = (V, V)^{1/2}$. Using the fact that $f \in C_0^\infty(M)$ or if $\partial M = \emptyset$, M is compact, integrate (3) over M using the volume ((9) of Section 1) to get

$$\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) \quad (4)$$
$$+ i \int_M \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}\right) = - \int_M \mathrm{Re}\left(\nabla_b f, \nabla_b(\Delta_b f)\right).$$

Integration by parts in the term on the right yields (see (5.4) in [9])

$$- \int_M \mathrm{Re}(\nabla_b f, \nabla_b(\Delta_b f)) = \frac{1}{2} \int_M |\Delta_b f|^2. \quad (5)$$

Combining (4) and (5) we get

$$\int_M \sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 + \int_M \left(\mathrm{Ric} + \frac{n-2}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) \quad (6)$$
$$+ i \int_M \sum_\alpha \left(f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}\right) = \frac{1}{2} \int_M |\Delta_b f|^2.$$

To handle the third integral on the left-hand side, we use Lemmas 4 and 5 of [9] (valid for real functions) according to which we have

$$i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) = \frac{2}{n}\int_M \Big(\sum_{\alpha,\beta} \big(|f_{\alpha\bar{\beta}}|^2 - |f_{\alpha\beta}|^2\big) - \mathrm{Ric}(\nabla_b f, \nabla_b f)\Big), \quad (7)$$

and

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) = &- \frac{4}{n}\int_M \big|\sum_\alpha f_{\alpha\bar{\alpha}}\big|^2 \\ &+ \frac{1}{n}\int_M |\Delta_b f|^2 \\ &+ \int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \quad (8)$$

Applying the Cauchy–Schwarz inequality to the first term on the right-hand side of (8), we get

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) \geq &- 4\int_M \sum_{\alpha,\beta} |f_{\alpha\bar{\beta}}|^2 \\ &+ \frac{1}{n}\int_M |\Delta_b f|^2 \\ &+ \int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \quad (9)$$

Multiply (7) by $1-c$ and (9) by c , $0 < c < 1$, and where c will eventually be chosen to be $1/(n+1)$, and add to get

$$\begin{aligned} i\int_M \sum_\alpha (f_{\overline{\alpha}} f_{\alpha 0} - f_\alpha f_{\bar{\alpha}0}) \geq &\ 2\ \frac{(1-c)}{n}\int_M \sum_{\alpha,\beta} \big(|f_{\alpha\bar{\beta}}|^2 - |f_{\alpha\beta}|^2\big) \\ &- 2\frac{(1-c)}{n}\int_M \mathrm{Ric}(\nabla_b f, \nabla_b f) \\ &- 4c\int_M \sum_{\alpha,\beta} |f_{\alpha\overline{\beta}}|^2 \\ &+ \frac{c}{n}\int_M |\Delta_b f|^2 + c\int_M \mathrm{Tor}(\nabla_b f, \nabla_b f). \end{aligned} \quad (10)$$

We now insert (10) into (6) and simplify. We have

$$\left(1-\frac{2(1-c)}{n}\right)\int_M \mathrm{Ric}(\nabla_b f,\nabla_b f)+$$
$$\left(\frac{(n-2)}{2}+c\right)\int_M \mathrm{Tor}(\nabla_b f,\nabla_b f)+$$
$$\left(1+\frac{2(1-c)}{n}-4c\right)\int_M \sum_{\alpha,\beta}|f_{\alpha\bar\beta}|^2+ \tag{11}$$
$$\left(1-\frac{2(1-c)}{n}\right)\int_M \sum_{\alpha,\beta}|f_{\alpha\beta}|^2 \le \left(\frac{1}{2}-\frac{c}{n}\right)\int_M |\Delta_b f|^2.$$

Let $c=1/(n+1)$. Then (11) becomes

$$\left(\frac{n-1}{n+1}\right)\left[\int_M \sum_{\alpha,\beta}\left(|f_{\alpha\beta}|^2+|f_{\alpha\bar\beta}|^2\right)+\int_M\left(\mathrm{Ric}+\frac{n}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)\right] \tag{12}$$
$$\le \left(\frac{n-1}{n+1}\right)\left(\frac{n+2}{2n}\right)\int_M |\Delta_b f|^2.$$

Since $n\ge 2$, $n-1>0$ and we can cancel the factor $\frac{n-1}{n+1}$ from both sides to get (1).

