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SURFACES WITH PRESCRIBED GAUSS CURVATURE

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1. Introduction. Let $S_g = (\mathbb{R}^2, g)$ denote a conformally flat surface over $\mathbb{R}^2$ with metric given by

$$\mathrm{d}s^2 = g^{ij}\,\mathrm{d}x_i\,\mathrm{d}x_j = e^{2u(x)}\big(\mathrm{d}x_1{}^2 + \mathrm{d}x_2{}^2\big), \tag{1.1}$$

where u is a real-valued function of the isothermal coordinates $x = (x_1, x_2) \in \mathbb{R}^2$. If u is given, the Gauss curvature function K for S_g is then explicitly given by

$$K(x) = -e^{-2u(x)}\Delta u(x), \tag{1.2}$$

where Δ is the Laplacian for the standard metric on $\mathbb{R}^2$. The quantity

$$\mathcal{K}(u) \equiv \int_{\mathbb{R}^2} K(x)e^{2u(x)}\,\mathrm{d}x, \tag{1.3}$$

where $\mathrm{d}x$ denotes Lebesgue measure on $\mathbb{R}^2$, is called the integral curvature of the surface (sometimes called total curvature). We say that S_g is a *classical* surface over $\mathbb{R}^2$ if $u \in C^2(\mathbb{R}^2)$. Clearly, $K \in C^0(\mathbb{R}^2)$ in that case. The inverse problem, namely, to prescribe K and to find a surface S_g pointwise conformal to $\mathbb{R}^2$ for which K is the Gauss curvature, renders (1.2) a semilinear elliptic partial differential equation (PDE) for the unknown function u. The problem of prescribing Gaussian curvature thus amounts to studying the existence, uniqueness or multiplicity, and classification of solutions u of (1.2) for the given K. A particularly interesting aspect of the classification problem is the question under which conditions radial symmetry of the prescribed Gauss curvature function K implies radial symmetry of the classical surface $S_g = (\mathbb{R}^2, g)$ and under which conditions radial symmetry is broken. Notice that the inverse problem may not have a solution. In particular, when considered on $\mathbb{S}^2$ instead of $\mathbb{R}^2$, there are so many obstructions to finding a solution u to (the analog of) (1.2) for the prescribed K that Nirenberg was prompted many years ago to raise the question: Which real-valued functions K are Gauss curvatures of some surface

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S_g over $\mathbb{S}^2$? For Nirenberg's problem, see [4], [6], [9], [10], [11], [33], [36], [38], [44], [45], [47], [48], [50], and [51]. For related works on other compact 2-manifolds, see, for example, [26] and [60]. In this work, we are interested in the prescribed Gauss curvature problem on $\mathbb{R}^2$. There is considerable literature on this problem, for example, [2], [3], [12], [17], [18], [19], [21], [22], [46], [49], and [59]. We study here the existence problem of surfaces for a large class of K via a novel approach. We also mention an existence-of-solutions result for a monotonically decreasing K that is unbounded below and positive at the origin. Moreover, we study the question of radial symmetry of classical surfaces, which correspond to classical solutions of (1.2), for monotonically decreasing K and for nonpositive K. The problem of nonpositive prescribed Gaussian curvature K is already fairly well understood (see [1], [21], [22], [52], [53], and [61]). In particular, [21, Theorem III] characterizes any S_g with compactly supported K and finite integral curvature uniquely by its integral curvature and by an entire harmonic function H to which u is asymptotic at infinity. If the entire harmonic function is constant and K radially symmetric, then u is radially symmetric, by uniqueness. [21, Theorem II] characterizes any S_g with $K \sim -C|x|^{-\ell}$ when $|x| \to \infty$, $\ell > 2$, and finite integral curvature uniquely by its integral curvature alone, so that u is radially symmetric if K is. [21, Theorem II] is extended in [22] to K satisfying an integrability condition and $C|x|^{-m} \leq |K(x)| \leq C|x|^m$ as $|x| \to \infty$.

Our Theorem 2.1 below generalizes [21, Theorem III] as well as Cheng and Ni's [21, Theorem II] and its sequel in [22] to a larger class of K satisfying mild integrability conditions without pointwise asymptotic bounds or even compact support for K. Our existence results follow as corollaries from our probabilistic Theorem 8.4 that applies to nonnegative K as well as nonpositive ones. We prove our Theorem 8.4 using the methods developed in [37], [8], [40], and [41]; see also [38]. For nonpositive radial K, the radial symmetry of u then follows from our uniqueness Theorem 2.2, which we prove in its dual version Theorem 9.1.

Prescribing Gaussian curvature K that is somewhere strictly positive is a much richer problem and less well understood. Existence results are available in [3], [18], [19], [46], and [59]; note [18] regarding [3]. The question of radial symmetry of u has been studied by various authors for decreasing K under various additional conditions, see [12], [15], [16], [17], and [54].

As already emphasized above, our Theorem 2.1 establishes existence of u also for nonnegative K, under mild integrability conditions on K rather than prescribed asymptotic behavior or pointwise bounds, as employed in [3], [18], [19], [46], and [59]. We also announce an existence result of a radial surface with positive integral curvature for a radial continuous K that is positive at the origin and diverges logarithmically to $-\infty$ as $|x| \to \infty$ (see Proposition 2.4). In our proof of Proposition 2.4, we actually do not prescribe K but, inspired by [39], we consider a *system of equations* whose solutions determine both K and u, and we use scattering theory and gradient flow techniques to control it. This system case is of independent interest, and details will be published elsewhere.

The radial symmetry of surfaces with K positive somewhere does not follow simply by uniqueness. In Section 2, we list various nonradial surfaces with radial Gauss curvature. We extract from this discussion a set of conditions on K and g which rule out the various nonradial surfaces we found. In particular, we demand that K be radial decreasing. We formulate a conjecture that, under this set of conditions, any classical surface for the corresponding prescribed Gauss curvature K is radially symmetric about some point.

We then state in Section 3, and subsequently prove in Sections 4–7, using the method of moving planes (see [30] and [49]), Theorems 3.2 and 3.3 on radial symmetry of classical surfaces. Our symmetry theorems require a slightly stronger set of conditions than formulated in our conjecture. However, our conditions are considerably weaker than those used in [12], [15], [16], and [17]. In particular, we impose no pointwise bounds near infinity on positive K. We also allow K to be unbounded below, but with some growth conditions near infinity, allowing logarithmic as well as power law growth of $|K|$. Our existence-of-solutions Theorem 2.1 and Proposition 2.4 establish that solutions exist under these conditions on K and thus verify that our radial symmetry theorems cover more cases than the earlier symmetry results listed above.

After submission of our work, existence results when K is positive somewhere and satisfies $0 \geq K(x) \geq -C|x|^{\ell}$ as $|x| \to \infty$, with $0 < \ell < 2$, appeared in [20]. Of these surfaces, those which also satisfy the hypotheses on K listed in Proposition 3.1 are radial symmetric by Theorems 3.2 and 3.3.

2. Broken symmetry and a symmetry conjecture. We say that S_g is radially symmetric about some point $x^* \in \mathbb{R}^2$ if the associated solution u of (1.2) satisfies $u(x-x^*) = u(\mathcal{R}(x-x^*))$ for any $\mathcal{R} \in SO(2)$. We say that u is nonradial if no such point exists. We now collect a list of examples of nonradial surfaces from which we extract conditions on K and g under which one can hope to assert the radial symmetry of u.

Clearly, u cannot be radially symmetric about some point if K is not radially symmetric about the same point. Without loss, we choose the point about which K is radially symmetric to be the origin; that is, we demand that

$$K(x) = K(\mathcal{R}x). \tag{2.1}$$

A few moments of reflection reveal that some further conditions on $K(x)$ and $u(x)$ are needed; for without further conditions, examples to nonradially symmetric surfaces having a Gauss curvature K satisfying (2.1) are readily found.

In particular, if K satisfying (2.1) is compactly supported, then solutions u of (1.2) that display some nonconstant entire harmonic behavior near infinity are asserted to exist (for nonpositive K) in [21, Theorem III]. Our first theorem, proved in Section 8, generalizes [21, Theorem III], as well as [21, Theorem II] and its extension in [22], to a much wider class of sufficiently ‘concentrated’ K that have well-defined sign.

We define the sign $\sigma(K)$ of the function K by: $\sigma(K) = +1$ if $K \not\equiv 0$, $K(x) \geq 0$ for all $x \in \mathbb{R}^2$; $\sigma(K) = -1$ if $K \not\equiv 0$, $K(x) \leq 0$ for all $x \in \mathbb{R}^2$; $\sigma(K) = 0$ if $K(x) \equiv 0$. For other K, $\sigma(K)$ does not exist.

Theorem 2.1. *Assume $K \in L^\infty(\mathbb{R}^2)$ has well-defined sign $\sigma(K)$. Furthermore, assume that for some entire harmonic function $H : \mathbb{R}^2 \mapsto \mathbb{R}$ and for all $0 < \gamma < 2$, K satisfies*

$$\int_{\mathrm{B}_1(y)} |y-x|^{-\gamma} |K(x)| e^{2H(x)} \, \mathrm{d}x \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty, \tag{2.2}$$

where $\mathrm{B}_R(y) \subset \mathbb{R}^2$ is the open ball of radius R centered at y. Given the same H, assume also that K satisfies

$$\int_{\mathbb{R}^2} |K(x)| e^{2H(x)} |x|^q \, \mathrm{d}x < \infty \tag{2.3}$$

for some $q > 0$. If $K \leq 0$, define

$$\kappa^*(K, H) = -2\pi \sup_{q>0} \big\{ q : (2.3) \text{ is true} \big\}. \tag{2.4}$$

Then, for any such K, H, and any κ satisfying

$$\kappa \in \begin{cases} (\kappa^*, 0) & \text{if } K \not\equiv 0,\ K \leq 0; \\ \{0\} & \text{if } K \equiv 0; \\ (0, 4\pi) & \text{if } K \not\equiv 0,\ K \geq 0, \end{cases} \tag{2.5}$$

there exists a solution $u = U_{H,\kappa} \in W^{2,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$ of (1.2) for the prescribed Gaussian curvature function K, having integral curvature

$$\mathcal{K}\big(U_{H,\kappa}\big) = \kappa \tag{2.6}$$

and having asymptotic behavior given by

$$U_{H,\kappa}(x) = H(x) - \frac{\kappa}{2\pi} \ln|x| + o\big(|\ln|x||\big) \quad \text{as } |x| \longrightarrow \infty. \tag{2.7}$$

Moreover, if $K \in C^{0,\alpha}$, then $U_{H,\kappa}$ is a classical solution. If $K \in C^{0,\alpha}$ also satisfies (2.1), and H is nonconstant, then $U_{H,\kappa}$ generates a classical surface which is asymptotic to a nonradial entire harmonic surface, hence breaking radial symmetry.

We remark that if $K \in C^{0,\alpha}$ satisfying (2.1) is also decreasing, then all the conclusions of Theorem 2.1 hold without imposing (2.2).

Surfaces that are asymptotic to some nonradial entire harmonic surface (entire harmonic surfaces for $K \equiv 0$) can be eliminated by the mild integrability condition

$$u^+ \in L^1\big(\mathrm{B}_R(y), \mathrm{d}x\big), \quad \text{uniformly in } y, \tag{2.8}$$

where $u^+(x) := \max\{u(x), 0\}$. For the category of $K \leq 0$ covered in Theorem 2.1, which, in addition, satisfy (2.1), condition (2.8) already eliminates *all* nonradial solutions u of (1.2) with finite integral curvature. Indeed, we have the following theorem.

Theorem 2.2. *Under the hypotheses stated in Theorem 2.1, if $K \leq 0$, then $U_{H,\kappa}$ is unique. Moreover, if $K \leq 0$ also satisfies (2.1), and u satisfies (2.8), then $U_{H,\kappa}$ is radially symmetric and decreasing.*

It remains to discuss K that are strictly positive somewhere. In that case, among the S_g that satisfy (2.1) and (2.8), one finds nonradial surfaces that are periodic about the origin of the Euclidean plane, having fundamental period $2\pi/n$, with $n > 1$. We illustrate this with the following examples, taken from [12] (see also [54]). For $x \neq 0$, we introduce the usual polar coordinates (r, θ) of x; that is, $r = |x| > 0$ and $\tan\theta = x_2/x_1$, with $\theta \in [0, 2\pi)$. Let $\mathbb{N}$ denote the natural numbers. For $n \in \mathbb{N}$, let $K(x) = K^{(n)}(x)$, with

$$K^{(n)}(x) = 4n^2|x|^{2(n-1)}. \tag{2.9}$$

Clearly, $K^{(n)} \in C^\infty(\mathbb{R}^2)$. Let $y \in \mathbb{R}^2$ be chosen arbitrarily, except that $y \neq 0$, and let θ_0 be the polar angle coordinate of y. Let $\zeta \in \mathbb{R}$. Then $u(\,.\,) = U_\zeta^{(n)}(\,.\,; y)$, with

$$U_\zeta^{(n)}(x; y) = -\ln\left(1 - 2\frac{|x|^n}{|y|^n}\cos\big(n(\theta - \theta_0)\big)\tanh\zeta + \frac{|x|^{2n}}{|y|^{2n}}\right) - \ln\big(|y|^n\cosh\zeta\big), \tag{2.10}$$

is a $C^\infty(\mathbb{R}^2)$ solution of (1.2) for the Gaussian curvature function (2.9). The integral curvature of the surface described by (2.10) is given by

$$\mathcal{K}\Big(U_\zeta^{(n)}(x; y)\Big) = 4\pi n, \tag{2.11}$$

independently of ζ and y. For $\zeta = 0$ and all $n \in \mathbb{N}$, the solution (2.10) is radially symmetric about the origin. For $\zeta \neq 0$, if $n = 1$ so that (2.9) reduces to a constant, $K^{(1)} = 4$, the solution (2.10) is periodic about the origin with fundamental period 2π, yet it is radially symmetric about and decreasing away from the point $x^* = \tanh(\zeta)y$. For $\zeta \neq 0$ and $n > 1$, in which cases $K^{(n)}$ increases monotonically with $|x|$, the solution (2.10) is periodic about the origin with a fundamental period $2\pi/n$, hence nonradial about any point; see Figure 1.

This last family of nonradial surfaces is eliminated by admitting only monotonically decreasing radial K, that is, those K satisfying

$$K(x) \leq K(y), \quad \text{whenever } |x| \geq |y|. \tag{2.12}$$

Among the S_g that satisfy (2.1), (2.8), and (2.12), we still find nonradial surfaces, namely, when $K(x) = K_0$, with

$$K_0 = \text{const} > 0, \tag{2.13}$$

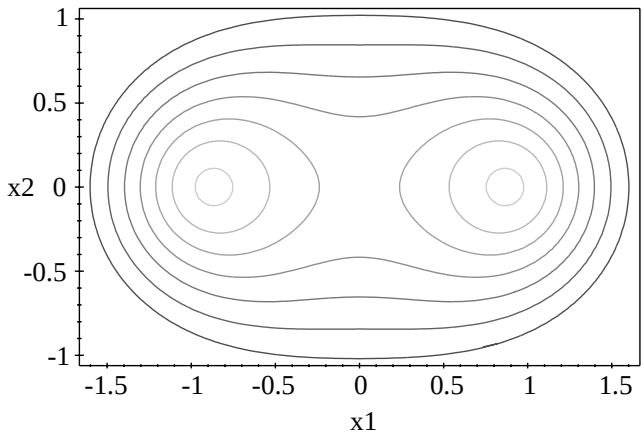

Figure 1. Level curves $e^{2u(x)} = 2^a$, $a \in \{-5, -4, \ldots, 0, 1\}$, for u given by (2.10) with $n = 2$, $|y| = 1$, $\theta_0 = 0$, and $\zeta = 1$. $\max e^{2u} \approx 2.57$ is taken at the centers of the two islands. For $|x|$ large, the conformal factor $e^{2u(x)} \sim C|x|^{-8}$ and the level curves become circular

in which case (1.2) is the conformally invariant Liouville equation [42]. Besides the radially symmetric entire solutions obtained with n=1 in (2.10), this equation has entire classical solutions that are periodic along a Cartesian coordinate direction. Let $y \in \mathbb{R}^2$ be an arbitrary fixed point, and let $v \in \mathbb{R}^2$ and $v' \in \mathbb{R}^2$ be two fixed vectors that are orthogonal with respect to Euclidean inner product; that is, $\langle v, v' \rangle = 0$, having identical lengths given by $|v| = |v'| = K_0^{1/2}$. Let $\zeta \in \mathbb{R}$. Then $u(\,.\,) = U_\zeta(\,.\,; y)$, with

$$U_\zeta(x; y) = -\ln\big(\cosh(\zeta)\cosh\langle v, x-y\rangle - \sinh(\zeta)\sin\langle v', x-y\rangle\big), \tag{2.14}$$

is a nonradial $C^\infty(\mathbb{R}^2)$ solution of (1.2) for the Gauss curvature function (2.13) (see also [12]). For $\zeta = 0$, the solution is translation invariant along v', while for $\zeta \neq 0$, it is periodic along v' with period $2\pi/\sqrt{K_0}$. See Figure 2.

Since $\exp(2U_\zeta(\,.\,; y)) \notin L^p(\mathbb{R}^2, \mathrm{d}x)$ for all p except $p = \infty$, the surface corresponding to (2.14) has integral curvature $\mathcal{K}(u) = +\infty$, as does any surface that is periodic or invariant along a fixed direction.

To rule out translation invariant surfaces and those that are periodic along a fixed Cartesian direction of the Euclidean plane, we could impose the integrability condition $\int \exp(2u(x))\,\mathrm{d}x < \infty$. However, it suffices to impose the milder, and more natural, restriction that the surface's Gauss curvature is absolutely integrable; that is,

$$\int_{\mathbb{R}^2} |K(x)| e^{2u(x)}\,\mathrm{d}x < \infty, \tag{2.15}$$

which reduces to $\int \exp(2u(x))\,\mathrm{d}x < \infty$ if $K = \text{const} > 0$.

