On the positive mass theorem for CR manifolds

Several Complex Variables and Complex Geometry

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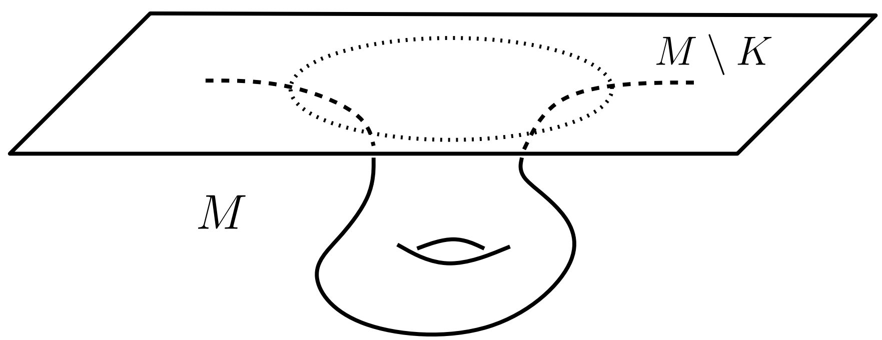
Asymptotically flat (Riemannian) manifolds

A manifold (M^3, g) is said to be *asymptotically flat* if $\exists\ K \subseteq\subseteq M$ s.t. $M \setminus K$ is diffeo. to $\mathbb{R}^3 \setminus B_1(0)$ and s.t., under this diffeo.

$$g_{ij} = \left(1 + \frac{M}{r}\right)^4 \delta_{ij} + h_{ij}, \qquad \partial^l h_{ij} = O(|x|^{-2-l}), \quad l = 0, 1, 2.$$



In general relativity these manifolds represent time-slices of static spacetimes where gravity is present.

Einstein equation

The structure of spacetime according to general relativity is governed by Einstein's equation

$$E_{ab} := R_{ab} - \frac{1}{2}R_g g_{ab} = T_{ab}.$$

Here R_{ab} is the Ricci tensor, R_g the scalar curvature, and T_{ab} the stress-energy tensor, generated by matter.

In vacuum ($T_{ab} \equiv 0$), this equation has variational structure, with Euler-Lagrange functional given by

$$\mathcal{A}(g) := \int_M R_g dV_g \qquad \text{Einstein-Hilbert functional.}$$

In fact, one has

$$\frac{d}{dg}(R_g dV_g)[h] = -\left(h^{ij}E_{ij} + \nabla^* \zeta\right) dV_g;$$

$$\zeta = -\left(\nabla^* h + \nabla(\mathrm{tr}_g h)\right) = \left({h_{jk}}^{,k} - {h^k}_{k,j}\right) dx^j.$$

The mass of an asymptotically flat manifold

If we consider variations which preserve asymptotic flatness, then the divergence term has a role (flux at infinity), and

$$\frac{d}{dg}(\mathcal{A}(g) + m(g))[h] = \int_M h^{ij} E_{ij} dV_g.$$

The quantity $m(g)$, called *ADM mass*, is defined as

$$m(g) := \lim_{r\to\infty} \oint_{S_r} \left({g_{jk}}^{,k} - {g^k}_{k,j} \right) \nu^j d\sigma.$$

Example: Schwartzschild metric. It describes a static black hole of total mass $m = m_{ADM}$. The expression is

$$\left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\xi^2\right).$$

At $r = \frac{m}{2}$ there is a minimal surface, representing a *event horizon*.

The positive mass theorem

Theorem ([Schoen-Yau, '79]) If $R_g \geq 0$ then $m(g) \geq 0$. In case $m(g) = 0$, then (M, g) is isometric to $(\mathbb{R}^3, dx^2)$.

Physically, this means that a positive local energy density implies a positive global energy for the system.