We now establish (2) using some results by Li–Luk [11] and [4]. When $n=1$, identity (6) becomes

$$\int_M |f_{1\bar1}|^2+|f_{11}|^2+\int_M\left(\mathrm{Ric}-\frac{1}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f) \tag{13}$$
$$+\,i\int_M\left(f_{10}f_{\bar1}-f_{\bar10}f_1\right)=\frac{1}{2}\int_M|\Delta_b f|^2.$$

By (3.8) in [11] we have

$$i\int_M\left(f_{01}f_{\bar1}-f_{0\bar1}f_1\right)=-\int_M f_0^2.$$

Moreover, by (3.6) in [11] we also have

$$i\left(f_{10}f_{\bar1}-f_{\bar10}f_1\right)=i\left(f_{01}f_{\bar1}-f_{0\bar1}f_1\right)+\mathrm{Tor}(\nabla_b f,\nabla_b f),$$

and combining the last two identities we get

$$i\int_M\left(f_{10}f_{\bar1}-f_{\bar10}f_1\right)=-\int_M f_0^2+\int_M\mathrm{Tor}(\nabla_b f,\nabla_b f). \tag{14}$$

Substituting (14) into (13) we obtain

$$\int_M |f_{1\bar1}|^2+|f_{11}|^2+\int_M\left(\mathrm{Ric}+\frac{1}{2}\mathrm{Tor}\right)(\nabla_b f,\nabla_b f)-\int_M f_0^2 \tag{15}$$
$$=\frac{1}{2}\int_M|\Delta_b f|^2.$$

Next, we use (3.4) in [4],

$$\int_M f_0^2 = \int_M |\Delta_b f|^2 + 2\int_M \mathrm{Tor}(\nabla_b f, \nabla_b f) - \frac{1}{2}\int_M P_0 f \cdot f. \qquad (16)$$

Finally, substitute (16) into (15) and simplify to get

$$\int_M |f_{1\bar{1}}|^2 + |f_{11}|^2 + \int_M \left(\mathrm{Ric} - \frac{3}{2}\mathrm{Tor}\right)(\nabla_b f, \nabla_b f) + \frac{1}{2}\int_M P_0 f \cdot f = \frac{3}{2}\int_M |\Delta_b f|^2.$$

Assuming $P_0 \geq 0$ we obtain (2). □

We now wish to make some remarks about our theorem:

(a) It is shown in [7] that on the Heisenberg group the constant $(n+2)/2n$ is sharp. Since the Heisenberg group is a pseudo-Hermitian manifold with Ric $\equiv 0$ and Tor $\equiv 0$, we easily conclude that our theorem is sharp and contains the result proved in [7].

(b) We notice that when we consider manifolds such that Ric$+(n/2)$Tor $>$ 0, then for $n \geq 2$, in general we have the strict inequality

$$\sum_{\alpha,\beta} \int_M |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 < \frac{n+2}{2n}\int_M |\Delta_b f|^2.$$

On the Heisenberg group Ric $\equiv 0$, Tor $\equiv 0$, and the constant $(n+2)/2n$ is achieved by a function with fast decay [7]. Thus, the Heisenberg group is, in a sense, extremal for inequality (1) in Theorem 1. A similar remark holds for inequality (2).

(c) The hypothesis on the Paneitz operator in the case $n = 1$ in our theorem is satisfied on manifolds with zero torsion. A result from [2] shows that, if the torsion vanishes, the Paneitz operator is non-negative.

(d) We note that Chiu [4] shows how to perturb the standard pseudo-Hermitian structure in $\mathbb{S}^3$ to get a structure with non-zero torsion, for which $P_0 > 0$ and Ric $- (3/2)$Tor > 1. To get such a structure, let θ be the contact form associated to the standard structure on $\mathbb{S}^3$. Fix g a smooth function on $\mathbb{S}^3$. For $\epsilon > 0$ consider

$$\tilde{\theta} = e^{2f}\theta, \text{ where } f = \epsilon^3 \sin(\frac{g}{\epsilon}). \qquad (17)$$

Since the sign of the Paneitz operator is a CR invariant and θ has zero torsion, we conclude by [2] that the CR Paneitz operator $\tilde{P}_0$ associated to $\tilde{\theta}$ satisfies $\tilde{P}_0 > 0$. Furthermore, following the computation in Lemma (4.7) of [4], we easily have for small ϵ that

$$\mathrm{Ric} - \frac{3}{2}\mathrm{Tor} \geq (2 + O(\epsilon))\, e^{-2f} \geq 1 \geq 0.$$

Thus, the hypothesis of the case $n = 1$ in our theorem are met, and for such $(M, \tilde{\theta})$ we have, for $f \in C^\infty(M)$, the estimate

$$\int_M |f_{11}|^2 + |f_{1\bar{1}}|^2 \, dV \leq \frac{3}{2} \int_M |\Delta_b f|^2 \, dV.$$

(e) Compact pseudo-Hermitian 3-manifolds with negative Webster curvature may be constructed by considering the co-sphere bundle of a compact Riemann surface of genus g, $g \geq 2$. Such a construction is given in [3].