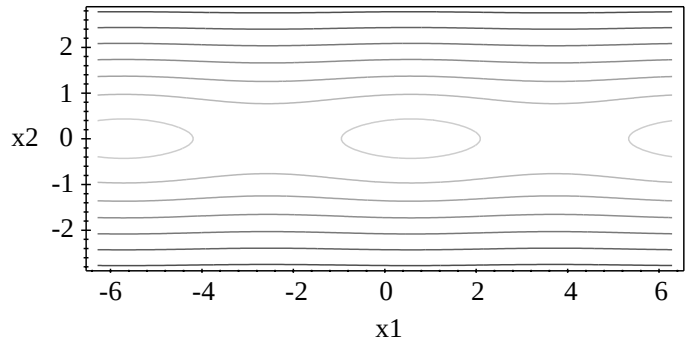

Figure 2. Level curves $e^{2u} = 2^a$, $a \in \{-6, -5, \dots, 0\}$, with u given by (2.14), with $\zeta = 1$, $y = -v'$, $K_0 = 1$, $x_1 = \langle x, v' \rangle$, and $x_2 = \langle x, v \rangle$. $\max e^{2u} \approx 1.22$ is taken at the centers of the islands. For $|\langle v, x \rangle|$ large, $e^{2u(x)} \sim Ce^{-|\langle v,x \rangle|}$ and level curves become straight lines.

We summarize the various conditions on S_g as follows.

Definition 2.3. For each $K \in C^{0,\alpha}(\mathbb{R}^2)$ satisfying (2.12), we denote by $\mathbf{S}_K$ the set of classical surfaces S_g, with Gauss curvature K being absolutely integrable, (2.15), and with metric (1.1) satisfying (2.8).

Notice that there exist K for which the set $\mathbf{S}_K$ is empty. Thus, since K satisfies (2.12), no entire solutions of (1.2) exist if $K < 0$ everywhere (see [52]). In particular, entire solutions in $\mathbb{R}^2$ with $K = constant < 0$ do not exist (see [1], [52], and [61]). Moreover, if $K(x) \sim -C|x|^p$ for $p \geq 2$ (irrespective of whether $K(x) \leq 0$ for $|x| < R$ or not), then it follows from an easy application of Pokhozaev's identity that $\mathbf{S}_K$ is empty.

On the other hand, if $K \geq 0$ everywhere, then there are plenty of radially symmetric surfaces in $\mathbf{S}_K$, which follows from Theorem 2.1 with $H \equiv \mathrm{constant}$. Furthermore, we note that $\mathbf{S}_K$ is not empty for certain radial K that are unbounded below, for the following proposition.

Proposition 2.4. *There exist continuous $K(x)$ satisfying (2.12) and $K(x) \sim -C \ln|x|$ as $|x| \to \infty$ for which $\mathbf{S}_K$ contains radial surfaces with finite positive integral curvature.*

The proof of Proposition 2.4, which uses ideas from scattering theory similar to those in [39], together with gradient flow techniques, is of independent interest and will be published elsewhere.

All known examples of surfaces in $\mathbf{S}_K$ are radially symmetric, and we could not conceive of any counterexample to radial symmetry. Hence, we conjecture that all surfaces in $\mathbf{S}_K$ are radially symmetric. More precisely, our conjecture reads as follows.

Conjecture 2.5. *Any classical surface $S_g \in \mathbf{S}_K$ is equipped with a radially symmetric nonexpansive metric, in the sense that the conformal factor e^{2u} is radially symmetric and decreasing about some point.*

Presumably, Conjecture 2.5 can even be widened to include certain K that are not everywhere decreasing (see [12] and [54] for examples). However, currently it seems not clear how to prove even Conjecture 2.5 without some additional technical conditions. In the ensuing sections, we will first state and then prove radial symmetry theorems for $\mathbf{S}_K$ under conditions that are weaker than those used in previous theorems, yet slightly stronger than those stated in Conjecture 2.5. In the next section, we state precisely our main symmetry results, assess the territory covered by them, and also compare them to existing results.

3. Symmetry theorems for radial decreasing K. To state our new symmetry results for $K \in C^{0,\alpha}(\mathbb{R}^2)$ satisfying (2.12), we define

$$\kappa_*(K) = \pi \inf \left\{ q > 0 : \int_{\mathbb{R}^2} |K(x)|(1+|x|)^{-q}\,\mathrm{d}x < \infty \right\}. \tag{3.1}$$

The significance of $\kappa_*(K)$ is that of an explicit lower bound to the integral curvature.

Proposition 3.1. *Let $K \in C^{0,\alpha}(\mathbb{R}^2)$ satisfy (2.12). If K is unbounded below, then let K also satisfy one of the following two conditions, either (1) there exists some $C > 0$ such that*

$$|K(x)| \le C \inf_{y \in \mathrm{B}_1(x)} |K(y)| \quad \text{as } |x| \longrightarrow \infty, \tag{3.2}$$

uniformly in x (this condition is satisfied, e.g., if $K \sim -C|x|^\ell$, any $\ell > 0$); or (2) there exist some finite $P \ge 1$ and $C > 0$ such that

$$|K(x)| \le C\big|\ln|x|\big|^P \quad \text{as } |x| \longrightarrow \infty. \tag{3.3}$$

Let K be the Gauss curvature function for a surface $S_g \in \mathbf{S}_K$. Then the integral curvature of S_g is bounded below by

$$\mathcal{K}(u) \ge \kappa_*(K). \tag{3.4}$$

We now state two theorems on radial symmetry of surfaces in $\mathbf{S}_K$, distinguishing the cases $\mathcal{K}(u) > \kappa_*(K)$ and $\mathcal{K}(u) = \kappa_*(K)$. Theorem 3.2 verifies Conjecture 2.5, under the hypotheses of Proposition 3.1, for all integral curvatures $\mathcal{K}(u) > \kappa_*(K)$. By Proposition 3.1, this covers the spectrum of potential integral curvature values all the way down to its lower bound (3.4), but not including it. This signals that the borderline case $\mathcal{K}(u) = \kappa_*(K)$ is critical. The critical case $\mathcal{K}(u) = \kappa_*(K)$ is dealt with in Theorem 3.3, where we assert the radial symmetry and decrease of u under an additional hypothesis that is mildly stronger than (2.15).

Theorem 3.2 (the sub-critical case). *Under the assumptions stated in Proposition 3.1, all surfaces $S_g \in \mathbf{S}_K$ with integral curvature $\mathcal{K}(u) > \kappa_*(K)$ are equipped with a radially symmetric, nonexpansive metric (1.1); that is, there exists a point $x^* \in \mathbb{R}^2$ such that u in (1.1) is radially symmetric and decreasing about x^*,*

$$u(x-x^*) \le u(y-x^*), \quad \textit{whenever } |x-x^*| \ge |y-x^*|. \tag{3.5}$$

Moreover, if $K \not\equiv \text{const}$, then $x^* = 0$, and if $K \equiv \text{const}$, then x^* is arbitrary.

Theorem 3.3 (the critical case). *Under the assumptions stated in Proposition 3.1, a surface $S_g \in \mathbf{S}_K$ having integral curvature $\mathcal{K}(u) = \kappa_*(K)$ is equipped with a radially symmetric, nonexpansive metric (1.1) (in the sense of (3.5)) provided*

$$\int_{\mathbb{R}^2} \big|\ln|x|\big|^2 |K(x)| e^{2u(x)}\,\mathrm{d}x < \infty. \tag{3.6}$$

In that case, if $K \not\equiv 0$, then $x^* = 0$, and if $K \equiv 0$, then x^* is arbitrary.

With reference to Conjecture 2.5, the foremost question now is how much of $\mathbf{S}_K$ is actually covered by Theorems 3.2 and 3.3, and how much remains uncharted territory. A priori speaking, Theorems 3.2 and 3.3 leave us anywhere in between the following extreme scenarios. In the best conceivable case, all surfaces with critical integral curvature satisfy (3.6), and then Theorems 3.2 and 3.3 taken together would prove Conjecture 2.5 completely. In the worst conceivable case, all surfaces have critical integral curvature, and none satisfies (3.6); thus, Theorems 3.2 and 3.3 would be empty. To assess the situation, we need to address the question whether for any K there exists a critical surface S_g such that inequality (3.4) is an equality, and if so, whether any such critical S_g satisfies (3.6). Notice that (3.6) is only needed for those K for which there exists a critical surface, that is, a surface for which (3.4) is an equality.

Inequality (3.4) is certainly an equality in the trivial case $K \equiv 0$, where we have $\mathcal{K}(u) = 0 = \kappa_*(0)$. Of course, (3.6) is trivially satisfied when $K \equiv 0$, hence this case is covered by Theorem 3.3.

If $K(x) \not\equiv 0$ decreases to zero at least as $C|x|^{-2-\epsilon}$, possibly having compact support, then $\mathcal{K}(u) > 0$, by (1.3), while $\kappa_*(K) = 0$. Obviously inequality (3.4) is strict in these cases, hence Theorem 3.2 covers all possible surfaces for each such K. We remark that by Theorem 2.1, with $H \equiv \text{constant}$, it follows for such a decreasing K that surfaces do exist for all integral curvature values in the open interval $(0, 4\pi)$. Together with $\mathcal{K}(u) > 0$, this implies for these K that $\kappa_*(K) = 0$ is the infimum to the set of integral curvatures for surfaces $S_g \in \mathbf{S}_K$.

For Gauss curvature functions $K = K_0 > 0$, with K_0 a constant, we have $\kappa_*(K_0) = 2\pi$, while $\mathcal{K}(u) = 4\pi$ for all solutions u of (1.2), (2.8), (2.12), and (2.15) (see [13] and [15]). Not only is inequality (3.4) strict in these cases, but $\kappa_*(K_0)$ is not even the best constant in the sense of an optimal lower bound to the integral curvature. Clearly, the cases $K = K_0 > 0$, with K_0 a constant, are entirely covered by Theorem 3.2.

The situation seems less clear when, as $|x| \to \infty$, K behaves like $C|x|^{-p}$ or like $-C|x|^p$, with $p < 2$. In these cases, explicit existence statements of surfaces in $\mathbf{S}_K$ with critical curvature $\mathcal{K}(u) = \kappa_*(K)$ seem currently not available.

We remark that surfaces with critical curvature $\mathcal{K}(u) = \kappa_*(K)$ do exist when $K \leq 0$ and $K(x) \sim -|x|^{-\ell}$ as $|x| \to \infty$, with $\ell > 2$. While these surfaces are radially symmetric by a uniqueness argument, it is nevertheless quite interesting to register that they do *not* satisfy (3.6)! The metric (1.1) of these surfaces is equipped with a conformal factor e^{2U}, where U is the maximal solution of Cheng and Ni (see [21, Theorem II, p. 723]). Cheng and Ni's result signals the possible existence of surfaces with critical curvature in $\mathbf{S}_K$ to which our Theorem 3.3 does not apply.

We summarize this state of affairs with the following list of interesting open questions.

Open Problems 3.4. Do there exist radially decreasing $K \not\equiv 0$ for which there exist solutions of (1.2), (2.8), and (2.12), with $\mathcal{K}(u) = \kappa_*(K)$? If the answer to the previous question is positive, is (3.6) a genuine condition, in the sense that there exist surfaces in $\mathbf{S}_K$ violating (3.6)? In case the answer to that question is also positive, is Conjecture 2.5 false for some of these surfaces?

Incidentally, the above discussion also points to a related open question, which, though less directly relevant to our inquiry into radial symmetry, is an interesting problem in itself. To this extent, we introduce the notion of a least integrally curved surface in $\mathbf{S}_K$, and, with an eye toward the above discussion, also the notion of when such a surface is critical.

Definition 3.5. A surface $S_g \in \mathbf{S}_K$ is called *least integrally curved* if $\mathcal{K}(u) = \kappa(K)$, where $\kappa(K)$ is defined as the infimum of the set of integral curvatures for which there exists a surface $S_g \in \mathbf{S}_K$, given K. A least integrally curved surface is called *critical* if $\kappa(K) = \kappa_*(K)$.

Open Problems 3.6. Find and classify all K for which there exists a least integrally curved surface in $\mathbf{S}_K$; with reference to Problems 3.4, determine which of those surfaces are critical!

We now return to the question of radial symmetry and to our strategy of proof for Theorems 3.2 and 3.3. We use the technique of the moving planes (see [30] and [43]), adapted to the setting in two-dimensional Euclidean space (where it is proper to rather speak of moving lines) so that it is possible to move in the lines from "spatial infinity." Due to the logarithmic divergence of solutions at infinity, this part is more delicate than in higher dimensions, in particular when $K > 0$. Various authors before have applied this method to the problem under consideration here. Hence, before we enter the details of our proof, we briefly explain in which way Theorems 3.2 and 3.3 go beyond existing results.

Radial symmetry of surfaces with strictly positive, *constant* Gauss curvature function (2.13) and finite integral curvature (2.15) was proven by Chen and Li [15]. In

[15], a radial "comparison function" was invented that made it possible to overcome the "problem at infinity." In this case, the result allows one to compute explicitly all surfaces, which are given by (2.10) with $n = 1$. This result was also obtained, with two different alternate methods, in [23] and [13].

In [12], the method of [15] was extended to a wider class of surfaces with monotone decreasing, bounded Gauss curvature functions, given certain integrability conditions. The following was proven in [12].

Theorem 3.7. *Let K be the bounded Gauss curvature function of a classical surface S_g, with metric given by (1.1), and assume that (2.8), (2.15), and (2.12) are satisfied. Let K^+ denote the positive part of K. Then any surface S_g whose integral curvature satisfies*

$$\mathcal{K}(u) > \pi\left(3+\limsup_{|x|\to\infty}\frac{\ln K^+(x)}{\ln|x|}\right), \tag{3.7}$$

is radial; more precisely, there exists a point $x^* \in \mathbb{R}^2$ such that (3.5) holds.

Remark 3.8. The proof of Theorem 3.7 is contained in [12, proof of Theorem P1].

Clearly, Theorem 3.7 falls short of proving Conjecture 2.5 because K is assumed bounded in Theorem 3.7 and, furthermore, because there exist surfaces with radially decreasing and bounded Gauss curvature function whose integral curvatures $\mathcal{K}(u)$ violate (3.7). For instance, consider the special case of (2.3), where $K > 0$ satisfies the growth condition

$$\lim_{|x|\to\infty}\frac{\ln K(x)}{\ln|x|} = -m < -2. \tag{3.8}$$

Theorem 3.7 asserts the radial symmetry of surfaces with $\mathcal{K}(u) > \pi(3-m)^+$ ((3.7) with 'lim sup' now 'lim'). Surfaces with integral curvature in the interval $0 < \mathcal{K}(u) \le \pi(3-m)^+$, which, by Theorem 2.1, exist for $m \in (2,3)$, are not covered by Theorem 3.7.

On the other hand, by Proposition 3.1, $\kappa_*(K) = 0$ for $K > 0$ satisfying (2.12) and (2.3), while $\mathcal{K}(u) > 0$ because of $K > 0$. Hence, Theorem 3.2 applies and asserts the radial symmetry of all surfaces in $\mathbf{S}_K$ with nonnegative radially decreasing Gauss curvature functions K satisfying (2.3), including, as a special case, the K that satisfy (3.8).

Closer inspection of the proof of Theorem 3.7 (see Remark 3.8) reveals that the origin of the 3π in (3.7), versus the 2π that is required to cover all surfaces for the K satisfying (3.8), traces back to our using the comparison function of [15]. That comparison function, while well-suited for constant and for certain monotonically decreasing Gauss curvature functions K, does not suit radially decreasing K in general.

One main technical innovation of the present paper is the systematic construction of a new, radial comparison function which proves itself nearly optimal for handling

the problem at infinity. We also obtain better control of solutions u of (1.2) near infinity, which allows us to forgo some technical contraptions used in [12].

Other, heuristic, comparison functions have been explored in the literature. Chen and Li [16] use a translation invariant comparison function rather than a radial one, and they require the stronger conditions that $e^{2u} \in L^1(\mathbb{R}^2)$, thereby restricting integral curvatures to $\mathcal{K}(u) > 2\pi$, to prove that all corresponding surfaces with strictly positive, radially symmetric decreasing K are given by radially symmetric and decreasing solutions u of (1.2). This result of [16] is contained in our Theorems 3.2 and 3.3. Furthermore, it intersects with, but does not subsume, due to its stronger conditions on u, [12, Theorem 3.11]. For example, consider the Gauss curvature function $K(x) = K_\gamma(x)$, with

$$K_\gamma(x) = 4\gamma \exp\big(2(1-\gamma)U_0^{(1)}(x;y)\big), \tag{3.9}$$

where $U_0^{(1)}(x;y)$ is the special case $\zeta = 0$ and $n = 1$ in (2.10), with $y \neq 0$ arbitrary, and $0 < \gamma \leq 1$. All K_γ are radially decreasing, and we have

$$K_\gamma(x) \sim C|x|^{-4(1-\gamma)}.$$

Clearly,

$$u(x) = \gamma U_0^{(1)}(x;y) \tag{3.10}$$

is a radial, decreasing solution of (1.2) for K given by (3.9). A classical radial surface described by (3.10) has integral curvature

$$\int_{\mathbb{R}^2} K_\gamma(x) e^{2\gamma U_0^{(1)}(x;y)}\,\mathrm{d}x = \gamma 4\pi \in (0, 4\pi], \tag{3.11}$$

independently of y. When $\gamma \leq 1/2$, our examples (3.10) violate Chen and Li's condition that $e^{2u} \in L^1$. Nevertheless, for K given by (3.9), solutions of (1.2) that satisfy (2.8) and (2.15) also satisfy condition (3.7) in Theorem 3.7, irrespective of γ, hence radial symmetry follows by Theorem 3.7 (cf., also [12, Theorem V2]). Incidentally, $\kappa_*(K_\gamma) = 2\pi(2\gamma - 1)^+ < \gamma 4\pi$, and so none of these surfaces is critical. Hence, the radial symmetry of these surfaces follows by Theorem 3.2 as well. Finally, a nonsymmetric comparison function (a sum of a radial and a translation invariant function) is used in [17] to prove the radial symmetry of surfaces with radially decreasing Gauss curvature function K, having finite integral curvature, under stronger conditions on K than in Theorems 3.2 and 3.3, namely, that K be strictly positive and decay slower than exponentially.

This concludes our discussion of the radial symmetry theorems. The next three sections are devoted to the proof of Theorems 3.2 and 3.3. In Section 8, we prove Theorem 2.1, and in Section 9, we prove Theorem 2.2.