A simplified model In Newtonian gravity, the gravitational potential is described by the Poisson equation. If

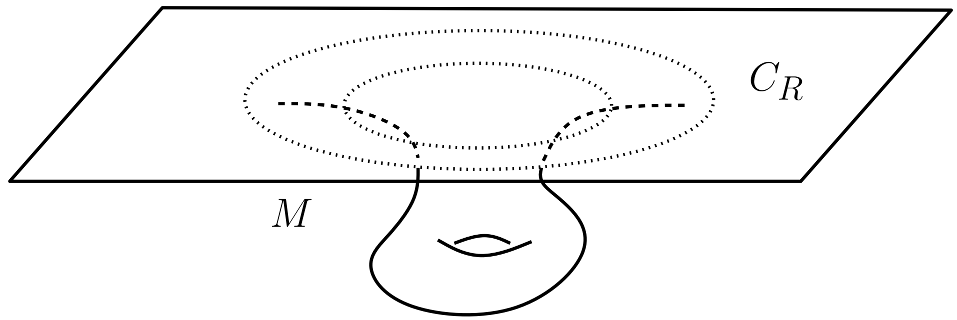
$$\Delta f = \rho \in C_c(\mathbb{R}^3) \qquad \Longrightarrow \qquad f(y) \simeq_\infty \frac{A}{|y|}; \quad A = \int_{\mathbb{R}^3} \rho.$$

The issue is that R_g and $m(g)$ are nonlinear in the metric.

• The above theorem has a fundamental role in the study of the Yamabe problem (more details later).

Idea of the proof

The main argument relies on constructing an asymptotically planar minimal surface in M. This is done by solving Plateau problems on larger and larger circles C_R of (asymptotic) radius R.



- By contradiction, if $m(g) < 0$ there is a uniform control on the height and it is possible to pass to the limit as $R \to +\infty$ and find a <u>stable</u> minimal surface.

- On the other hand $R_g \geq 0$ implies instability by the second variation formula for the area.

• The argument works for every dimension $n \leq 7$: needs regularity for minimal surfaces.

Witten's approach

On *spin manifolds* E.Witten used Dirac's equation to find an alternative proof. He solved for

$$(*) \quad \bar{D}\psi := \sum_i e_i \cdot \nabla_{e_i}\psi = 0; \qquad \psi \to \psi_0 \text{ as } |x| \to \infty.$$

The square of the $\bar{D}$ operator satisfies *Lichnerowitz's formula*

$$\bar{D}^2 = \nabla^*\nabla + \frac{1}{4}R_g.$$

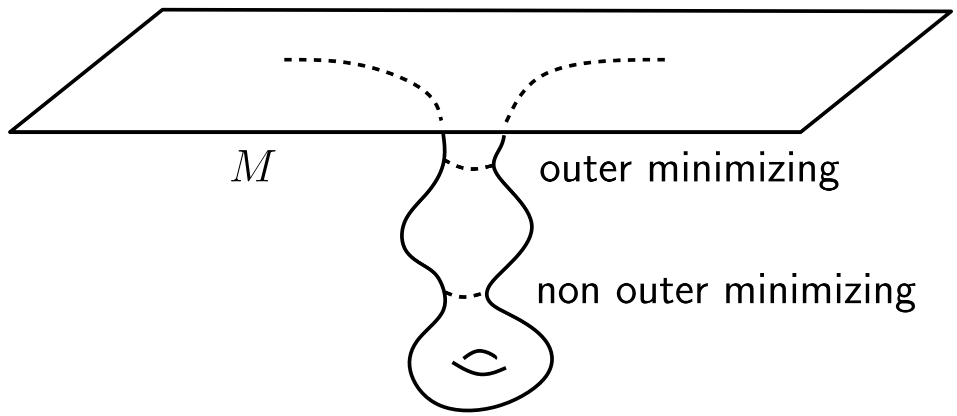
Integrating $(*)$ by parts, the mass appears in the boundary terms

$$m(g) = c_n \int_M \left(|\nabla\psi|^2 + \frac{1}{4}R_g|\psi|^2 \right) dV_g.$$

The last formula allows also to characterize $\mathbb{R}^n$ as the unique asymptotically flat space with ($R_g \geq 0$ and) zero mass

Some extensions

There are also quantitative lower bounds on the mass. One is *Penrose's inequality*, concerning *outer minimizing surfaces*



If A is the total area of the outer minimizing surfaces, Penrose's inequality asserts that $m \geq \sqrt{\frac{A}{16\pi}}$. The inequality was proved in [Huisken-Ilmanen, '97], [Bray, '01] using geometric flows.