3 Applications to PDE

For applications to subelliptic PDE it is helpful to restate our main result Theorem 1 in its real version. We set

$$X_i = \mathrm{Re}(Z_i) \text{ and } X_{i+n} = \mathrm{Im}(Z_i)$$

for $i = 1, 2, \ldots, n$. The horizontal gradient of a function is the vector field

$$\mathfrak{X}(f) = \sum_{i=1}^{2n} X_i(f) X_i.$$

Its sub-Laplacian is given by

$$\Delta_{\mathfrak{X}} f = \sum_{i=1}^{2n} X_i X_i(f),$$

and the horizontal second derivatives are the $2n \times 2n$ matrix

$$\mathfrak{X}^2 f = (X_i X_j(f)) \, .$$

For f real we have the following relationships:

$$\nabla_b f = \mathfrak{X}(f) + i \left(\sum_{i=1}^{n} X_i(f) X_{i+n} - X_{i+n}(f) X_i \right),$$

$$\Delta_b f = 2\, \Delta_{\mathfrak{X}} f,$$

and

$$\sum_{\alpha,\beta} |f_{\alpha\beta}|^2 + |f_{\alpha\bar{\beta}}|^2 = 2 \sum_{i,j} |X_i X_j(f)|^2 = 2|\mathfrak{X}^2 f|^2,$$

where the expression on the right is the Hilbert–Schmidt norm of the tensor taken by viewing the Levi form as a Riemannian metric on H.

Theorem 2 *Let M^{2n+1} be a strictly pseudo-convex pseudo-Hermitian manifold. When M is non-compact, assume that $f \in C_0^\infty(M)$. When M is compact with $\partial M = \emptyset$, we may assume $f \in C^\infty(M)$. When f is real valued and $n \geq 2$, we have*

$$\int_M |\mathfrak{X}^2 f|^2 + \int_M \frac{1}{2}\left(Ric + \frac{n}{2}\,Tor\right)(\nabla_b f, \nabla_b f) \leq \frac{(n+2)}{n}\int_M |\Delta_{\mathfrak{X}} f|^2. \quad (1)$$

When $n = 1$ assume that the CR Paneitz operator $P_0 \geq 0$. For $f \in C_0^\infty(M)$ we then have

$$\int_M |\mathfrak{X}^2 f|^2 + \int_M \frac{1}{2}\left(Ric - \frac{3}{2}\,Tor\right)(\nabla_b f, \nabla_b f) \leq 3\int_M |\Delta_{\mathfrak{X}} f|^2. \quad (2)$$

Let $A(x) = (a_{ij}(x))$ a $2n \times 2n$ matrix. Consider the second order linear operator in non-divergence form

$$\mathcal{A}u(x) = \sum_{i,j=1}^{2n} a_{ij}(x) X_i X_j u(x), \quad (3)$$

where coefficients $a_{ij}(x)$ are bounded measurable functions in a domain $\Omega \subset M^{2n+1}$. Cordes [5] and Talenti [17] identified the optimal condition expressing how far $\mathcal{A}$ can be from the identity and still be able to understand (3) as a perturbation of the case $A(x) = I_{2n}$, when the operator is just the sub-Laplacian. This is called Cordes condition, roughly says that all eigenvalues of A must cluster around a single value.

Definition 1 *([5],[17], [7]) We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that*

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2n} a_{ij}^2(x) \leq \frac{1}{2n-1+\varepsilon}\left(\sum_{i=1}^{2n} a_{ii}(x)\right)^2 \quad (4)$$

for a.e. $x \in \Omega$.

Let $c_n = \frac{(n+2)}{n}$ for $n \geq 2$ and $c_1 = 3$ be the constants in the right-hand sides of the equations of Theorem 2. We can now adapt the proof of Theorem 2.1 in [7] to get

Theorem 3 *Let M^{2n+1} be a strictly pseudo-convex pseudo-Hermitian manifold such that $Ric + \frac{n}{2}\,Tor \geq 0$ if $n \geq 2$ and $Ric - \frac{3}{2}\,Tor \geq 0$, $P_0 \geq 0$ if $n = 1$. Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{(1-\varepsilon)c_n} < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in C_0^\infty(\Omega)$ we have the a priori estimate*

$$\|\mathfrak{X}^2 u\|_{L^2} \leq \sqrt{1 + \frac{2}{n}}\,\frac{1}{1-\gamma}\,\|\alpha\|_{L^\infty}\|\mathcal{A}u\|_{L^2}\,, \quad (5)$$

where

$$\alpha(x) = \frac{\langle A(x), I\rangle}{||A(x)||^2} = \frac{\sum_{i=1}^{2n} a_{ii}(x)}{\sum_{i,j=1}^{2n} a_{ij}^2(x)}\,.$$