4. Asymptotics. To prepare the proofs of our theorems, we need to gather some facts about the asymptotic behavior of the solutions u of (1.2). In the following lemma, $X = x/|x|^2$ denotes the Kelvin transform of x.

Lemma 4.1. *Let u be a classical solution of (1.2) satisfying (2.8). Assume that (2.15) holds and that K satisfies (2.12). Then u satisfies the integral equation*

$$u(x) - u(0) = -\frac{\mathcal{K}(u)}{2\pi} \ln|x| - \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|X - Y| K(y) e^{2u(y)} \, \mathrm{d}y \tag{4.1}$$

for all x.

Proof. By hypothesis, K is monotone decreasing. We distinguish the cases with $K \geq 0$ from those where $K < 0$ for $|x| > R$.

In case K becomes negative somewhere, say, for $|x| > R$, then outside the disk $\mathrm{B}_R(0)$ the function u is subharmonic, and so is u^+. Hence, for concentric disks $\mathrm{B}_{1/2}(y)$ and $\mathrm{B}_1(y)$, we have

$$\|u^+\|_{L^\infty(\mathrm{B}_{1/2}(y))} \leq C \|u^+\|_{L^1(\mathrm{B}_1(y))} \tag{4.2}$$

for some constant C that is independent of y. Our hypothesis (2.8) guarantees that the right-hand side in (4.2) is bounded by a constant, hence we have a uniform L^∞ bound for u^+ outside a disk, and this implies a uniform L^∞ bound for u^+ in all $\mathbb{R}^2$.

In case $K \geq 0$, since K is decreasing, and we are assuming that u is a classical solution so that K is continuous, we automatically have $K \in L^\infty$. Then, by examining Brezis and Merle [7, Theorem 2] (see also [14]), we again conclude that u^+ is uniformly bounded above.

With $u^+ \in L^\infty$, we now proceed, as in the proof of [12, Lemma 1, p. 224], to get

$$u(x) = u(0) - \frac{1}{2\pi} \int_{\mathbb{R}^2} \big(\ln|x-y| - \ln|y|\big) K(y) e^{2u(y)} \, \mathrm{d}y. \tag{4.3}$$

Pulling out the contribution $\propto \ln|x|$ from the integral, noting that

$$\ln \frac{|x-y|}{|x||y|} = \ln \left| \frac{x}{|x|^2} - \frac{y}{|y|^2} \right|, \tag{4.4}$$

and recalling the definition of the Kelvin transform, gives us (4.1). □

Proof of Proposition 3.1. Let $|x| \geq 4$. We define, for given x, the set

$$D_x = \left\{ y : \frac{|x|}{2} \leq |y| \leq 2|x| \text{ and } |x-y| \leq 4 \right\} \tag{4.5}$$

and split $\mathbb{R}^2$ accordingly into $\mathbb{R}^2 = D_x \cup D_x^C$, where D_x^C is the complement of D_x in $\mathbb{R}^2$. Moreover, for $y \in D_x^C$, we use the decomposition $D_x^C = E_x \cup F_x \cup G_x$, with

$$E_x = \{y : 2|y| \le |x|\}, \tag{4.6}$$

$$F_x = \{y : |y| \ge 2|x|\}, \tag{4.7}$$

$$G_x = \{y : |y| \le 2|x| \le 4|y| \text{ and } |x-y| \ge 4\}. \tag{4.8}$$

Recall (4.4). Let $\mathbf{I}_\Lambda$ denote the indicator function of the set Λ. It is now readily verified that, with positive generic constants C,

$$\big|\ln|X-Y|\big| = \left|\ln\frac{|x-y|}{|x||y|}\right| \le \begin{cases} C\ln|x| + C\big|\ln|x-y|\big|; & y \in D_x, \\ C + C\big|\ln|y|\big|\mathbf{I}_{E_x} + C\big|\ln|x|\big|\mathbf{I}_{F_x\cup G_x}; & y \in D_x^C. \end{cases} \tag{4.9}$$

In each of these regions, the corresponding inequality in (4.9) follows by an application of the triangle inequality, paying attention to the a priori bounds on x, y, and $x-y$.

Thus, with positive generic constants C,

$$\begin{aligned} &\frac{1}{\ln|x|}\int_{\mathbb{R}^2}\left|\ln\frac{|x-y|}{|x||y|}\right||K(y)|e^{2u(y)}\,\mathrm{d}y \\ &\quad\le \frac{C}{\ln|x|}\int_{|y|\le 1}\big|\ln|y|\big|\,\mathrm{d}y + C\int_{1\le|y|\le|x|/2}\frac{\big|\ln|y|\big|}{\ln|x|}|K(y)|e^{2u(y)}\,\mathrm{d}y \\ &\qquad + C\int_{|y|\ge 2|x|}|K(y)|e^{2u(y)}\,\mathrm{d}y + C\int_{|y-x|\le 4}\frac{\big|\ln|x-y|\big|}{\ln|x|}|K(y)|e^{2u(y)}\,\mathrm{d}y. \end{aligned} \tag{4.10}$$

The first term on the right obviously goes to zero as $|x| \to \infty$. The second integral on the right goes to zero as $|x| \to \infty$, by the dominated convergence theorem and because $Ke^{2u} \in L^1(\mathbb{R}^2)$. The third integral on the right goes to zero as $|x| \to \infty$ because $Ke^{2u} \in L^1(\mathbb{R}^2)$. For the fourth integral on the right, we need to distinguish two cases, (i) $K \in L^\infty$ and (ii) $K \notin L^\infty$. As for case (i), since $u^+ \in L^\infty$, we have $Ke^{2u} \in L^\infty$, and so

$$\begin{aligned} \frac{1}{\ln|x|}\int_{|y-x|\le 4}\big|\ln|x-y|\big||K(y)|e^{2u(y)}\,\mathrm{d}y &\le \frac{C}{\ln|x|}\int_{|y-x|\le 4}\big|\ln|x-y|\big|\,\mathrm{d}y \\ &\le \frac{C}{\ln|x|} \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty. \end{aligned} \tag{4.11}$$

As for case (ii), since then $K(x) < 0$ for $|x| > R$, we have

$$-\Delta u(x) = K(x)e^{2u(x)} \le 0 \quad \text{for } |x| > R, \tag{4.12}$$

hence $u(x)$ is subharmonic for $|x| > R$. Thus, for $|x_0| \geq R+1$, we have

$$u(x_0) \leq \frac{1}{\pi} \int_{\mathrm{B}_1(x_0)} u(y)\,\mathrm{d}y. \tag{4.13}$$

By Jensen's inequality [34],

$$e^{2u(x_0)} \leq \frac{1}{\pi} \int_{\mathrm{B}_1(x_0)} e^{2u(y)}\,\mathrm{d}y, \tag{4.14}$$

hence

$$\big|K(x_0)\big|e^{2u(x_0)} \leq \frac{1}{\pi} \int_{\mathrm{B}_1(x_0)} \big|K(x_0)\big|e^{2u(y)}\,\mathrm{d}y. \tag{4.15}$$

Now, by hypothesis, either (3.2) or (3.3) holds. If (3.2) holds, then $|K(x_0)| \leq C|K(y)|$ for all y in $\mathrm{B}_1(x_0)$, hence

$$\int_{\mathrm{B}_1(x_0)} \big|K(x_0)\big|e^{2u(y)}\,\mathrm{d}y \leq C \int_{\mathrm{B}_1(x_0)} |K(y)|e^{2u(y)}\,\mathrm{d}y \leq C, \tag{4.16}$$

where the second estimate holds by (2.15). It follows once again that $Ke^{2u} \in L^\infty$, and so we are back to (4.11). If (3.3) holds, then, writing $|K| = |K|^{1/p}|K|^{1/q}$ with $p = P$, $1/p + 1/q = 1$, we have, by Hölder's inequality [34],

$$\begin{aligned}\int_{|y-x|\leq 4} &\big|\ln|x-y|\big||K(y)|e^{2u(y)}\,\mathrm{d}y \\ &\leq \left(\int_{|y-x|\leq 4} |K(y)|e^{2qu(y)}\,\mathrm{d}y\right)^{1/q} \left(\int_{|y-x|\leq 4} \big|\ln|x-y|\big|^p |K(y)|\,\mathrm{d}y\right)^{1/p}.\end{aligned} \tag{4.17}$$

Since $u^+ \in L^\infty$, and since (3.3) holds, we now have

$$\begin{aligned}\frac{1}{\ln|x|}\int_{|y-x|\leq 4} &\big|\ln|x-y|\big||K(y)|e^{2u(y)}\,\mathrm{d}y \\ &\leq C\left(\int_{|y-x|\leq 4} |K(y)|e^{2u(y)}\,\mathrm{d}y\right)^{1/q} \left(\int_{|y-x|\leq 4} \big|\ln|x-y|\big|^p\,\mathrm{d}y\right)^{1/p} \longrightarrow 0 \\ &\quad \text{as } |x| \longrightarrow \infty,\end{aligned} \tag{4.18}$$

and this completes the estimates on the third integral in (4.10).

In total, by Lemma 4.1 and our estimates on the last integral in (4.1), we conclude that for any ϵ there exists a $C(\epsilon)$ and $R(\epsilon)$ such that

$$e^{2u(x)} \leq C|x|^{-(\mathcal{K}(u)/\pi)+\epsilon} \quad \text{for } |x| > R(\epsilon). \tag{4.19}$$

Recalling now the definition of κ_*, Proposition 3.1 follows. □

Lemma 4.2. *Let u be a classical solution of (1.2) satisfying (2.8). Assume that (2.15) holds and that K satisfies (2.12). Moreover, if K is unbounded below, assume that either (3.2) or (3.3) holds. Finally, if $\mathcal{K}(u) = \kappa_*(K)$, let (3.6) be satisfied. Then, uniformly in x,*

$$\lim_{|x|\to\infty}\left(u(x) - u(0) + \frac{1}{2\pi}\mathcal{K}(u)\ln|x|\right) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\ln|y|K(y)e^{2u(y)}\,\mathrm{d}y. \tag{4.20}$$

Proof. By Proposition 3.1, $\mathcal{K}(u) \geq \kappa_*(K)$. If $\mathcal{K}(u) = \kappa_*(K)$, then (3.6) is satisfied, by hypothesis, and this implies that $\int_{\mathbb{R}^2}\ln|y|K(y)e^{2u(y)}\,\mathrm{d}y$ exists. If $\mathcal{K}(u) > \kappa_*(K)$, then, by (4.19) and the definition of $\kappa_*(K)$, the existence of $\int_{\mathbb{R}^2}\ln|y|K(y)e^{2u(y)}\,\mathrm{d}y$ follows once again. By inspecting the estimates of the proof of Proposition 3.1, we now conclude, once again by dominated convergence, that

$$\lim_{X\to 0}\int_{\mathbb{R}^2}\ln|X-Y|K(y)e^{2u(y)}\,\mathrm{d}y = -\int_{\mathbb{R}^2}\ln|y|K(y)e^{2u(y)}\,\mathrm{d}y. \tag{4.21}$$

Lemma 4.2 follows. □

5. Global results. With the help of Lemma 4.2, and noting (2.12) and (2.15), we now see that the asymptotic behavior of u implies that the integral curvature $\mathcal{K}(u)$ of $S_g \in \mathbf{S}_K$ is strictly positive if $K(x) < 0$ for $|x| > R$. In addition, it follows trivially from the definition of $\mathcal{K}(u)$ that $\mathcal{K}(u) \geq 0$ if $K \geq 0$, with equality holding if and only if $K \equiv 0$. We summarize this as the following lemma.

Lemma 5.1. *Let u be a classical solution of (1.2) satisfying (2.8). Assume (2.15) holds. In addition, assume that K satisfies (2.12). If $\mathcal{K}(u) = \kappa_*(K)$, let (3.6) be satisfied. Then the integral curvature $\mathcal{K}(u)$ of S_g is positive,*

$$\int_{\mathbb{R}^2} K(x)e^{2u(x)}\,\mathrm{d}x \geq 0, \tag{5.1}$$

with "=" holding if and only if $K \equiv 0$.

We also need an angular average of u. In the following, we set $r = |x|$, and we identify points in $\mathbb{R}^2$ with points in $\mathbb{C}$. We define the radial function

$$\overline{u}(r) = \frac{1}{2\pi}\int_0^{2\pi} u\left(re^{i\theta}\right)\mathrm{d}\theta, \tag{5.2}$$

which is well defined for all $r \geq 0$ because u is a classical solution. Similarly, we define $\overline{K}(r)$. Notice that $\overline{K}(|x|) = K(x)$.

Lemma 5.2. *Let u be a classical solution of (1.2) satisfying (2.8) and (2.15), with K satisfying (2.12). If $\mathcal{K}(u) = \kappa_*(K)$, let (3.6) be satisfied. Let $\overline{u}$ be defined by (5.2).*

Then there exists a positive constant $c(u) < \infty$ such that

$$\left|u(x) - \overline{u}(|x|)\right| \leq c(u) \tag{5.3}$$

for all x, and $c(u)$ is the smallest such c.

Proof. For $|x| \leq R$, the statement is trivial, since u is a classical solution. For $|x| > R$, the statement follows from Lemma 4.2. □

Lemma 5.3. *Let u be a classical solution of (1.2) satisfying (2.8) and (2.15), with K satisfying (2.12). Let $\overline{u}$ be defined by (5.2). Then we have*

$$\int_{\mathbb{R}^2} |K(x)| e^{2\overline{u}(|x|)}\, \mathrm{d}x < \infty. \tag{5.4}$$

If (3.6) holds, then we also have

$$\int_{\mathbb{R}^2} (\ln|x|)^2 |K(x)| e^{2\overline{u}(|x|)}\, \mathrm{d}x < \infty. \tag{5.5}$$

Proof. By Jensen's inequality,

$$e^{\overline{u}(r)} \leq \frac{1}{2\pi} \int_0^{2\pi} e^{u(re^{i\theta})}\, \mathrm{d}\theta. \tag{5.6}$$

Upon multiplying (5.6) by $2\pi r|\overline{K}(r)|$ and then integrating over r, we get

$$\int_0^{2\pi} \int_0^{\infty} \left|\overline{K}(r)\right| e^{2\overline{u}(r)} r\, \mathrm{d}r\, \mathrm{d}\theta \leq \int_{\mathbb{R}^2} |K(x)| e^{2u(x)}\, \mathrm{d}x, \tag{5.7}$$

which now shows that (5.4) holds because of (2.15). Similarly, if (3.6) holds, then we can multiply (5.6) by $2\pi r(\ln r)^2|\overline{K}(r)|$ and subsequently integrate the result over r to get

$$\int_0^{2\pi} \int_0^{\infty} \left|\overline{K}(r)\right| e^{2\overline{u}(r)} (\ln r)^2 r\, \mathrm{d}r\, \mathrm{d}\theta \leq \int_{\mathbb{R}^2} |K(x)| e^{2u(x)} (\ln|x|)^2\, \mathrm{d}x, \tag{5.8}$$

which shows that (5.5) now holds because of (3.6). □

6. The comparison function. In this section, we construct a comparison function for u, a classical solution of (1.2) satisfying (2.8) and (2.15), with K satisfying (2.12). In case $\mathcal{K}(u) = \kappa_*(K)$, we assume that (3.6) is satisfied. Recall that $\overline{u}$ is defined by (5.2).

We first introduce a function $g : [0, \infty) \to \mathbb{R}$, given by

$$g(r) = r \int_r^{\infty} \left|\overline{K}(s)\right| e^{2\overline{u}(s)} s (\ln s)^2\, \mathrm{d}s - r \ln r \int_r^{\infty} \left|\overline{K}(s)\right| e^{2\overline{u}(s)} s \ln s\, \mathrm{d}s, \tag{6.1}$$

if $r > 0$, while $g(0)$ is given by continuous extension to $r = 0$. Notice that g is well-defined for $r \geq 0$, for, by Lemma 5.3, the integrals are well-defined for all $r \geq 0$, and $r \ln r$ has a removable singularity at $r = 0$.

Lemma 6.1. *The function g defined in (6.1) is the unique $C^2(\mathbb{R}^+)$ solution of the inhomogeneous Euler equation*

$$r^2 g''(r) - r g'(r) + g(r) = \left|\overline{K}(r)\right| e^{2\overline{u}(r)} r^3 \ln r, \tag{6.2}$$

under the asymptotic condition

$$g(r) = o(r) \quad \text{as } r \longrightarrow \infty. \tag{6.3}$$

Furthermore, g is eventually positive,

$$g(r) \geq 0 \quad \text{if } r > 1, \tag{6.4}$$

and g vanishes at $r = 0$,

$$g(0) = 0. \tag{6.5}$$

Proof. Inserting (6.1) into (6.2), one verifies that (6.1) is a particular solution of (6.2). Moreover, since $|K| \geq 0$ and $r < s$, when $r > 1$ we have the bounds $0 < (\ln r)(\ln s) < (\ln s)^2$, which imply

$$0 \leq g(r) \leq r \int_r^\infty \left|\overline{K}(s)\right| e^{2\overline{u}(s)} s (\ln s)^2 \, \mathrm{d}s \quad \text{for } r > 1. \tag{6.6}$$

The first inequality in (6.6) states positivity (6.4), and both together prove (6.3), for clearly

$$0 \leq \lim_{r\to\infty} \frac{g(r)}{r} \leq \lim_{r\to\infty} \int_r^\infty \left|\overline{K}(s)\right| e^{2\overline{u}(s)} s (\ln s)^2 \, \mathrm{d}s = 0, \tag{6.7}$$

the last step as a consequence of Lemma 5.3. Moreover, since $g(0)$ is defined by $g(0) = \lim_{r\to 0} g(r)$, (6.5) holds because of Lemma 5.3 and $r \ln r \to 0$ for $r \to 0$.

The general solution of (6.2) is obtained by adding to this particular solution the general solution of the homogeneous problem $Ar + Br \ln r$, with A, B constants. By (6.3), we conclude that $A = B = 0$, and thus also uniqueness is shown. □

Let $\alpha > 0$, and define $R(\alpha)$ as the smallest $R > 0$ such that $r - \alpha g(r) > e$ for all $r > R$. By (6.5), (6.3), and the continuity of $r \mapsto r - \alpha g(r)$, it follows that a positive $R(\alpha)$ exists and that $R(\alpha) - \alpha g(R(\alpha)) = e$. We now introduce the family of radial functions $f_\alpha : \mathbb{R}^2 \setminus \mathrm{B}_{R(\alpha)} \to \mathbb{R}$, given by

$$f_\alpha(x) = \ln\big(|x| - \alpha g(|x|)\big). \tag{6.8}$$

Clearly, $f_\alpha(x) > 1$ for $|x| > R(\alpha)$, and $f_\alpha(x) = 1$ for $|x| = R(\alpha)$. We also introduce

$$\alpha^*(u) = 2e^{2c(u)}, \tag{6.9}$$

where $c(u)$ is defined in Lemma 5.2.