Energia positiva

Conformal blow-up of a compact manifold

Motivated by the study of Yamabe's problem, one obtains asymptotically flat manifolds by conformal blow-up of compact ones.

Consider the *conformal Laplacian*

$$\varphi \quad \mapsto \quad L_g\varphi := -8\Delta\varphi + R_g\varphi.$$

This operator rules the conformal transformation law of scalar curvature, and behaves nicely under conformal changes of metric.

If the first eigenvalue of L_g is positive, its Green's function $G(x,y)$ is also positive: choosing any $p \in M$, consider the metric

$$\tilde{g}(x) = G(x,p)^4 g(x).$$

Since $G(x,p) \simeq d(x,p)^{-1}$ near p, one can show that $(M \setminus \{p\}, \tilde{g})$ is an asymptotically flat manifold.

CR manifolds: notation

Consider a three dimensional CR manifold endowed with a contact structure ξ and a CR structure $J : \xi \to \xi$ such that $J^2 = -1$.

We assume that there exists a global choice of contact form θ which annihilates ξ and for which $\theta \wedge d\theta$ is always nonzero. The *Reeb vector field* is the unique vector field T for which

$$\theta(T) \equiv 1; \qquad T \lrcorner\, d\theta = 0.$$

Given J as above, we have locally a vector field Z_1 such that

(1) $\quad JZ_1 = iZ_1; \qquad JZ_{\overline{1}} = -iZ_{\overline{1}} \qquad$ where $\qquad Z_{\overline{1}} = \overline{(Z_1)}.$

We also define $(\theta, \theta^1, \theta^{\overline{1}})$ as the dual triple to $(T, Z_1, Z_{\overline{1}})$, so that

$$d\theta = ih_{1\overline{1}}\theta^1 \wedge \theta^{\overline{1}} \qquad \text{for some } h_{1\overline{1}} > 0 \quad (\text{w.l.o.g. } h_{1\overline{1}} \equiv 1).$$

The connection 1-form ω_1^1 and the torsion $A_{\overline{1}}^1$ are uniquely determined by the equations

$$\begin{cases} d\theta^1 = \theta^1 \wedge \omega_1^1 + A_{\overline{1}}^1 \theta \wedge \theta^{\overline{1}}; \\ \omega_1^1 + \omega_{\overline{1}}^{\overline{1}} = 0. \end{cases}$$

The *Webster curvature* is then defined by the formula

$$d\omega_1^1 = W\,\theta^1 \wedge \theta^{\overline{1}} \pmod{\theta}.$$

Example: the *Heisenberg group* $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$

$$\begin{cases} \overset{\circ}{\theta} = dt + iz d\overline{z} - i\overline{z} dz; \\ \overset{\circ}{\theta}{}^1 = \sqrt{2} dz; \\ \overset{\circ}{\theta}{}^{\overline{1}} = \sqrt{2} d\overline{z}, \end{cases} \qquad \begin{cases} \overset{\circ}{T} = \frac{\partial}{\partial t}; \\ \overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial z} + i\overline{z} \frac{\partial}{\partial t} \right); \\ \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial \overline{z}} - iz \frac{\partial}{\partial t} \right). \end{cases}$$

ξ_0 on $\mathbb{H}^1$ is spanned by real and imaginary parts of $\overset{\circ}{Z}_1$. The standard CR structure verifies $J_0 : \xi_0 \to \xi_0$ by $J_0 \overset{\circ}{Z}_1 = i \overset{\circ}{Z}_1$.