Proof We start from formula (2.7) in [7] which gives

$$\int_{\Omega} |\Delta_{\mathfrak{X}} u(x) - \alpha(x)\mathcal{A}u(x)|^2 \, dx \le (1-\varepsilon) \int_{\Omega} |\mathfrak{X}u|^2 dx.$$

We now apply Theorem 2 to get

$$\int_{\Omega} |\Delta_{\mathfrak{X}} u(x) - \alpha(x)\mathcal{A}u(x)|^2 \, dx \le (1-\varepsilon) c_n \int_{\Omega} |\Delta_{\mathfrak{X}} f|^2.$$

The theorem then follows as in [7]. □

Remark. The hypothesis of Theorem 2, $n \ge 2$, can be weakened to assume only a bound from below,

$$\mathrm{Ric} + \frac{n}{2}\mathrm{Tor} \ge -K, \text{ with } K > 0,$$

to obtain estimates of the type

$$\int_M |\mathfrak{X}^2 f|^2 \le \frac{(n+2)}{n} \int_M |\Delta_{\mathfrak{X}} f|^2 + 2K \int_M |\mathfrak{X} f|^2. \tag{6}$$

A similar remark applies to the case $n = 1$.

We finish this paper by indicating how the *a priori* estimate of Theorem 3 can be used to prove regularity for p-harmonic functions in the Heisenberg group $\mathcal{H}^n$ when p is close to 2. We follow [7], where full details can be found. Recall that, for $1 < p < \infty$, a p-harmonic function u in a domain $\Omega \subset \mathcal{H}^n$ is a function in the horizontal Sobolev space

$$W^{1,p}_{\mathfrak{X},\mathrm{loc}}(\Omega) = \{u \colon \Omega \mapsto \mathbb{R} \text{ such that } u, \mathfrak{X}u \in L^p_{\mathrm{loc}}(\Omega)\}$$

such that

$$\sum_{i=1}^{2n} X_i \left(|\mathfrak{X}u|^{p-2} \, X_i u \right) = 0 \,, \text{ in } \Omega \tag{7}$$

in the weak sense. That is, for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} |\mathfrak{X}u(x)|^{p-2} (\mathfrak{X}u(x), \mathfrak{X}\phi(x) \, dx = 0. \tag{8}$$

Assume for the moment that u is a smooth solution of (7). We can then differentiate to obtain

$$\sum_{i,j=1}^{2n} a_{ij} \, X_i X_j u = 0 \,, \text{ in } \Omega, \tag{9}$$

where

$$a_{ij}(x) = \delta_{ij} + (p-2)\frac{X_i u(x)\, X_j u(x)}{|\mathfrak{X} u(x)|^2}\,.$$

A calculation shows that this matrix satisfies the Cordes condition (4) precisely when

$$p-2 \in \left(\frac{n - n\sqrt{4n^2+4n-3}}{2n^2+2n-2}\,,\; \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2} \right). \tag{10}$$

In the case $n = 1$ this simplifies to

$$p-2 \in \left(\frac{1-\sqrt{5}}{2}\,,\; \frac{1+\sqrt{5}}{2} \right).$$

We then deduce *a priori* estimates for $\mathfrak{X}^2 u$ from Theorem 3. To apply the Cordes machinery to functions that are only in $W^{1,p}_{\mathfrak{X}}$, we need to know that the second derivatives $\mathfrak{X}^2 u$ exist. This is done in the Euclidean case by a standard difference quotient argument applied to a regularized p-Laplacian. In the Heisenberg case this would correspond to proving that solutions to

$$\sum_{i=1}^{2n} X_i \left(\left(\frac{1}{m} + |\mathfrak{X} u|^2 \right)^{\frac{p-2}{2}} X_i u \right) = 0 \tag{11}$$

are smooth. Contrary to the Euclidean case (where solutions to the regularized p-Laplacian are C^∞-smooth), in the subelliptic case this is known only for $p \in [2, c(n))$, where $c(n) = 4$ for $n = 1, 2$, and $\lim_{n\to\infty} c(n) = 2$ (see [13]). The final result will combine the limitations given by (10) and $c(n)$.

Theorem 4 *(Theorem 3.1 in [7]) For*

$$2 \le p < 2 + \frac{n + n\sqrt{4n^2+4n-3}}{2n^2+2n-2}$$

we have that p-harmonic functions in the Heisenberg group $\mathcal{H}^n$ are in $W^{2,2}_{\mathfrak{X},loc}(\Omega)$.

At least in the one-dimensional case $\mathcal{H}^1$ one can also go below $p = 2$. See Theorem 3.2 in [7]. We also note that when p is away from 2, for example, $p > 4$ nothing is known regarding the regularity of solutions to (7) or its regularized version (11) unless we assume *a priori* that the length of the gradient is bounded below and above

$$0 < \frac{1}{M} \le |\mathfrak{X} u| \le M < \infty.$$

See [1] and [13].

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