Lemma 6.2. *Given u, $\alpha > \alpha^*(u)$, the function f_α defined in (6.8) satisfies the partial differential inequality*

$$\Delta f_\alpha(x) + 2K(x)e^{2u(x)} f_\alpha(x) < 0 \tag{6.10}$$

for all x satisfying $|x| > \max\{1, R(\alpha)\}$.

Proof. In the following, $g'(r) = \partial_r g(r)$, and so on. Recall that $r = |x|$.
By explicit calculation we find

$$\begin{aligned}
\frac{\Delta f_\alpha(x)}{f_\alpha(x)} &= \frac{\alpha\big(-r^2 g''(r) + rg'(r) - g(r)\big) + \alpha^2\big(rg(r)g''(r) - rg'(r)^2 + g(r)g'(r)\big)}{r\big(r-\alpha g(r)\big)^2 \ln\big(r-\alpha g(r)\big)} \\
&= -\frac{\alpha\big|\overline{K}(r)\big|e^{2\overline{u}(r)}}{\big(1-\alpha g(r)/r\big)\big(1+\ln\big(1-\alpha g(r)/r\big)/\ln r\big)} \\
&\quad - \frac{\alpha^2\big(g(r)-rg'(r)\big)^2}{r^2\big(r-\alpha g(r)\big)^2 \ln\big(r-\alpha g(r)\big)} \\
&< -\alpha|K(x)|e^{2\overline{u}(|x|)} \quad \text{for } r > R(\alpha),
\end{aligned} \tag{6.11}$$

the last step by the facts that $r > 1$ and $\alpha g(r) > 0$ for $r > 1$, and $r > R(\alpha)$ and $1-\alpha g/r > 1/r$ for $r > R(\alpha)$. By (6.11), Lemma 5.2, and $\alpha > \alpha^*(u)$ defined in (6.9), we now have

$$\begin{aligned}
\Delta f_\alpha(x) + 2K(x)e^{2u(x)} f_\alpha(x) &< -\big(\alpha|K(x)|e^{2\overline{u}(|x|)} - 2K(x)e^{2u(x)}\big) f_\alpha(|x|) \\
&\le -\big(\alpha|K(x)|e^{-2c(u)} - 2K(x)\big)e^{2u(x)} f_\alpha(|x|) \\
&\le 0
\end{aligned} \tag{6.12}$$

for all x satisfying $|x| > \max\{1, R(\alpha)\}$. □

7. Proof of symmetry Theorems 3.2 and 3.3. In the following, we always understand that $S_g \in \mathbf{S}_K$, that u is the associated solution of (1.2), and that (3.6) is assumed to be satisfied in case that $\mathcal{K}(u) = \kappa_*(K)$. Moreover, if K is unbounded below, it is also assumed that either (3.2) or (3.3) holds.

By Lemma 4.2, $u(x) \to -\infty$ as $|x| \to \infty$. Therefore, and since u is a classical solution, u has a global maximum, say, at x^*. Since K satisfies (2.12), if $x \mapsto u(x)$ solves (1.2), then so does $x \mapsto u(\mathcal{R}(x))$ for any $\mathcal{R} \in SO(2)$. Therefore, after at most a rotation, we can assume that our solution u has a global maximum at the point $x^* = (-|x^*|, 0)$, with $|x^*| \ge 0$.

We now introduce the family of straight lines

$$T_\lambda = \big\{x \in \mathbb{R}^2 \mid x_1 = \lambda\big\} \tag{7.1}$$

and the half-plane "left of T_λ,"

$$\Sigma_\lambda = \{x : x_1 < \lambda\}. \tag{7.2}$$

We denote the reflection of x at T_λ by

$$x^{(\lambda)} = (2\lambda - x_1, x_2). \tag{7.3}$$

Lemma 7.1. *For $x_1 \le \lambda \le 0$, and in particular for $x \in \Sigma_\lambda$, with $\lambda \le 0$, we have*

$$K(x) \le K\left(x^{(\lambda)}\right). \tag{7.4}$$

Proof. K satisfies (2.12). □

We next introduce $u_\lambda(x) = u(x^{(\lambda)})$, and also

$$v_\lambda(x) = u_\lambda(x) - u(x). \tag{7.5}$$

Clearly, v_λ is well defined on $\mathbb{R}^2$.

Lemma 7.2. *For all $\lambda \in \mathbb{R}$, v_λ vanishes on T_λ and at infinity; that is,*

$$\lim_{|x|\to\infty} v_\lambda(x) = 0 \tag{7.6}$$

uniformly in $|x|$.

Proof. Notice that on T_λ we have $x^{(\lambda)} = x$, hence $v_\lambda(x) = 0$ for $x \in T_\lambda$. The vanishing of v_λ at infinity is a consequence of Lemma 4.2. □

We next resort to our comparison function f_α. We pick any $\alpha > \alpha^*(u)$ and introduce the function $w_\lambda : \Sigma_\lambda \cup T_\lambda \to \mathbb{R}$, defined as

$$w_\lambda(x) = \begin{cases} \dfrac{v_\lambda(x)}{f_\alpha(x)}, & |x| \ge R(\alpha), \\[2ex] v_\lambda(x), & |x| \le R(\alpha). \end{cases} \tag{7.7}$$

Notice that w_λ is twice continuously differentiable at all x with $|x| \ne R(\alpha)$, and is continuous as a function of $x \in \Sigma_\lambda \cup T_\lambda$, with any λ. It vanishes for $|x| \to \infty$ as well as for $x \in T_\lambda$. Therefore, if $w_\lambda(x) < 0$ for some $x \in \Sigma_\lambda$, then w_λ will have a global negative minimum in Σ_λ. Our next lemma allows us to initialize the moving planes argument and also to finalize it.

Lemma 7.3. *For each u, there exists an $R(u) > 0$ such that, if $x_* \in \Sigma_\lambda$ is a minimum point for w_λ, and $w_\lambda(x_*) < 0$, then $|x_*| < R(u)$, independently of λ.*

Proof. We begin by observing that, in the flat case $K \equiv 0$, $u = \text{const.}$, and $v_\lambda \equiv 0$ for all λ, so that the claim is trivially true.

In the nonflat case where $K \not\equiv 0$, we prove Lemma 7.3 by contradiction. Thus, assume that no such $R(u)$ exists. Then, for any R, we can find a $\lambda \leq 0$ such that $|x_*| > R$, where $x_* \in \Sigma_\lambda$ is a minimum point for w_λ with $w_\lambda(x_*) < 0$. In particular, we may choose $R > \max\{1, R(\alpha)\}$. At such a minimum point $x_* \in \Sigma_\lambda$, we have $\nabla w_\lambda(x_*) = 0$ and $\Delta w_\lambda(x_*) \geq 0$, and, of course, $w_\lambda(x_*) < 0$.

First, notice that the reflected function u_λ satisfies the PDE

$$-\Delta u_\lambda(x) = K\big(x^{(\lambda)}\big)e^{2u_\lambda(x)}. \tag{7.8}$$

Taking the difference between (7.8) and (1.2), we get

$$-\Delta v_\lambda(x) = K\big(x^{(\lambda)}\big)e^{2u_\lambda(x)} - K(x)e^{2u(x)}. \tag{7.9}$$

By the mean value theorem, there exists a number $\psi_\lambda(x)$ between $u(x)$ and $u(x^{(\lambda)})$ such that

$$e^{2u_\lambda(x)} - e^{2u(x)} = 2v_\lambda(x)e^{2\psi_\lambda(x)}. \tag{7.10}$$

By (7.10) and Lemma 7.1, we see that v_λ satisfies the partial differential inequality

$$\Delta v_\lambda(x) + 2K(x)e^{2\psi_\lambda(x)}v_\lambda(x) \leq 0, \tag{7.11}$$

for all $x \in \Sigma_\lambda$. With the help of (7.11), we now easily find that w_λ satisfies the partial differential inequality

$$\Delta w_\lambda(x) + 2\frac{\nabla f_\alpha(x)}{f_\alpha(x)} \cdot \nabla w_\lambda(x) + \left(\frac{\Delta f_\alpha(x)}{f_\alpha(x)} + 2K(x)e^{2\psi(x)}\right)w_\lambda(x) \leq 0, \tag{7.12}$$

for all $x \in \Sigma_\lambda$ for which $|x| > \max\{1, R(\alpha)\}$. Now, by assumption, $w_\lambda(x_*) < 0$, with $|x_*| > \max\{1, R(\alpha)\}$, and since $f_\alpha(x) > 1$ for $|x| > \max\{1, R(\alpha)\}$, we also have $v_\lambda(x_*) < 0$, and this means that $u_\lambda(x_*) < u(x_*)$. But then $\psi_\lambda(x_*) \leq u(x_*)$. Making use of this and of $\nabla w_\lambda(x_*) = 0$, from (7.12) we now obtain the inequality

$$\Delta w_\lambda(x_*) + \left(\frac{\Delta f_\alpha(x_*)}{f_\alpha(x_*)} + 2K(x_*)e^{2u(x_*)}\right)w_\lambda(x_*) \leq 0. \tag{7.13}$$

Using now Lemma 6.2, recalling that $\alpha > \alpha^*(u)$, in combination with $w_\lambda(x_*) < 0$, we see that (7.13) implies that $\Delta w_\lambda(x_*) < 0$. But this is a contradiction to $\Delta w_\lambda(x_*) \geq 0$.

Hence, w_λ has no strictly negative minimum outside the disk $B_{R(u)}$ with $R(u) = \max\{1, R(\alpha^*(u))\}$. This concludes the proof of Lemma 7.3. □

Corollary 7.4. *For each u, when $\lambda < -R(u)$, then $v_\lambda(x) \geq 0$ for $x \in \Sigma_\lambda$.*

Proof. Assume $v_\lambda(x_*) < 0$ for some $x_* \in \Sigma_\lambda$, with $\lambda < -R(u)$. Then, since $w_\lambda = v_\lambda/f_\alpha$ for all $x \in \Sigma_\lambda$ with $\lambda < -R(u)$, and since $f_\alpha > 1$ for all $x \in \Sigma_\lambda$ with $\lambda < -R(u)$, we conclude that $w_\lambda(x_*) < 0$ for $x_* \in \Sigma_\lambda$ with $\lambda < -R(u)$. But then,

since $w_\lambda \to 0$ as $|x| \to \infty$, and $w_\lambda = 0$ on T_λ, we see that w_λ attains a negative minimum for some $x_* \in \Sigma_\lambda$, with $\lambda < -R(u)$. This is a contradiction to Lemma 7.3. □

Recall the maximum principle (MP) and the Hopf maximum principle (HMP) (see [31]).

MP: *Let* $\Delta v(x) + \sum_i b_i(x)\partial_{x_i} v(x) + c(x)v(x) \le 0$ *in* $\Omega \subset \mathbb{R}^n$ *and* $v \ge 0$. *If* $v(\hat{x}) = 0$ *for at least one* $\hat{x} \in \mathrm{int}(\Omega)$, *then* $v \equiv 0$ *in all of* Ω.

HMP: *Under the same assumptions as in MP, if* $v \not\equiv 0$ *in* Ω, *and* $\partial\Omega$ *is smooth with* $v|_{\partial\Omega} \equiv 0$, *then* $\partial v/\partial\nu < 0$, *where* $\partial v/\partial\nu$ *is the exterior normal derivative on* $\partial\Omega$.

Notice that no sign condition is being imposed on $c(x)$ as the minimum of v is 0.

We are now ready for the moving lines. The arguments in our ensuing proof of Theorems 3.2 and 3.3 are a straightforward modification of those in the proof of [12, Theorem P1]. For the convenience of the reader, we give the complete argument instead of listing where to modify the arguments of [12].

Proof of Theorems 3.2 and 3.3. By Lemma 7.4, $v_\lambda(x) \ge 0$ for $\lambda < -R(u)$, independently of λ. We now slide the line T_λ to the right until we reach a critical value λ_0, which is the largest value of λ for which $v_\lambda(x) \ge 0$, $x \in \Sigma_\lambda$.

Claim A. *We have* $v_\lambda(x) > 0$ *for* $x \in \Sigma_\lambda$ *with* $\lambda < \lambda_0$, *and* $\partial_{x_1} u > 0$ *for* $x_1 < \lambda_0$.

Claim B. *We have* $\lambda_0 = -|x^*|$.

Proof of Claim A. We begin by establishing the first assertion in Claim A. Suppose, for $\lambda < \lambda_0$, that $v_\lambda(x) = 0$ at some point $x \in \Sigma_\lambda$. Since $v_\lambda(x) \ge 0$ for $x \in \Sigma_\lambda$, if $v_\lambda(x) = 0$, the minimum of $v_\lambda(x)$ is achieved in Σ_λ. Since (7.11) holds, and $v_\lambda(x) \ge 0$, we can apply the maximum principle and deduce $v_\lambda(x) \equiv 0$ in Σ_λ. This means for $\lambda = \lambda_0 - \delta$, some $\delta > 0$, that $u(\lambda_0 - 2\delta, x_2) = u(\lambda_0, x_2)$. But $v_\lambda(x) \ge 0$, thus $u(x^{(\lambda)}) \ge u(x)$, which implies $\partial_{x_1} u \ge 0$ for $x_1 \le \lambda_0$. This fact, together with the fact $u(\lambda_0 - 2\delta, x_2) = u(\lambda_0, x_2)$, yields $\partial_{x_1} u = 0$ for $\lambda_0 - 2\delta \le x_1 \le \lambda_0$. In particular, $\partial_{x_1} u = 0$ when $x_1 = \lambda_0 - 2\delta$. By the HMP and the MP, we have $v_\lambda \equiv 0$ if and only if $\partial_{x_1} v_\lambda = 0$ on T_λ. Now, $\partial_{x_1} v_\lambda = -2\partial_{x_1} u$ for $x_1 = \lambda$. But, since $\partial_{x_1} u = 0$ when $x_1 = \lambda_0 - 2\delta$, we see $\partial_{x_1} v_{\lambda_0 - 2\delta} = 0$ for $x_1 = \lambda_0 - 2\delta$ or, which is the same, on $T_{\lambda_0 - 2\delta}$. Now the HMP says $v_{\lambda_0 - 2\delta} \equiv 0$. We may repeat this procedure indefinitely and thus deduce that u is independent of x_1. This is a contradiction, and so the first assertion of Claim A is proved.

As for the second assertion of Claim A, note that since $v_\lambda > 0$ in Σ_λ for $\lambda < \lambda_0$, and $v_\lambda = 0$ on T_λ, by the HMP we get $\partial_{x_1} v_\lambda < 0$ on T_λ. Since for $x_1 = \lambda$, we have $\partial_{x_1} u = -(1/2)\partial_{x_1} v_\lambda$, we also have $\partial x_1(u) > 0$ for $x_1 = \lambda$, with $\lambda < \lambda_0$. So Claim A is proved. □

Proof of Claim B. From the second assertion in Claim A, we see u is strictly increasing for $x_1 < \lambda_0$. By a rotation, we had arranged that the maximum of u is at $(-|x^*|, 0)$. It follows that $\lambda_0 \le -|x^*|$. Thus, to prove Claim B, we need to rule out the case $\lambda_0 < -|x^*|$.

Assume $\lambda_0 < -|x^*|$. There are two possibilities. Either $v_{\lambda_0} \equiv 0$ or $v_{\lambda_0} \not\equiv 0$.

We will first rule out the case $\lambda_0 < -|x^*|$, and $v_{\lambda_0} \not\equiv 0$. Indeed, since $v_{\lambda_0}(x) \equiv 0$ for $x \in T_{\lambda_0}$ and for $|x| \to \infty$, but $v_{\lambda_0} \not\equiv 0$ in Σ_{λ_0}, and since v_{λ_0} satisfies (7.11), then by the MP we get $v_{\lambda_0} > 0$ for $x_1 < \lambda_0$. Hence, by the HMP, $\partial_{x_1} v_{\lambda_0} < 0$ when $x_1 = \lambda_0$. On the other hand, by definition of λ_0, there exists a sequence of numbers λ_k, decreasing to λ_0, such that $v_{\lambda_k} < 0$, hence also $w_{\lambda_k} < 0$, and $\lambda_0 < \lambda_k < -|x^*|$. Notice that w_{λ_k} is well defined for $\lambda_k < -|x^*|$. Let x_k be a minimum point for w_{λ_k}. Then $w_{\lambda_k}(x_k) < 0$ and $\nabla w_{\lambda_k}(x_k) = 0$. As Lemma 7.3 implies $|x_k| < R(u)$, independently of λ, there exists a subsequence $x_{k_j} \to x^*$ such that $\nabla w_{\lambda_0}(x^*) = 0$ and $w_{\lambda_0}(x^*) \le 0$ for $x^* = (A, B)$, $A \le \lambda_0$. This is a contradiction. Thus, our claim is proved in this case.

We now rule out the case $\lambda_0 < -|x^*|$ and $v_{\lambda_0} \equiv 0$. Indeed, in that case $u(x_{\lambda_0}) = u(x)$ for $x \in \Sigma_{\lambda_0}$. But u attains its maximum at $(-|x^*|, 0)$ and by Claim A, $\partial_{x_1} u > 0$ for $x_1 < \lambda_0$. Since $\lambda_0 < -|x^*|$, it follows that $\partial_{x_1} u = 0$ at $(|x^*| + 2\lambda_0, 0)$, which again is a contradiction. □

Thus, $\lambda_0 = -|x^*|$. Recall that λ_0 is the largest value of λ for which $v_\lambda(x) \ge 0$, $x \in \Sigma_\lambda$. Hence, $v_{-|x^*|}(x) \ge 0$ for $x \in \Sigma_{-|x^*|}$, thus $u_{-|x^*|}(x) \ge u(x)$.