Blow-up of a CR manifold

We will consider manifolds of *positive Webster class*, namely for which there exists a conformal $\hat{\theta} = u^2\theta$ with $W_{\hat{\theta}} > 0$.

This class can be characterized via the *conformal sublaplacian*

$$u \quad \mapsto \quad \mathcal{L}u := -4\Delta_b u + W u = -4(u_{1,\bar{1}} + u_{\bar{1}1}) + W\, u.$$

Positivity of the class is equivalent to positivity of $\lambda_1(\mathcal{L})$.

If $M(\xi, J)$ has positive Webster class, then as before the Green's function $\mathcal{G}(x, y)$ of $\mathcal{L}$ is positive, and for $p \in M$ we can consider the form $\hat{\theta} = \mathcal{G}(p, \cdot)^2\theta$. Correspondingly, $\hat{\theta}^1 = u(\theta^1 + 2i(\log u)^1)$.

This means that we are solving for

$$-4\Delta_b \mathcal{G}(p, \cdot) + W\mathcal{G}(p, \cdot) = \delta_p,$$

namely we get zero curvature outside p.

In CR normal coordinates $(z(x), t(x))$ at p ([Jerison-Lee, '111]) one has the following asymptotics for $\mathcal{G}$

$$\mathcal{G}(p,x) = \frac{1}{32\pi}\rho(x)^{-2} + A + o_x(1); \qquad \rho^4(x) = |z|^4 + t^2,$$

for some $A \in \mathbb{R}$, where $o_x(1) \to 0$ as $(z,t) \to 0$.

CR inversion

If (z,t) are CR normal coordinates in a neighborhood $\mathcal{U}$ of p, we define *inverted CR normal coordinates* (z_*,t_*) as

$$(2) \qquad z_* = \frac{z}{v}; \qquad t_* = -\frac{t}{|v|^2}; \qquad \text{on } \mathcal{U} \setminus \{p\},$$

where $v = t + i|z|^2$. Notice that $\rho_*(z_*,t_*) = \rho(z,t)^{-1}$.

In these coordinates the new forms become

$$\hat{\theta} = \left(1 + 4\pi A\rho_*^{-2} + O(\rho_*^{-3})\right)(\theta_0)_* + O(\rho_*^{-3})dz_* + O(\rho_*^{-3})d\bar{z}_*;$$

$$\begin{aligned}\hat{\theta}^1 &= \left(-2\sqrt{2}\pi A\frac{z_* v_*}{\rho_*^6} + O(\rho_*^{-5})\right)(\theta_0)_* + O(\rho_*^{-4})d\bar{z}_* \\ &+ \left(1 + 2\pi A\rho_*^{-2} + O(\rho_*^{-3})\right)\sqrt{2}dz_*.\end{aligned}$$

converging to the standard $\overset{\circ}{\theta}$ and $\overset{\circ}{\theta}{}^1$ as $\rho_* \to +\infty$.

Asymptotically flat pseudohermitian manifolds

Motivated by the above computations we introduce the

Definition A three dimensional pseudohermitian manifold (N, J, θ) is said to be asymptotically flat pseudohermitian if $N = N_0 \cup N_\infty$, with N_0 compact and N_∞ diffeomorphic to $\mathbb{H}^1 \setminus B_{\rho_0}$ in which (J, θ) is close to (J_0, θ_0) in the sense that

$$\theta = \left(1 + 4\pi A\rho^{-2} + O(\rho^{-3})\right)\theta_0 + O(\rho^{-3})dz + O(\rho^{-3})d\overline{z};$$

$$\theta^1 = O(\rho^{-3})\theta_0 + O(\rho^{-4})d\overline{z} + \left(1 + 2\pi A\rho^{-2} + O(\rho^{-3})\right)\sqrt{2}dz,$$

for some $A \in \mathbb{R}$ and a unitary coframe θ^1 in some system of coordinates (asymptotic coordinates). We also require $W \in L^1(N)$.