We may now repeat this argument by sliding the line T_λ in from $x_1 = \infty$ to get $u_{-|x^*|}(x) \le u(x)$. Putting the two inequalities together, we conclude that $u_{-|x^*|}(x) = u(x)$. This now implies that u is symmetric with respect to $T_{-|x^*|}$. Moreover, from the arguments involving the HMP, we see that any solution is also decreasing away from $T_{-|x^*|}$. Recall that $(-|x^*|, 0)$ is the point of global maximum of u.

Finally, if $x^* \neq 0$, then, since u satisfies (1.2), and since K is radially symmetric about $(0, 0)$, we conclude that K is a constant. But if K is a constant, then if $u(x)$ is a solution of (1.2), so is $u(x + x^*)$ for any fixed x^*. Thus, by a simple translation of the origin to x^*, we can assume that our solution is in fact symmetric with respect to, and decreasing away from, T_0. On the other hand, if $K \not\equiv$ const., then $x^* = 0$, and again our solution is symmetric with respect to, and decreasing away from, T_0. But if $x^* = 0$, then we can repeat our moving line argument with any other than the x_1-direction; thus, we come to the conclusion that u is symmetric about, and decreasing away from, any straight line through the origin. This now means that u is radially symmetric about and decreasing away from the origin, modulo a translation in case that $K =$ const.

This completes the proof of Theorems 3.2 and 3.3. □

8. Proof of existence theorem 2.1. We begin with the remark that, in the special case of identically vanishing Gauss curvature, our Theorem 2.1 is obviously true. Hence, in the rest of this section, we assume that the Gauss curvature is not identically zero.

In the following, we prove a probabilistic theorem which implies Theorem 2.1 as an immediate corollary. Incidentally, the proof also provides us with an algorithm

for the *construction* (in principle at least) of nonradial surfaces. We use the methods developed in [37] (see also [8]), [40], and [41]. For applications to Nirenberg's problem, see [38].

We first introduce some probabilistic notation and terminology. In the following, x_1, $x_2, \dots$ denote points in $\mathbb{R}^2$, not Cartesian components of x. Let $\mathbb{N}$ denote the natural numbers. For each $N \in \mathbb{N}$, we denote the probability measures on $\mathbb{R}^{2N}$ by $P(\mathbb{R}^{2N})$. For $\varrho^{(N)} \in P(\mathbb{R}^{2N})$, we denote the associated Radon measure by $\widehat{\varrho}^{(N)}$. A measure $\varrho^{(N)} \in P(\mathbb{R}^{2N})$ is called absolutely continuous with respect to a measure $\varpi^{(N)} \in P(\mathbb{R}^{2N})$, written $\mathrm{d}\varrho^{(N)} \ll \mathrm{d}\varpi^{(N)}$, if there exists a positive $d\varpi^{(N)}$-integrable function $f(x_1, \dots, x_N)$, called the density of $\varrho^{(N)}$ with respect to $\varpi^{(N)}$, such that $\mathrm{d}\varrho^{(N)} = f(x_1, \dots, x_N)\,\mathrm{d}\varpi^{(N)}$. By $P^s(\mathbb{R}^{2N})$, we denote the exchangeable probabilities, that is, the subset of $P(\mathbb{R}^{2N})$ whose elements are permutation symmetric in $x_1, \dots, x_N$. The nth marginal measure of $\varrho^{(N)} \in P^s(\mathbb{R}^{2N})$, $n < N$, is an element of $P^s(\mathbb{R}^{2n})$, given by

$$\varrho_n^{(N)}\big(\mathrm{d}x_1 \cdots \mathrm{d}x_n\big) = \int_{\mathbb{R}^{2N-2n}} \varrho^{(N)}\big(\mathrm{d}x_1 \cdots \mathrm{d}x_n\, \mathrm{d}x_{n+1} \cdots \mathrm{d}x_N\big). \tag{8.1}$$

By $\Omega \equiv (\mathbb{R}^2)^{\mathbb{N}}$, we denote the infinite Cartesian product of the exchangeable $\mathbb{R}^2$-valued infinite sequences. By $P^s(\Omega)$, we denote the permutation symmetric probability measures on Ω. The de Finetti-type result of Hewitt and Savage [35] states that each $\mu \in P^s(\Omega)$ is uniquely presentable as a convex superposition of product measures; that is, for each $\mu \in P^s(\Omega)$, there exists a unique probability measure $\nu(\mathrm{d}\varrho \mid \mu)$ on $P(\mathbb{R}^2)$, such that

$$\mu_n\big(\mathrm{d}x_1 \cdots \mathrm{d}x_n\big) = \int_{P(\mathbb{R}^2)} \nu\big(\mathrm{d}\varrho \mid \mu\big)\varrho^{\otimes n}\big(\mathrm{d}x_1 \cdots \mathrm{d}x_n\big), \quad n \in \mathbb{N}, \tag{8.2}$$

where $\varrho^{\otimes n}(\mathrm{d}x_1 \cdots \mathrm{d}x_n) \equiv \varrho(\mathrm{d}x_1) \otimes \cdots \otimes \varrho(\mathrm{d}x_n)$, and μ_n denotes the nth marginal measure of μ. For de Finetti's original work, see [29] (see also [27], [24], and [25]). We remark that (8.2) coincides with the extremal decomposition for the convex set $P^s(\Omega)$, an application of the Krein-Milman theorem. For details, see [35].

To $\varrho \in P(\mathbb{R}^2)$, we assign the energy

$$\mathscr{E}(\varrho) \equiv \frac{1}{2}\widehat{\varrho}^{\otimes 2}\big(\ln|x-y|\big) = \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-y|\varrho(\mathrm{d}x)\varrho(\mathrm{d}y), \tag{8.3}$$

whenever the integral on the right exists. We denote by $P_{\mathscr{E}}(\mathbb{R}^2)$ the subset of $P(\mathbb{R}^2)$ for which $\mathscr{E}(\varrho)$ exists. For $\mu \in P^s(\Omega)$, the mean energy of μ is defined as

$$e(\mu) = \frac{1}{2}\widehat{\mu}_2\big(\ln|x-y|\big), \tag{8.4}$$

whenever the integral on the right exists. The following proposition, proved in [57], characterizes the subset of $P^s(\Omega)$ for which (8.4) is well-defined.

Proposition 8.1. *The mean energy of μ, (8.4), is well-defined for those μ whose decomposition measure $\nu(\mathrm{d}\varrho \mid \mu)$ is concentrated on $P_{\mathscr{E}}(\mathbb{R}^2)$, and in that case given by*

$$e(\mu) = \int_{P_{\mathscr{E}}(\mathbb{R}^2)} \nu\big(\mathrm{d}\varrho \mid \mu\big)\mathscr{E}(\varrho). \tag{8.5}$$

Let $\Upsilon : \mathbb{R}^2 \to \mathbb{R}^+$ be an L^∞ function, $\Upsilon \not\equiv 0$. For some entire harmonic function H, which may be constant, and all $0 < \gamma < 2$, we assume Υ satisfies

$$\int_{\mathrm{B}_1(y)} \Upsilon(x) e^{2H(x)} |x-y|^{-\gamma}\,\mathrm{d}x \longrightarrow 0 \quad \text{as } |y| \longrightarrow \infty. \tag{8.6}$$

Moreover, we assume that for the same harmonic function H and some $q > 0$, Υ satisfies

$$\int_{\mathbb{R}^2} \Upsilon(x) e^{2H(x)} |x|^q\,\mathrm{d}x < \infty, \tag{8.7}$$

and we define

$$q^*(\Upsilon, H) = \sup\left\{q > 0 : \int_{\mathbb{R}^2} \Upsilon(x) e^{2H(x)} |x|^q\,\mathrm{d}x < \infty\right\}. \tag{8.8}$$

Given such H and Υ, we now define the a priori measure

$$\tau(\mathrm{d}x) = \Upsilon(x) e^{2H(x)}\,\mathrm{d}x \tag{8.9}$$

on $\mathbb{R}^2$. Since Υ satisfies (8.7), the integral

$$M^{(1)} = \int_{\mathbb{R}^2} \tau(\mathrm{d}x) \tag{8.10}$$

exists and is called the mass of τ. The probability measure associated to τ, given by

$$\mu^{(1)}(\mathrm{d}x) = \frac{1}{M^{(1)}}\tau(\mathrm{d}x), \tag{8.11}$$

is thus clearly absolutely continuous with respect to $\mathrm{d}x$.

For each $\varrho^{(N)}(\mathrm{d}x_1 \cdots \mathrm{d}x_N) \in P(\mathbb{R}^{2N})$, its entropy with respect to the probability measure $\mu^{(1)}(\mathrm{d}x_1) \otimes \cdots \otimes \mu^{(1)}(\mathrm{d}x_N) \equiv \mu^{(1)\otimes N}(\mathrm{d}x_1 \cdots \mathrm{d}x_N)$ is defined as

$$\mathscr{S}^{(N)}\big(\varrho^{(N)}\big) = -\int_{\mathbb{R}^{2N}} \ln\left(\frac{\mathrm{d}\varrho^{(N)}}{\mathrm{d}\mu^{(1)\otimes N}}\right)\varrho^{(N)}\big(\mathrm{d}x_1 \cdots \mathrm{d}x_N\big), \tag{8.12}$$

if $\varrho^{(N)}$ is absolutely continuous with respect to $\mathrm{d}\tau^{\otimes N}$, and provided the integral in (8.12) exists. In all other cases, $\mathscr{S}^{(N)}(\varrho^{(N)}) = -\infty$. In particular, if μ_n is the nth marginal measure of a $\mu \in P^s(\Omega)$, then the entropy of μ_n, $n \in \{1, \dots\}$, is given by $\mathscr{S}^{(n)}(\mu_n)$, with $\mathscr{S}^{(n)}$ defined as in (8.12) with $\varrho^{(n)} = \mu_n$. We also define $\mathscr{S}^{(0)}(\mu_0) = 0$.

For each $\mu \in P^s(\Omega)$, the sequence $n \mapsto \mathcal{S}^{(n)}(\mu_n)$ enjoys the following useful properties, proofs of which are found in [55, Section 2, proof of Proposition 1] (see also [28] and [37]).

Nonpositivity of $\mathcal{S}^{(n)}(\mu_n)$*:* For all n,

$$\mathcal{S}^{(n)}(\mu_n) \leq 0. \tag{8.13}$$

Monotonic decrease of $\mathcal{S}^{(n)}(\mu_n)$*:* If $n < n'$, then

$$\mathcal{S}^{(n')}(\mu_{n'}) \leq \mathcal{S}^{(n)}(\mu_n). \tag{8.14}$$

Strong subadditivity of $\mathcal{S}^{(n)}(\mu_n)$*:* For $n', n'' \leq n$, and with $\mathcal{S}^{(-m)}(\mu_{-m}) \equiv 0$ for $m > 0$,

$$\begin{aligned} \mathcal{S}^{(n)}(\mu_n) \leq{}& \mathcal{S}^{(n')}(\mu_{n'}) + \mathcal{S}^{(n'')}(\mu_{n''}) \\ &+ \mathcal{S}^{(n-n'-n'')}\big(\mu_{n-n'-n''}\big) - \mathcal{S}^{(n'+n''-n)}\big(\mu_{n'+n''-n}\big). \end{aligned} \tag{8.15}$$

As a consequence of the subadditivity (8.15) of $\mathcal{S}^{(n)}(\mu_n)$, the limit

$$s(\mu) = \lim_{n\to\infty} \frac{1}{n} \mathcal{S}^{(n)}(\mu_n) \tag{8.16}$$

exists whenever $\inf_n n^{-1}\mathcal{S}^{(n)}(\mu_n) > -\infty$; otherwise, $s(\mu) = -\infty$. The quantity $s(\mu)$ given in (8.16) is called the mean entropy of $\mu \in P^s(\Omega)$. The mean entropy is an affine function (see [55]). This entails the following useful representation, proved in [55].

Proposition 8.2. *The mean entropy of μ, (8.16), is given by*

$$s(\mu) = \int_{P(\mathbb{R}^2)} \nu\big(\mathrm{d}\varrho \mid \mu\big) \mathcal{S}^{(1)}(\varrho). \tag{8.17}$$

Next, identifying each $x_k \in \mathbb{R}^2$ with the corresponding $z_k \in \mathbb{C}$, we recall the definition of the alternant $\Delta^{(N)}(x_1, \ldots, x_N)$,

$$\Delta^{(N)}\big(x_1, \ldots, x_N\big) = \prod_{1 \leq i < j \leq N} \big(z_i - z_j\big). \tag{8.18}$$

Clearly,

$$\big|\Delta^{(N)}\big|\big(x_1, \ldots, x_N\big) = \prod_{1 \leq i < j \leq N} \big|x_i - x_j\big|. \tag{8.19}$$

We also recall the definition of $q^* > 0$ in (8.8) and define

$$\beta^*(\Upsilon, H) = -2q^*. \tag{8.20}$$

For $\beta \in (\beta^*, 4)$, and $N \in \mathbb{N}$, we now introduce the probability measure $\mu^{(N)}$ on $\mathbb{R}^{2N}$ by

$$\mu^{(N)}\big(\mathrm{d}x_1 \cdots \mathrm{d}x_N\big) \equiv \frac{1}{M^{(N)}(\beta)} \big|\Delta^{(N)}\big|^{-\beta/N}\big(x_1, \ldots, x_N\big) \prod_{1 \le \ell \le N} \tau(\mathrm{d}x_\ell) \tag{8.21}$$

if $N > 1$, and $\mu^{(N)} \equiv \mu^{(1)}$ given in (8.11) if $N = 1$. The next lemma asserts that (8.21) is well-defined for all $N \in \mathbb{N}$, and all $\beta \in (\beta^*, 4)$.

Lemma 8.3. *For all $\beta \in (\beta^*, 4)$, the measure (8.21) satisfies $\mathrm{d}\mu^{(N)} \ll \mathrm{d}\tau^{\otimes N}$. Moreover, for the associated density we have $\mathrm{d}\mu^{(N)}/\mathrm{d}\tau^{\otimes N} \in L^p(\mathbb{R}^{2N}, \mathrm{d}\tau^{\otimes N})$, with $p \in [1, \beta^*/\beta)$ when $\beta < 0$, $p \in [1, \infty]$ when $\beta = 0$, and $p \in [1, 4/\beta)$ when $\beta > 0$.*

Proof. First, if $\beta = 0$, or $N = 1$, the claim is obviously true.

If $N > 1$ and $\beta \in (\beta^*, 0)$, we make use of the inequality

$$\big|x_i - x_j\big| \le \big(|x_i| + 2\big)\big(|x_j| + 2\big), \tag{8.22}$$

valid for any two $x_i \in \mathbb{R}^2$ and $x_j \in \mathbb{R}^2$. Inequality (8.22) is a consequence of the triangle inequality $|x_i - x_j| \le |x_i| + |x_j|$, the fact that $|x| < |x| + 2$, and finally the fact that $r + s < sr$ when both $r > 2$ and $s > 2$. To verify this last inequality, use that $2 + r < 2r$ whenever $r > 2$, so that when $r > 2$ and $s > 2$, we have $r + s = r + 2 + \epsilon < 2r + \epsilon = (2+\epsilon)r - \epsilon r + \epsilon = sr - \epsilon(r-1) < sr$. With the help of (8.22), we now have for $\beta < 0$,

$$\begin{aligned} M^{(N)}(\beta) &= \int_{\mathbb{R}^{2N}} \big|\Delta^{(N)}\big|^{-\beta/N}\big(x_1, \ldots, x_N\big) \prod_{1 \le k \le N} \tau(\mathrm{d}x_k) \\ &\le \int_{\mathbb{R}^{2N}} \prod_{1 \le i \le N} \big(2 + |x_i|\big)^{-\beta/2} \tau(\mathrm{d}x_i) \\ &= \left(\int_{\mathbb{R}^2} \big(2 + |x|\big)^{-\beta/2} \tau(\mathrm{d}x) \right)^N. \end{aligned} \tag{8.23}$$

The last integral exists, by hypothesis (8.7). This proves $\mathrm{d}\mu^{(N)} \ll \mathrm{d}\tau^{\otimes N}$ for $\beta \in (\beta^*, 0)$.

If $N > 1$ and $\beta \in (0, 4)$, we use the inequality between arithmetic and geometric means (see [34]), permutation invariance (twice), and Hölder's inequality (see [34]), to get

$$\begin{aligned} M^{(N)}(\beta) &= \int_{\mathbb{R}^{2N}} \big|\Delta^{(N)}\big|^{-\beta/N}\big(x_1, \ldots, x_N\big) \prod_{1 \le k \le N} \tau(\mathrm{d}x_k) \\ &\le \int_{\mathbb{R}^{2N}} \frac{1}{N} \sum_{1 \le i \le N} \prod_{\substack{1 \le j \le N \\ (j \ne i)}} \big|x_i - x_j\big|^{-\beta/2} \prod_{1 \le k \le N} \tau(\mathrm{d}x_k) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^{2(N-1)}} \prod_{2\le j\le N} \left|x_1 - x_j\right|^{-\beta/2} \tau\left(\mathrm{d}x_j\right) \tau(\mathrm{d}x_1) \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |x-y|^{-\beta/2} \tau(\mathrm{d}y) \right)^{N-1} \tau(\mathrm{d}x) \qquad (8.24)\\
&\le \left(\sup_x \int_{\mathbb{R}^2} |x-y|^{-\beta/2} \tau(\mathrm{d}y) \right)^{N-1} M^{(1)}.
\end{aligned}
$$

In the last step, we used that for $\beta \in (0,4)$, we have

$$
\int_{\mathbb{R}^2} |x-y|^{-\beta/2} \tau(\mathrm{d}y) < M^{(1)} + \Psi_\beta(x), \qquad (8.25)
$$

with $\Psi_\beta : x \mapsto \int_{B_1(x)} |x-y|^{-\beta/2} \tau(\mathrm{d}y) \in C^0(\mathbb{R}^2)$ (because $\Upsilon \in L^\infty$) and $\Psi_\beta(x) \to 0$ for $|x| \to \infty$ (by hypothesis (8.6)). This proves $\mathrm{d}\mu^{(N)} \ll \mathrm{d}\tau^{\otimes N}$ for $\beta \in (0,4)$.

By repeating now the same chains of estimates with $p\beta$ in place of β, one concludes that $\mathrm{d}\mu^{(N)}/\mathrm{d}\tau^{\otimes N} \in L^p(\mathbb{R}^{2N}, \mathrm{d}\tau^{\otimes N})$ for all $p \in [1, 4/\beta)$ when $\beta > 0$, respectively, all $p \in [1, \beta^*/\beta)$ when $\beta < 0$. □

We now come to the main theorem of this section. It addresses the limiting behavior of $\mu_n^{(N)}$ as $N \to \infty$, with n arbitrary but fixed.

Theorem 8.4. *The sequence of probability measures $N \mapsto \mu_n^{(N)}(\mathrm{d}x_1 \cdots \mathrm{d}x_n)$ is the union of weakly convergent subsequences in the sense that there exist disjoint sequences $\mathbf{E}_\ell = \{N_\ell(k)\}_{k\in\mathbb{N}}$, $\mathbf{E}_\ell \cap \mathbf{E}_{\ell'} = \emptyset$, for $\ell \ne \ell'$, such that for each ℓ, the map $k \mapsto \mu_n^{(N_\ell(k))}(\mathrm{d}x_1 \cdots \mathrm{d}x_n)$ converges weakly in the sense of probability measures, with densities with respect to $\mathrm{d}\tau^{\otimes n}$ converging weakly in $L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$, for all $p \in [1,\infty)$.*

Let μ_n^ℓ denote the weak limit point of such a subsequence. Then there exists a unique $\mu^\ell \in P^s(\Omega)$ (of which μ_n^ℓ is the nth marginal), and μ^ℓ has its decomposition measure $\nu(\mathrm{d}\varrho \mid \mu^\ell)$ concentrated on the subset of $P(\mathbb{R}^2) \cap \bigcup_{p>1} L^p(\mathbb{R}^2, \mathrm{d}\tau)$, whose elements minimize the functional

$$
\mathcal{F}_\beta(\varrho) = \beta \mathcal{E}(\varrho) - \mathcal{S}^{(1)}(\varrho). \qquad (8.26)
$$

Remark 8.5. Notice that Theorem 8.4 asserts that $\mathcal{F}_\beta$ does have a minimizer $\varrho_\beta \in P_{\mathcal{E}}$. If it can be shown that (8.26) has a unique minimizer, say, ϱ_β, then in fact we have convergence to a product measure,

$$
\lim_{N\to\infty} \mu_n^{(N)}\left(\mathrm{d}x_1 \cdots \mathrm{d}x_n\right) = \bigotimes_{1\le k\le n} \varrho_\beta(\mathrm{d}x_k), \qquad (8.27)
$$

weakly in $P(\mathbb{R}^{2n}) \cap L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$ for any $p \in [1,\infty)$.

Before we prove Theorem 8.4, we show that Theorem 2.1 is a corollary of Theorem 8.4.

Proof of Theorem 2.1. Assume that all hypotheses of Theorem 8.4 are fulfilled. Then (8.26) has a solution for all $\beta \in (\beta^*, 4)$. The minimizers of (8.26) are of the form $\varrho(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$, with ρ satisfying the Euler-Lagrange equation

$$\rho(x) = \frac{\Upsilon(x)\exp\left(-\beta\int_{\mathbb{R}^2}\ln|x-y|\rho(y)\,\mathrm{d}y + 2H(x)\right)}{\int_{\mathbb{R}^2}\Upsilon(x)\exp\left(-\beta\int_{\mathbb{R}^2}\ln|x-y|\rho(y)\,\mathrm{d}y + 2H(x)\right)\mathrm{d}x}. \tag{8.28}$$

Recall that $\Upsilon \geq 0$, by hypothesis. If $\beta \in (0,4)$, we now identify Υ with a (positive) Gauss curvature function, $K \equiv \Upsilon$, and if $\beta \in (\beta^*, 0)$, we identify $-\Upsilon$ with a (negative) Gauss curvature function, $K \equiv -\Upsilon$. In either case, K satisfies the hypotheses of Theorem 2.1. We also identify $\beta\pi$ with the integral Gauss curvature,

$$\kappa = \beta\pi, \tag{8.29}$$

and we notice that $\beta^*\pi = \kappa^*$, defined in (2.4).

We now pick a corresponding solution of (8.28), say, $\rho_{H,\beta}$, which exists by Theorem 8.4. With the help of this $\rho_{H,\beta}$, we define, for all $x \in \mathbb{R}^2$, the function

$$U_{H,\kappa}(x) = H(x) - \frac{\beta}{2}\int_{\mathbb{R}^2}\ln|x-y|\rho_{H,\beta}(y)\,\mathrm{d}y + U_0, \tag{8.30}$$

the constant U_0 being uniquely determined by the requirement that

$$\int_{\mathbb{R}^2} K(x)e^{2U_{H,\kappa}(x)}\,\mathrm{d}x = \kappa. \tag{8.31}$$

By Theorem 8.4, $\rho_{H,\beta} \in L^p(\mathbb{R}^2, \mathrm{d}\tau)$ for all $p \in [1,\infty)$, hence $U_{H,\kappa} \in W^{2,p}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}$. With $\Delta\ln|x-y| = 2\pi\delta(x-y)$, it now follows that $u(x) = U_{H,\kappa}(x)$ is a distributional solution of (1.2) for the prescribed Gauss curvature function K, with K satisfying (2.2) and (2.3), and u satisfying the asymptotics (2.7).

For the subset of $K \in C^{0,\alpha}(\mathbb{R}^2)$, we can bootstrap to $U_{H,\kappa} \in C^{2,\alpha}(\mathbb{R}^2)$ by using elliptic regularity, thus obtaining an entire classical solution of (1.2). For the further subset of K satisfying also (2.1), this classical solution obviously breaks the radial symmetry if $H \not\equiv \mathrm{constant}$. Finally, for the further subset of K satisfying (2.12), a straightforward estimate shows that (2.2) is redundant.

This concludes the proof of Theorem 2.1. □

We now prepare the proof of Theorem 8.4. Let $\Pi(\mathbb{R}^{2N})$ denote the subset of $P(\mathbb{R}^{2N})$ whose elements are absolutely continuous with respect to $\mathrm{d}\tau^{\otimes N}$, having density $\mathrm{d}\varrho^{(N)}/\mathrm{d}\tau^{\otimes N} \in \bigcup_{p>1} L^p(\mathbb{R}^{2N}, \mathrm{d}\tau^{\otimes N})$. On $\Pi(\mathbb{R}^{2N})$, we define the functional

$$\mathcal{F}^{(N)}_\beta\left(\varrho^{(N)}\right) = \beta\widehat{\varrho}^{(N)}\left(\ln\left|\Delta^{(N)}\right|\right) - N\mathcal{S}^{(N)}\left(\varrho^{(N)}\right). \tag{8.32}$$

Lemma 8.6. *For each $\beta \in (\beta^*, 4)$, the functional (8.32) takes its unique minimum*

at the probability measure (8.21); that is,

$$\min_{\varrho^{(N)}\in\Pi(\mathbb{R}^{2N})} \mathscr{F}_\beta^{(N)}\big(\varrho^{(N)}\big) = \mathscr{F}_\beta^{(N)}\big(\mu^{(N)}\big). \tag{8.33}$$

Moreover,

$$\mathscr{F}_\beta^{(N)}\big(\mu^{(N)}\big) = -N\ln\widehat{\mu}^{(1)\otimes N}\big(\big|\Delta^{(N)}\big|^{-\beta/N}\big). \tag{8.34}$$

For $\beta \geq 4$, and for $\beta < \beta^*$, (8.32) is unbounded below.

Proof. Since $\ln|\Delta^{(N)}| \in L^p(\mathbb{R}^{2N}, \mathrm{d}\tau^{\otimes N})$ for all $p \in [1,\infty)$ by Lemma 8.3, the integral $\mathscr{F}_\beta^{(N)}(\mu^{(N)})$ is well-defined for $\beta \in (\beta^*, 4)$. The identity (8.34) is readily verified by explicit computation. In turn, the Gibbs variational principle (8.33) is just convex duality [56], verified by the standard convexity argument (cf. [28, proof of Proposition I.4.1]). Thus, rewriting (8.32) as

$$\mathscr{F}_\beta^{(N)}\big(\varrho^{(N)}\big) = \int_{\mathbb{R}^{2N}} \ln\left(\frac{\mathrm{d}\varrho^{(N)}}{\mathrm{d}\mu^{(N)}}\right)\frac{\mathrm{d}\varrho^{(N)}}{\mathrm{d}\mu^{(N)}}\,\mathrm{d}x_1\cdots\mathrm{d}x_N \tag{8.35}$$

and using now $x\ln x \geq x-1$, with equality if and only if $x=1$, we find that $\mathscr{F}_\beta^{(N)}(\varrho^{(N)}) \geq 0$, with equality holding if and only if $\varrho^{(N)} = \mu^{(N)}$. This proves Lemma 8.6 for $\beta \in (\beta^*, 4)$.

Now, let $\beta \geq 4$, or $\beta < \beta^*$. Assume that $M^{(N)}(\beta)$ is finite. Then, by (8.34) and by the Gibbs variational principle (8.33), we have $\min_\varrho \mathscr{F}_\beta^{(N)}(\varrho^{(N)}) = -N\ln M^{(N)}(\beta)$. However, a simple scaling argument shows that $M^{(N)}(\beta \geq 4) > C$ for any C, and similarly we have $M^{(N)}(\beta < \beta^*) > C$ for any C, by definition of β^*. This verifies the unboundedness below of (8.32) for $\beta \geq 4$ and $\beta < \beta^*$. □

Lemmas 8.6 and 8.3 entail the following lemma.

Lemma 8.7. *The function $\beta \mapsto F(\beta)$ defined by*

$$F(\beta) \equiv \inf_{\varrho\in\Pi(\mathbb{R}^2)} \mathscr{F}_\beta(\varrho) \tag{8.36}$$

is continuous for all $\beta \in (\beta^*, 4)$.

Proof. Gibbs' variational principle (8.33) evaluated with a trial product measure $\varrho^{(N)} = \varrho^{\otimes N} \in P(\mathbb{R}^{2N})$, with $\varrho \in P(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \mathrm{d}\tau)$ for some $p > 1$, gives us

$$\frac{1}{N^2}\mathscr{F}_\beta^{(N)}\big(\mu^{(N)}\big) \leq \frac{1}{N^2}\mathscr{F}_\beta^{(N)}\big(\varrho^{\otimes N}\big) = \left(1-\frac{1}{N}\right)\beta\mathscr{E}(\varrho) - \mathscr{S}^{(1)}(\varrho) \tag{8.37}$$

for all $\varrho \in P(\mathbb{R}^2) \cap L^p(\mathbb{R}^2, \mathrm{d}\tau)$, $p > 1$, and $N > 1$. Now, by (8.23) and (8.24), the left-hand side in (8.37) is uniformly bounded below. Letting $N \to \infty$ in (8.37), we

obtain a lower bound for $\mathcal{F}_\beta(\varrho)$, uniformly over $P(\mathbb{R}^2)\cap L^p(\mathbb{R}^2, \mathrm{d}\tau)$, $p > 1$, for each $\beta \in (\beta^*, 4)$. Thus,

$$\beta\mathcal{E}(\varrho) - \mathcal{S}^{(1)}(\varrho) \geq \limsup_{N\to\infty} \frac{1}{N^2}\mathcal{F}_\beta^{(N)}\big(\mu^{(N)}\big) \geq \liminf_{N\to\infty} \frac{1}{N^2}\mathcal{F}_\beta^{(N)}\big(\mu^{(N)}\big) \geq f_0(\beta), \tag{8.38}$$

with

$$f_0(\beta) = \begin{cases} -\ln \sup\limits_x \displaystyle\int_{\mathbb{R}^2} |x-y|^{-\beta/2}\mu^{(1)}(\mathrm{d}y) & \text{for } \beta \geq 0, \\ -\ln \displaystyle\int_{\mathbb{R}^2} \big(2+|x|\big)^{-\beta/2}\mu^{(1)}(\mathrm{d}x) & \text{for } \beta \leq 0. \end{cases} \tag{8.39}$$

Recalling (8.26), this proves that $\mathcal{F}_\beta$ is bounded below for $\beta \in (\beta^*, 4)$.

Having a lower bound, continuity of F now follows from the definition of F. Assume that F is discontinuous at $\beta_0 \in (\beta^*, 4)$. Without loss of generality, we can assume $F(\beta_0^-) > F(\beta_0^+)$. (The reverse case $F(\beta_0^-) < F(\beta_0^+)$ is treated essentially verbatim.) Now let $\beta = \beta_0 + \epsilon$. Clearly, for each ϵ we can find a minimizing sequence $\{\varrho_k\}_{k\in\mathbb{N}}$ (depending on ϵ) such that $\mathcal{F}_{\beta_0+\epsilon}(\varrho_k) < F(\beta^+) + \delta$ if $k > M(\delta)$. Pick a sufficiently small δ and select a $\varrho_* \in \{\varrho_k\}_{k>M(\delta)}$. Insert this ϱ_* into $\mathcal{F}_{\beta_0-\epsilon}$. Using $\mathcal{F}_\beta = \beta\mathcal{E} - S^{(1)}$, we find, for any ϵ and δ,

$$\begin{aligned} F(\beta_0 - \epsilon) &\leq \mathcal{F}_{\beta_0-\epsilon}(\varrho_*) \\ &= \mathcal{F}_{\beta_0+\epsilon}(\varrho_*) - 2\epsilon\mathcal{E}(\varrho_*) \\ &\leq F(\beta_0+\epsilon) + \delta - 2\epsilon\mathcal{E}(\varrho_*). \end{aligned} \tag{8.40}$$

Letting $\epsilon \to 0$ and $\delta \to 0$, we obtain $F(\beta_0^-) \leq F(\beta_0^+)$, which is a contradiction. □

Taking the infimum over ϱ in (8.38) and noting Lemma 8.7, gives the following proposition.

Proposition 8.8. *For all $\beta \in (\beta^*, 4)$,*

$$\limsup_{N\to\infty} \frac{1}{N^2}\mathcal{F}_\beta^{(N)}\big(\mu^{(N)}\big) \leq F(\beta). \tag{8.41}$$

Proposition 8.8 is complemented by a sharp estimate in the opposite direction.

Proposition 8.9. *For all $\beta \in (\beta^*, 4)$,*

$$\liminf_{N\to\infty} \frac{1}{N^2}\mathcal{F}_\beta^{(N)}\big(\mu^{(N)}\big) \geq F(\beta). \tag{8.42}$$

To prove Proposition 8.9, we need to prove that the sequence of the nth marginal measures $\mu_n^{(N)}$ is not "leaking at ∞" as $N \to \infty$. When $\beta > 0$, we also need to

show that the sequences of the densities $\mathrm{d}\mu_n^{(N)}/\mathrm{d}\tau^{\otimes n}$ of these marginal measures are uniformly in $L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$ for $N > N_n(\beta)$. However, since it gives a priori regularity, we prove uniform L^p bounds for all $\beta \in (\beta^*, 4)$. We remark that when Υ is radially symmetric decreasing, or has compact support, then many of the following proofs simplify considerably, some to trivialities. However, since we work with a minimal set of assumptions on Υ, it is unavoidable that the ensuing estimates become somewhat more technical.

We begin by deriving bounds on the expected value of $\ln|\Delta^{(N)}|$ with respect to $\mu^{(N)}$ which, using permutation symmetry, can be written in terms of $\mu_2^{(N)}$,

$$\widehat{\mu}^{(N)}\big(\ln\big|\Delta^{(N)}\big|\big) = N(N-1)\frac{1}{2}\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big). \tag{8.43}$$

Lemma 8.10. *For each $\beta \in (\beta^*, 4)$, there exist constants $\underline{C}(\beta)$ and $\overline{C}(\beta)$, independent of N, such that for all $N \geq 2$, we have the estimates*

$$\overline{C}(\beta) \geq \beta\widehat{\mu}^{(1)\otimes 2}\big(\ln|x-y|\big) \geq \beta\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big) \geq \underline{C}(\beta). \tag{8.44}$$

Proof. The first inequality in (8.44) is implied by our hypotheses (8.6) and (8.7) that enter our definitions of τ (8.9) and $\mu^{(1)}$ (8.11).

To obtain the second inequality, we study the functions $\beta \mapsto f_N(\beta)$, $N > 1$, given by

$$f_N(\beta) = -\frac{2}{N-1}\ln\widehat{\mu}^{(1)\otimes N}\Big(\big|\Delta^{(N)}\big|^{-\beta/N}\Big) \tag{8.45}$$

for $\beta \in (\beta^*, 4)$. Jensen's inequality [34] with respect to $\mu^{(1)\otimes N}$ applied in (8.45) gives us

$$f_N(\beta) \leq \beta\widehat{\mu}^{(1)\otimes 2}\big(\ln|x-y|\big). \tag{8.46}$$

On the other hand, $N(N-1)f_N(\beta) = 2\mathcal{F}_\beta^{(N)}(\mu^{(N)})$. Therefore, by Lemma 8.6, (8.35), definition (8.34), and the negativity of $\mathcal{S}^{(N)}$ (see (8.13)), we have

$$f_N(\beta) = \beta\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big) - \frac{2}{N-1}\mathcal{S}^{(N)}\big(\mu^{(N)}\big) \geq \beta\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big). \tag{8.47}$$

The second estimate in (8.44) is proved.

To prove the third estimate in (8.44), we note that for any $\beta \in (\beta^*, 4)$, there exists a small $\epsilon > 0$ such that $(1+\epsilon)\beta \in (\beta^*, 4)$. By Jensen's inequality with respect to $\mu^{(N)}$,

$$M^{(N)}\big((1+\epsilon)\beta\big) \geq M^{(N)}(\beta)\exp\bigg(-\frac{1}{2}(N-1)\epsilon\beta\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big)\bigg). \tag{8.48}$$

Dividing (8.48) by ${M^{(1)}}^N$, taking the logarithm, and then multiplying by $-2/(N-1)$

gives

$$f_N\big((1+\epsilon)\beta\big) \le f_N(\beta) + \epsilon\beta\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big). \tag{8.49}$$

Now, $f_N(\beta)$ is bounded above and below independently of N, $N > 1$, for

$$2F(\beta) \ge \big(1-N^{-1}\big) f_N(\beta) \ge 2 f_0(\beta), \tag{8.50}$$

$\beta \in (\beta^*, 4)$, and since $1-N^{-1} \to 1$. The first inequality in (8.50) is Proposition 8.8; the second is (8.38). With the help of (8.50), from (8.49) we now obtain, for $N > 1$,

$$\begin{aligned} \beta\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big) &\ge \frac{1}{\epsilon}\big(f_N\big((1+\epsilon)\beta\big) - f_N(\beta)\big) \\ &\ge \frac{2}{\big(1-N^{-1}\big)\epsilon}\big(f_0\big((1+\epsilon)\beta\big) - F(\beta)\big) \\ &\ge \underline{C}(\beta) \end{aligned} \tag{8.51}$$

uniformly in N, for all $\beta \in (\beta^*, 4)$. □

We next prove a hybrid bound, which for $N = 1$ reduces to the first inequality in (8.44).

Lemma 8.11. *For each $\beta \in (\beta^*, 4)$, $N \ge 1$, there is an N-independent $\widetilde{C}(\beta)$ such that*

$$\beta\widehat{\mu}^{(1)} \otimes \widehat{\mu}_1^{(N)}\big(\ln|x-y|\big) \le \widetilde{C}(\beta). \tag{8.52}$$

Proof. For $\beta = 0$, the statement is obvious.

For $\beta \in (\beta^*, 0)$, we have

$$\begin{aligned} \beta\widehat{\mu}^{(1)} \otimes \widehat{\mu}_1^{(N)}\big(\ln|x-y|\big) &\le \widehat{\mu}_1^{(N)}\bigg(\int_{B_1(x)} \beta\ln|x-y|\mu^{(1)}(\mathrm{d}y)\bigg) \\ &\le \widehat{\mu}_1^{(N)}\big(\widetilde{C}(\beta)\big) \\ &= \widetilde{C}(\beta). \end{aligned} \tag{8.53}$$

The first estimate in (8.53) is obvious, since $\beta \in (\beta^*, 0)$. The second estimate follows from the fact that $\Psi_{\log} : x \mapsto \int_{B_1(x)} \ln|x-y|\mu^{(1)}(\mathrm{d}y) \in C^0(\mathbb{R}^2)$ (because $\Upsilon \in L^\infty$), with $\Psi_{\log}(x) \to 0$ as $|x| \to \infty$ (by $|\ln|x-y|| < |x-y|^{-\gamma}$ on $B_1(x)$, $\gamma \in (0,2)$, followed by (8.6)).

For $\beta \in (0, 4)$, we use (8.22) to estimate

$$\widehat{\mu}^{(1)} \otimes \widehat{\mu}_1^{(N)}\big(\ln|x-y|\big) \le \widehat{\mu}^{(1)}\big(\ln(2+|x|)\big) + \widehat{\mu}_1^{(N)}\big(\ln(2+|y|)\big). \tag{8.54}$$

By (8.7),

$$\widehat{\mu}^{(1)}\big(\ln(2+|x|)\big) = C_1 < \infty. \tag{8.55}$$

As to estimating $\widehat{\mu}_1^{(N)}(\ln(2+|y|))$, if $\beta \in (0,2)$, we can pick $p \in (1, 2/\beta)$ and apply Hölder's inequality with respect to $\tau(\mathrm{d}x_1)$ followed by obvious L^∞ estimates to get the upper bound $\widehat{\mu}_1^{(N)}(\ln(2+|y|)) \leq C(\beta)M^{(N-1)}(\beta')/M^{(N)}(\beta)$, where $\beta' = (1-N^{-1})\beta$ and where

$$C(\beta) = \left(\int_{\mathbb{R}^2} \big(\ln(2+|y|)\big)^{p^*} \tau(\mathrm{d}y)\right)^{1/p^*} \sup_N \sup_{x\in\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |x-y|^{-p\beta'} \tau(\mathrm{d}y)\right)^{1/p} < \infty. \tag{8.56}$$

We subsequently estimate the ratio of M's uniformly in N in the manner done below, but when $\beta \in [2,4)$, Hölder's inequality does not lead to L^∞ functions and so this road is then blocked. However, noting that for $q \in (0, q^*)$ we have, by (8.7),

$$\int_{\mathbb{R}^2} \exp\big(q\ln(2+|y|)\big)\tau(\mathrm{d}y) = C_2 < \infty, \tag{8.57}$$

we can use convex duality (see [56]) for "exp" to get, for any $q \in (0, q^*)$ and all $\beta \in (0,4)$,

$$\begin{aligned} &\widehat{\mu}_1^{(N)}\big(\ln(2+|y|)\big) - \frac{M^{(N-1)}(\beta')}{M^{(N)}(\beta)} \int_{\mathbb{R}^2} \exp\big(q\ln(2+|y|)\big)\tau(\mathrm{d}y) \\ &\quad \leq -\frac{1}{q}\big(1+\ln q + \beta'\widehat{\mu}_2^{(N)}\big(\ln|x-y|\big)\big) \\ &\quad \leq C^*(\beta). \end{aligned} \tag{8.58}$$

In (8.58), $C^*(\beta)$ is independent of N, by Lemma 8.10. Hence, it now remains to estimate $M^{(N-1)}(\beta')/M^{(N)}(\beta)$ from above uniformly in N, for each $\beta \in (0,4)$. To carry out this last step, we regularize $M^{(N)}$ and prove an N-independent upper bound on the "regularized ratio of M's" which is independent of the regularization parameter.

We regularize $\ln|x-y|$ by $-V_\epsilon(x,y) \equiv \pi^{-2}\epsilon^{-4}\int_{B_\epsilon(x)}\int_{B_\epsilon(y)} \ln|\xi-\eta|\,\mathrm{d}\xi\,\mathrm{d}\eta$. Let $\mathcal{H}_\epsilon$ denote the Hilbert space obtained by completing the $C_0^\infty(\mathbb{R}^2)$ functions with vanishing integral, $\int_{\mathbb{R}^2} f(x)\,\mathrm{d}x = 0$, with respect to the positive definite inner product

$$\langle f, f\rangle_\epsilon \equiv N^{-1}\beta \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)V_\epsilon(x,y)f(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{8.59}$$

If $B^1 \equiv B_{1/\sqrt{\pi}}(0)$ denotes the disk of area 1 centered at the origin, and $\delta_y(x)$ is the Dirac measure on $\mathbb{R}^2$ concentrated at y, we note that

$$\delta_y^\sharp(x) \equiv \delta_y(x) - \chi_{B^1}(x) \in \mathcal{H}_\epsilon. \tag{8.60}$$

Accordingly,

$$\delta_{(N)}^\sharp(x) \equiv \sum_{k=1}^N \delta_{x_k}^\sharp(x) \in \mathcal{H}_\epsilon \tag{8.61}$$

as well. We now define

$$W_\epsilon(x) \equiv \int_{B^1} V_\epsilon(x,y)\,\mathrm{d}y - \frac{1}{2}\int_{B^1}\int_{B^1} V_\epsilon(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{8.62}$$

and write

$$\tau(\mathrm{d}x) = e^{\beta W_\epsilon(x)}\widetilde{\tau}(\mathrm{d}x). \tag{8.63}$$

Note that, unless $q^* > \beta$, $\widetilde{\tau}$ does not have finite mass, but that does *not* cause a problem. We write $M_\epsilon^{(N)}(\beta)$ for $M^{(N)}(\beta)$ with $-\ln|x-y|$ replaced by $V_\epsilon(x,y)$. With (8.59) to (8.63), we have the identity

$$M_\epsilon^{(N)}(\beta) = e^{-1/2}\beta V_\epsilon(0,0)\int_{\mathbb{R}^{2N}} e^{(1/2)\langle \delta_{(N)}^\sharp, \delta_{(N)}^\sharp\rangle_\epsilon}\prod_{\ell=1}^{N}\widetilde{\tau}(\mathrm{d}x_\ell). \tag{8.64}$$

We now use Gaussian functional integrals (see [32]) to rewrite (8.64). Minlos's theorem (see [32]) asserts that $N^{-1}\beta V_\epsilon(x,y)$ is the covariance "matrix" of a Gaussian probability measure with mean zero; that is, there exists a Gaussian average Ave(.) on a space of linear functionals Φ on $\mathscr{H}_\epsilon$, with $\mathrm{Ave}(\phi(x)) = 0$ and $\mathrm{Ave}(\phi(x)\phi(y)) = N^{-1}\beta V_\epsilon(x,y)$, where $\phi(x)$ is shorthand for $\Phi(\delta_x^\sharp)$. Using the generating function (see [32])

$$\mathrm{Ave}\left(e^{\Phi(f)}\right) = e^{(1/2)\langle f,f\rangle_\epsilon}, \tag{8.65}$$

with $f = \delta_{(N)}^\sharp$ given in (8.61), then integrating over $\mathbb{R}^{2N}$ with respect to $\widetilde{\tau}^{\otimes N}$, we obtain

$$M_\epsilon^{(N)}(\beta) = e^{-(1/2)\beta V_\epsilon(0,0)}\,\mathrm{Ave}\left(\left(\int_{\mathbb{R}^2} e^{\phi(x)}\tilde{\tau}(\mathrm{d}x)\right)^N\right). \tag{8.66}$$

Jensen's inequality in the form $\langle F^N\rangle \geq \langle F^{N-1}\rangle^{N/(N-1)}$ applied to the right-hand side of (8.66) now gives, in terms of the M_ϵ's,

$$M_\epsilon^{(N)}(\beta) \geq M_\epsilon^{(N-1)}(\beta')\left(M_\epsilon^{(N-1)}(\beta')\right)^{1/(N-1)} \tag{8.67}$$

for all ϵ. Hence, we let $\epsilon \to 0$, and then $N \to \infty$ to obtain

$$\begin{aligned}\limsup_{N\to\infty}\frac{M^{(N-1)}(\beta')}{M^{(N)}(\beta)} &\leq \limsup_{N\to\infty}\left(M^{(N-1)}(\beta')\right)^{-1/(N-1)}\\ &\leq \frac{1}{M^{(1)}}e^{\inf_\varrho \mathscr{F}_\beta(\varrho)}\\ &\leq \frac{1}{M^{(1)}}e^{\mathscr{F}_\beta(\mu^{(1)})}\\ &= \frac{1}{M^{(1)}}\exp\left(\frac{1}{2}\beta\widehat{\mu}^{(1)\otimes 2}\left(\ln|x-y|\right)\right).\end{aligned} \tag{8.68}$$

By Lemma 8.10, the right-hand side of (8.68) exists and is obviously N-independent. Combining (8.57), (8.58), and (8.68), we have

$$\widehat{\mu}_1^{(N)}\big(\ln(2+|x|)\big) \le \widetilde{C}_2(\beta) \tag{8.69}$$

independently of N. By (8.54), (8.55), and (8.69), and setting $\widetilde{C}(\beta) = \widetilde{C}_1(\beta)+\widetilde{C}_2(\beta)$, Lemma 8.11 is proved also for $\beta \in (0,4)$. □

We now prepare for uniform L^p bounds.

Lemma 8.12. *For each $n \in \mathbb{N}$, $\beta \in (\beta^*, 4)$, there exist $N_n(\beta) \in \mathbb{N}$ and $C(n,\beta) > 0$, such that for $N > N_n$, the Radon-Nikodym derivative of $\mu_n^{(N)}$ with respect to $\tau^{\otimes n}$ is bounded by*

$$\frac{\mathrm{d}\mu_n^{(N)}}{\mathrm{d}\tau^{\otimes n}}\big(x_1,\dots,x_n\big) \le C(n,\beta)\big|\Delta^{(n)}\big|^{-\beta/N}\big(x_1,\dots,x_n\big). \tag{8.70}$$

Proof. When $\beta = 0$, this is trivial.

When $\beta \neq 0$, we begin by writing

$$\frac{\mathrm{d}\mu_n^{(N)}}{\mathrm{d}\tau^{\otimes n}}\big(x_1,\dots,x_n\big) = \frac{1}{M^{(N)}(\beta)} G\big(x_1,\dots,x_n\big)\big|\Delta^{(n)}\big|^{-\beta/N}\big(x_1,\dots,x_n\big), \tag{8.71}$$

where

$$G\big(x_1,\dots,x_n\big) = \int_{\mathbb{R}^{2(N-n)}} \prod_{1\le i\le n<j\le N} \big|x_i - x_j\big|^{-\beta/N} \prod_{n<k<\ell\le N} \big|x_k - x_\ell\big|^{-\beta/N} \tau(\mathrm{d}x_j). \tag{8.72}$$

Let [[.]] denote integer part. We define

$$N_n(\beta) = \begin{cases} n\left[\!\left[\dfrac{2\beta^*-\beta}{\beta^*-\beta}\right]\!\right] & \text{if } \beta \in (\beta^*, 0), \\[2ex] n\left[\!\left[\dfrac{8-\beta}{4-\beta}\right]\!\right] & \text{if } \beta \in (0,4). \end{cases} \tag{8.73}$$

Given $\beta \in (\beta^*, 4)$, let $N > N_n(\beta)$. Then, by Hölder's inequality,

$$\begin{aligned} G\big(x_1,\dots,x_n\big) \le{}& \left(\int_{\mathbb{R}^{2(N-n)}} \prod_{1\le i\le n<j\le N} \big|x_i - x_j\big|^{-\beta/2n} \tau(\mathrm{d}x_j)\right)^{2n/N} \\ &\times \left(\int_{\mathbb{R}^{2(N-n)}} \prod_{n<i<j\le N} \big|x_i - x_j\big|^{-\beta/(N-2n)} \prod_{n<k\le N} \tau(\mathrm{d}x_k)\right)^{1-2n/N}. \end{aligned} \tag{8.74}$$

As for the first factor on the right-hand side of (8.74), permutation symmetry gives

$$\int_{\mathbb{R}^{2(N-n)}} \prod_{1\le i\le n<j\le N} \left|x_i - x_j\right|^{-\beta/2n} \tau\left(\mathrm{d}x_j\right) = \left(\int_{\mathbb{R}^2} \prod_{1\le i\le n} \left|x_i - x\right|^{-\beta/2n} \tau(\mathrm{d}x)\right)^{N-n}. \tag{8.75}$$

By the arithmetic-geometric mean inequality and permutation invariance, we have

$$\int_{\mathbb{R}^2} \prod_{1\le i\le n} \left|x_i - x\right|^{-\beta/2n} \tau(\mathrm{d}x) \le \frac{1}{n} \sum_{1\le i\le n} \int_{\mathbb{R}^2} \left|x_i - x\right|^{-\beta/2} \tau(\mathrm{d}x). \tag{8.76}$$

For the right-hand side of (8.76), we have the estimates

$$\frac{1}{n} \sum_{1\le i\le n} \int_{\mathbb{R}^2} \left|x_i - x\right|^{-\beta/2} \tau(\mathrm{d}x) \le \begin{cases} \sup_y \int_{\mathbb{R}^2} |y-x|^{-\beta/2} \tau(\mathrm{d}x) & \text{if } \beta \ge 0, \\ C_n \int_{\mathbb{R}^2} (2+|x|)^{-\beta/2} \tau(\mathrm{d}x) & \text{if } \beta < 0, \end{cases} \tag{8.77}$$

where $C_n = \max_{i\in\{1,\dots,n\}} (2+|x_i|)^{|\beta|/2}$. By (8.75), (8.76), and (8.77), the first term on the right-hand side of (8.74) is bounded by the $2n(1-n/N)$th power of the right-hand side of (8.77), hence uniformly with respect to N.

As for the second factor on the right-hand side of (8.74), we split off the $(-2n/N)$th power. We set $\alpha(N) = (N-n)/(N-2n)$. Since $N > N_n$, we have $1 < \alpha(N) < 4/\beta$ if $\beta > 0$, and $1 < \alpha(N) < \beta^*/\beta$ if $\beta < 0$. We also have $\alpha(N) \to 1$ as $N \to \infty$. Proceeding as in the proof of Lemma 8.7, we find that

$$\begin{aligned} &\limsup_{N\to\infty} \left(\int_{\mathbb{R}^{2N-2n}} \prod_{n<i<j\le N} \left|x_i - x_j\right|^{-\beta/(N-2n)} \prod_{n<k\le N} \tau(\mathrm{d}x_k)\right)^{-2n/N} \\ &\quad = \limsup_{N\to\infty} \left(M^{(N-n)}\left(\alpha(N)\beta\right)\right)^{-2n/N} \\ &\quad \le \left(\frac{e^{F(\beta)}}{M^{(1)}}\right)^{2n}, \end{aligned} \tag{8.78}$$

which implies an N-independent bound on $(M^{(N-n)}(\alpha(N)\beta))^{-2n/N}$. Feeding (8.75), (8.76), (8.77), and (8.78) back into (8.74), we see that $G(x_1,\dots,x_n) \le CM^{(N-n)}(\alpha(N)\beta)$. This already proves that the density (8.71) will eventually (i.e., for N large) be in any $L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$, $p < \infty$. To prove that $\mathrm{d}\mu_n^{(N)}/\mathrm{d}\tau^{\otimes n} \in L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$ uniformly in N, it remains to estimate the ratio $M^{(N-n)}(\alpha(N)\beta)/M^{(N)}(\beta)$ from above, independently of N.

We once again can apply the Gaussian functional integral method used in the proof of Lemma 8.11. Since $\alpha(N)\beta$ occurs in the argument of $M^{(N-n)}$ instead of

$(1-nN^{-1})\beta$, besides Jensen's inequality (now pulling a power $N/(N-n)$ out of the average), we also need a "change of covariance formula" (see [32]). However, having proved Lemmas 8.10 and 8.11 already, a more direct way follows.

Using Jensen's inequality twice in a self-explanatory way, we obtain

$$\begin{aligned}\frac{M^{(N-n)}\big(\alpha(N)\beta\big)}{M^{(N)}(\beta)} \le \frac{1}{M^{(1)^n}} \exp\left(\frac{n(n-1)}{N}\frac{1}{2}\beta\widehat{\mu}^{(1)\otimes 2}\big(\ln|x-y|\big)\right)\\ \times \exp\left(n\left(1-\frac{n}{N}\right)\beta\widehat{\mu}^{(1)}\otimes\widehat{\mu}_1^{(N-n),\alpha}\big(\ln|x-y|\big)\right)\\ \times \exp\left(-n\left(1-\frac{n+1}{N}\right)\alpha(N)\beta\widehat{\mu}_2^{(N-n),\alpha}\big(\ln|x-y|\big)\right),\end{aligned} \tag{8.79}$$

where $\widehat{\mu}^{(N-n),\alpha}$ stands for (8.21) with $\alpha(N)\beta$ in place of β. The first exponential factor on the right-hand side of (8.79) is bounded above uniformly in N because $\beta\widehat{\mu}^{(1)\otimes 2}(\ln|x-y|) \le \overline{C}(\beta)$ independently of N, by the first inequality in Lemma 8.10; as for the second exponential factor on the right-hand side of (8.79), by re-identifying $N \to N-n$ and $\beta \to \alpha\beta$, the N-independent upper bound in Lemma 8.11 gives $\beta\widehat{\mu}^{(1)}\otimes\widehat{\mu}_1^{(N-n),\alpha}(\ln|x-y|) \le \widetilde{C}(\alpha(N)\beta)$. Since $\alpha(N)\to 1$ as $N\to\infty$, the second exponential factor on the right-hand side of (8.79) is bounded above uniformly in N. As for the third exponential factor on the right-hand side of (8.79), since $\beta^* < \alpha(N)\beta < 4$, and since (8.44) holds for all $\beta \in (\beta^*, 4)$, by Lemma 8.10 we have $\beta\widehat{\mu}_2^{(N-n),\alpha}(\ln|x-y|) \ge \underline{C}(\alpha(N)\beta)$. Again, since $\alpha(N)\to 1$, we now see that also the third exponential factor on the right-hand side of (8.79) is bounded above uniformly in N. This proves Lemma 8.12. □

Lemma 8.12 establishes that for each triple $n \in \mathbb{N}$, $\beta \in (\beta^*, 4)$, $p \in [1,\infty)$ there exists a $\widetilde{N}_n(\beta, p)$ ($> N_n(\beta)$) such that $\mathrm{d}\mu_n^{(N)}/\mathrm{d}\tau^{\otimes n} \in L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$ uniformly in N when $N > \widetilde{N}_n(\beta, p)$. Hence, the sequence $N \mapsto \mu_n^{(N)}$ is $L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$-weakly compact when $N > \widetilde{N}_n(\beta, p)$, for each $p \in [1,\infty)$.

However, a weak L^p limit point of $\mu_n^{(N)}$ need not be a probability measure. Since $\mathbb{R}^2$ is unbounded, some partial mass of the marginals $\mu_n^{(N)}$ of (8.21) could escape to infinity when $N\to\infty$. We now show that this does not happen by proving tightness of the sequences. Recall (see [5]) that the sequence of probability measures $\mu_n^{(N)}$ is *tight* if, for each $\epsilon \ll 1$, there exists $R(\epsilon)$ such that $\mu_n^{(N)}(\mathrm{B}^n_{R(\epsilon)}) > 1-\epsilon$, independent of N, where $\mathrm{B}^n_R \subset \mathbb{R}^{2n}$ is the n-fold Cartesian product of the ball $\mathrm{B}_R \subset \mathbb{R}^2$ that is centered at the origin, having radius R.

Lemma 8.13. *For each n, the sequence $\{\mu_n^{(N)}\}_{N\ge n}$ given by (8.21) is tight.*

Proof. Since our marginal measures are permutation symmetric and consistent, in the sense that $\mu_n^{(N)}(\mathrm{d}x^n) = \mu_m^{(N)}(\mathrm{d}x^n \otimes \mathbb{R}^{2(m-n)})$ for $m > n$, it suffices to prove tightness for $n = 1$.

It follows from the definition of $\mu^{(1)}$ that the map $y \mapsto h(y) \equiv \int_{\mathbb{R}^2} \ln|y-x|\mu^{(1)}(\mathrm{d}x) + C$ is continuous and independent of N. The constant C is chosen so that $h(y) > 0$. Moreover, we have $h(y) \to \infty$ as $|y| \to \infty$, uniformly in y. Therefore, and by Lemma 8.11, for each positive $\epsilon \ll 1$, we can find $R(\epsilon)$, independent of N, such that for all N,

$$\inf_{x_1 \notin \mathrm{B}_{R(\epsilon)}} h(x_1) \geq \frac{1}{\epsilon}\widehat{\mu}_1^{(N)}(h(x_1)). \tag{8.80}$$

Let $\mathbf{I}_\Lambda$ denote the indicator function of the set Λ. We then have the chain of estimates

$$\begin{aligned}\widehat{\mu}_1^{(N)}(h(x_1)) &\geq \widehat{\mu}_1^{(N)}\big(h(x_1)\mathbf{I}_{\mathbb{R}^2\setminus \mathrm{B}_R(\epsilon)}\big)\\ &\geq \frac{1}{\epsilon}\widehat{\mu}_1^{(N)}(h(x_1))\widehat{\mu}_1^{(N)}\big(\mathbf{I}_{\mathbb{R}^2\setminus \mathrm{B}_R(\epsilon)}\big)\\ &= \frac{1}{\epsilon}\widehat{\mu}_1^{(N)}(h(x_1))\Big(1-\mu_1^{(N)}\big(\mathrm{B}_{R(\epsilon)}\big)\Big).\end{aligned} \tag{8.81}$$

Dividing (8.81) by $\epsilon^{-1}\widehat{\mu}_1^{(N)}(h(x_1))$ and resorting terms slightly gives us

$$\mu_1^{(N)}\big(\mathrm{B}_{R(\epsilon)}\big) \geq 1-\epsilon, \tag{8.82}$$

independent of N. The proof is complete. □

To prove Proposition 8.9, we also need a lower bound on the mean entropy.

Lemma 8.14. *For each $\beta \in (\beta^*, 4)$, there exists a $C(\beta)$, independent of N, such that*

$$\frac{1}{N}\mathcal{S}^{(N)}\big(\mu^{(N)}\big) \geq C(\beta). \tag{8.83}$$

Proof. By definition (8.32) of $\mathcal{F}_\beta^{(N)}(\varrho^{(N)})$,

$$\frac{1}{N}\mathcal{S}^{(N)}\big(\mu^{(N)}\big) = \beta\frac{1}{N^2}\widehat{\mu}^{(N)}\big(\ln\big|\Delta^{(N)}\big|\big) - \frac{1}{N^2}\mathcal{F}_\beta^{(N)}\big(\mu^{(N)}\big). \tag{8.84}$$

The bound (8.83) now follows from Proposition 8.8, (8.43), and Lemma 8.10. □

Proof of Proposition 8.9. By Lemma 8.13, the sequence of probability measures $\{\mu_n^{(N)} \mid N = n, n+1, \dots\}$ is tight in $P(\mathbb{R}^{2n})$ for all n. Therefore (see [5]) we can select a subsequence $k \mapsto N_\ell(k) \in \mathbb{N}$, $k \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, $\mu_n^{(N_\ell(k))} \rightharpoonup \mu_n^\ell \in P(\mathbb{R}^{2n})$, as $k \to \infty$. Since the marginals are consistent (in the sense defined above in the proof of tightness), by Kolmogorov's existence theorem (see [5, p. 228 ff.] and [28, p. 301 ff.]), the infinite family of marginals $\{\mu_n^\ell\}_{n\in\mathbb{N}}$ now defines a unique $\mu^\ell \in P^s(\Omega)$. Furthermore, for $\beta^* < \beta < 4$, we have as a corollary of Lemma 8.12 that, for any n and any $p \in [1,\infty)$, the sequence $\{\mu_n^{(N)} \mid N = n, n+1, \dots\}$ is eventually in a ball $\{g : \|g\|_{L^p(\mathbb{R}^{2n})} \leq T\}$, where $T(n,\beta,p)$ is independent of N. Therefore, as

$k \to \infty$, after at most selecting a subsubsequence (also denoted by $k \mapsto N_\ell(k) \in \mathbb{N}$, $k \in \mathbb{N}$), we have that $\mathrm{d}\mu_n^{(N_\ell(k))}/\mathrm{d}\tau^{\otimes n} \rightharpoonup \mathrm{d}\mu_n^\ell/\mathrm{d}\tau^{\otimes n}$, weakly in $L^p(\mathbb{R}^{2n}, \mathrm{d}\tau^{\otimes n})$, any $p \in [1, \infty)$.

We first study convergence of energy. By (8.43), we have

$$\frac{1}{N_\ell(k)^2}\widehat{\mu}^{(N_\ell(k))}\big(\ln\big|\Delta^{(N_\ell(k))}\big|\big) = \left(1 - \frac{1}{N_\ell(k)}\right)\frac{1}{2}\widehat{\mu}_2^{(N_\ell(k))}\big(\ln|x-y|\big). \tag{8.85}$$

Since $\ln|x-y| \in L^q(\mathbb{R}^4, \mathrm{d}\tau^{\otimes 2})$, $1/q + 1/p = 1$, by weak $L^p(\mathbb{R}^4, \mathrm{d}\tau^{\otimes 2})$ convergence of $\mu_2^{(N_\ell(k))}$,

$$\frac{1}{2}\widehat{\mu}_2^{(N_\ell(k))}\big(\ln|x-y|\big) \longrightarrow \frac{1}{2}\widehat{\mu}_2^\ell\big(\ln|x-y|\big) = e\big(\mu^\ell\big). \tag{8.86}$$

Furthermore, since $1 - N_\ell(k)^{-1} \to 1$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \frac{1}{N_\ell(k)^2}\widehat{\mu}^{(N_\ell(k))}\big(\ln\big|\Delta^{(N_\ell(k))}\big|\big) = e\big(\mu^\ell\big). \tag{8.87}$$

We now turn to the entropy. We define $m = N_\ell(k) - [\![N_\ell(k)/n]\!]n$. By subadditivity (8.15) and negativity (8.13) we have, for any $n < N_\ell(k)$,

$$\begin{aligned}\frac{1}{N_\ell(k)}\mathcal{S}^{(N_\ell(k))}\big(\mu^{(N_\ell(k))}\big) &\le \frac{1}{N_\ell(k)}\left[\!\!\left[\frac{N_\ell(k)}{n}\right]\!\!\right]\mathcal{S}^{(n)}\big(\mu_n^{(N_\ell(k))}\big) + \frac{1}{N_\ell(k)}\mathcal{S}^{(m)}\big(\mu_m^{(N_\ell(k))}\big)\\ &\le \frac{1}{N_\ell(k)}\left[\!\!\left[\frac{N_\ell(k)}{n}\right]\!\!\right]\mathcal{S}^{(n)}\big(\mu_n^{(N_\ell(k))}\big).\end{aligned} \tag{8.88}$$

Clearly, $N_\ell(k)^{-1}[\![N_\ell(k)/n]\!] \to n^{-1}$. Moreover, for each n, weak upper semicontinuity of $\mathcal{S}^{(n)}$ (see [57]) gives us

$$\limsup_{k\to\infty} \mathcal{S}^{(n)}\big(\mu_n^{(N_\ell(k))}\big) \le \mathcal{S}^{(n)}\big(\mu_n^\ell\big).$$

Therefore, for all n,

$$\limsup_{k\to\infty} \frac{1}{N_k}\mathcal{S}^{(N_\ell(k))}\big(\mu^{(N_\ell(k))}\big) \le \frac{1}{n}\mathcal{S}^{(n)}\big(\mu_n^\ell\big). \tag{8.89}$$

Recalling (8.16) and Lemma 8.14, we see that $s(\mu)$ exists. Hence, $n \to \infty$ in (8.89) gives

$$\limsup_{k\to\infty} \frac{1}{N_\ell(k)}\mathcal{S}^{(N_\ell(k))}\big(\mu^{(N_\ell(k))}\big) \le s\big(\mu^\ell\big) \tag{8.90}$$

for each convergent subsequence $\mu^{(N_\ell(k))} \rightharpoonup \mu^\ell$.

Pulling the estimates (8.87) and (8.90) together, we find, for any $\beta \in (\beta^*, 4)$,

$$\liminf_{k\to\infty} \frac{1}{N_\ell^2(k)} \mathscr{F}_\beta^{(N_\ell(k))}\big(\mu^{(N_\ell(k))}\big) \geq \beta e\big(\mu^\ell\big) - s\big(\mu^\ell\big). \tag{8.91}$$

Recalling Propositions 8.1 and 8.2, and finally using Lemma 8.7, we have

$$\beta e\big(\mu^\ell\big) - s\big(\mu^\ell\big) = \int_{P(\mathbb{R}^2)} \nu\big(\mathrm{d}\varrho \mid \mu^\ell\big)\mathscr{F}_\beta(\varrho) \geq F(\beta). \tag{8.92}$$

By (8.92) and (8.91), the proof of Proposition 8.9 is complete. □

We remark that, when $\beta < 0$, Proposition 8.9 can be proved without L^p estimates. Indeed, when $\beta < 0$, then (8.91) follows already, with (8.87) replaced by

$$\limsup_{k\to\infty} \frac{1}{2}\widehat{\mu}_2^{(N_\ell(k))}\big(\ln|x-y|\big) \leq \frac{1}{2}\widehat{\mu}_2^\ell\big(\ln|x-y|\big) = e\big(\mu^\ell\big), \tag{8.93}$$

which holds by the weak upper semicontinuity of $\ln|x-y|$ and the weak convergence of $\mu_2^{(N)}$ in the sense of measures (see [37] and [41]). Also, the entropy estimate in the proof of Proposition 8.9 holds by just such weak convergence of $\mu_2^{(N)}$. However, without L^p estimates, one loses the a priori information on the regularity of the solutions of (8.28).

We now prove our main existence theorem.

Proof of Theorem 8.4. Combining Propositions 8.8 and 8.9, we conclude that

$$\lim_{N\to\infty} \frac{1}{N^2}\mathscr{F}_\beta^{(N)}\big(\mu^{(N)}\big) = F(\beta). \tag{8.94}$$

Recalling (8.91) and (8.92), we see that (8.94) implies

$$\int_{P(\mathbb{R}^2)} \nu\big(\mathrm{d}\varrho \;\big|\; \mu^\ell\big)\mathscr{F}_\beta(\varrho) = F(\beta) \tag{8.95}$$

for every limit point μ^ℓ of $\mu^{(N)}$. Equation (8.95) in turn implies that the decomposition measure $\nu(\mathrm{d}\varrho \mid \mu^\ell)$ is concentrated on the minimizers of $\mathscr{F}_\beta(\varrho)$; for otherwise we would have

$$\int_{P(\mathbb{R}^2)} \nu\big(\mathrm{d}\varrho \;\big|\; \mu^\ell\big)\mathscr{F}_\beta(\varrho) > F(\beta),$$

by Lemma 8.7, which contradicts (8.95). The proof of Theorem 8.4 is complete. □

We now are also in the position to vindicate our Remark 8.5. By the tightness and weak L^p compactness, the sequence $\{\mu^{(N)}, N = 1, 2, \dots\}$ is a union of weakly convergent subsequences in L^p. If the minimizer ϱ_β is unique, the set of limit points of $\{\mu^{(N)}, N \in \mathbb{N}\}$ consists of the single product measure $\mu = \varrho_\beta^{\otimes\mathbb{N}}$. □

9. Proof of uniqueness Theorem 2.2 for $K \leq 0$. We conclude this paper with a proof of Theorem 2.2. We do this by proving the dual version, that is, uniqueness of solutions of (8.28) when $\beta < 0$.

Theorem 9.1. *For $\beta < 0$, the solution $\rho_{\beta,H}$ of the fixed point equation (8.28) is unique.*

Proof. We introduce operator notation for (8.28), thus

$$\rho = \mathcal{P}(\rho), \tag{9.1}$$

where $\mathcal{P}$ indicates that the right side is a probability density over $\mathbb{R}^2$. Now, assume that for given $\beta < 0$ and H entire harmonic, two solutions of (9.1) exist, say, $\rho^{(1)}$ and $\rho^{(2)}$. Then $\rho_{2,1} \equiv \rho^{(2)} - \rho^{(1)} \in H_0^{-1}(\mathbb{R}^2)$. In particular, $\int_{\mathbb{R}^2} \rho_{2,1}\,\mathrm{d}x = 0$, and

$$-\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_{2,1}(x) \ln|x-y| \rho_{2,1}(y)\,\mathrm{d}x\,\mathrm{d}y \geq 0, \tag{9.2}$$

with equality holding if and only if $\rho_{2,1} \equiv 0$ (cf. [58]).

For $\lambda \in [0,1]$, we define the interpolation density $\rho_\lambda = \rho^{(1)} + \lambda\rho_{2,1}$. Expected value with respect to $\mathcal{P}(\rho_\lambda)$ is denoted by

$$\langle f \rangle(\lambda) = \int_{\mathbb{R}^2} f(x)\mathcal{P}(\rho_\lambda)(x)\,\mathrm{d}x. \tag{9.3}$$

We use (8.28) for one of the $\rho_{2,1}$ in the left-hand side of (9.2) and, with the abbreviation

$$U_{2,1}(x) = \int_{\mathbb{R}^2} \ln|x-y|\rho_{2,1}(y)\,\mathrm{d}y, \tag{9.4}$$

find that

$$\begin{aligned} -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} &\rho_{2,1}(x)\ln|x-y|\rho_{2,1}(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} U_{2,1}(x)\big(\mathcal{P}\big(\rho^{(2)}\big) - \mathcal{P}\big(\rho^{(1)}\big)\big)(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} U_{2,1}(x)\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{P}(\rho_\lambda)(y)\mathrm{d}\lambda\,\mathrm{d}x\,\mathrm{d}y \\ &= \beta\int_0^1 \big\langle \big(U_{2,1} - \langle U_{2,1}\rangle(\lambda)\big)^2\big\rangle(\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{9.5}$$

Since $\beta < 0$, the last term in (9.5) is less than or equal to zero, and it vanishes if and only if $U_{2,1} \equiv \mathrm{constant}$. By (9.5) and (9.2), we conclude that $\rho_{2,1} \equiv 0$. Uniqueness is proved. □

Corollary 9.2. *If $\beta < 0$, $H \equiv$ const, and Υ is radially symmetric, then the unique solution $\rho_{\beta,H}$ of (8.28) is radially symmetric as well. Gauss's theorem then implies that the corresponding solution of (1.2), $U_{H,\kappa}$ given in (8.30), is radially decreasing.* □

The proof of Corollary 9.2 is trivial. Theorem 9.1 and Corollary 9.2 prove Theorem 2.2.

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