A notion of CR-mass

Given a one-parameter family of CR structures $J(s)$, we have

$$\dot{J} = 2E = 2E_{11}\theta^1 \otimes Z_{\bar{1}} + 2E_{\bar{1}\bar{1}}\theta^{\bar{1}} \otimes Z_1.$$

If $W(s)$ is the corresponding Webster curvature, then

$$\begin{aligned}
&\frac{d}{ds}_{|s=0} \int_N R(s)\, \theta \wedge d\theta = \int_N \dot{R}\, \theta \wedge d\theta \\
= \ & -\int_N d\left(E_{11,1}\, \theta \wedge \theta^1\right) + \text{conj.} - \int_N \left(A_{11}E_{\overline{11}} + \text{conj.}\right) \theta \wedge d\theta \\
= \ & \oint_\infty i\dot{\omega}_1^1 \wedge \theta - \int_N \left(A_{11}E_{\overline{11}} + \text{conj.}\right) \theta \wedge d\theta.
\end{aligned}$$

This formula leads us to the following

Definition Let N be an asymptotically flat manifold. We define

$$m(J,\theta) := i\oint_\infty \omega_1^1 \wedge \theta := \lim_{\Lambda \to +\infty} i \oint_{S_\Lambda} \omega_1^1 \wedge \theta,$$

where $S_\Lambda = \{\rho = \Lambda\}$.

The Paneitz operator

The CR Paneitz operator P is defined by

$$P\varphi := 4(\varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1)^1 + \text{ conj..}$$

Let $\tilde{P}_3\varphi := \varphi_{\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^1$. The CR pluriharmonic functions are characterized by $\tilde{P}_3\varphi = 0$ ([**?**]) compatibility (see [**?**]). Moreover, for the contact form change $\theta = e^{2f}\widehat{\theta}$ one has

$$P_{\widehat{\theta}}\varphi = e^{4f}P_\theta\varphi$$

• We call P *nonnegative* if $\langle\varphi, P\varphi\rangle_{L^2} \geq 0$ for all φ.

The Paneitz operator enters in the assumptions of the following embeddability theorem. Recall that $\Box_b u := -2u_{,\bar{1}1}$

Theorem ([Chanillo-Chiu-Yang, '10]) Let M be a compact 3D CR manifold. If $P \geq 0$ and $W > 0$, then every eigenvalue $\lambda \neq 0$ of $\Box_b$ is greater or equal to $\min_M W$. In particular range($\Box_b$) is closed. Moreover, M can be embedded into $\mathbb{C}^N$ for some $N \in \mathbb{N}$.

An integral formula for the mass

Proposition Let (N, J, θ) be an asymptotically flat pseudohermitian manifold. Let $\beta : N \to \mathbb{C}$ be such that

$$\beta = \bar{z} + \beta_{-1} + O(\rho^{-2+\varepsilon}) \quad \text{and} \quad \Box_b \beta = O(\rho^{-4}) \qquad \text{near } \infty,$$

where β_{-1} is a term with the homogeneity of ρ^{-1} satisfying

$$(\beta_{-1})_{,\bar{1}} = -2\sqrt{2}\pi A \frac{1}{\rho^2} - \frac{\sqrt{2}A}{|z|^2 + it},$$

and where $\varepsilon \in (0,1)$. Then one has

$$\begin{aligned} \frac{2}{3} m(J,\theta) &= -\int_N |\Box_b \beta|^2 \theta \wedge d\theta + 2 \int_N |\beta_{,\overline{11}}|^2 \theta \wedge d\theta \\ &+ 2 \int_N W |\beta_{,\bar{1}}|^2 \theta \wedge d\theta + \frac{1}{2} \int_N \bar{\beta} P \beta \, \theta \wedge d\theta. \end{aligned}$$

The proof uses integration by parts. The asymptotics on β_{-1} arises from trying to annihilate $\Box_b \beta$

On the asymptotics of β

Proposition Let

Solvability of $\Box_b \beta = 0$

Proposition Let

Positive mass theorem

Massa = Green

Proposition Let

The zero mass case

Suppose that $m(J,\theta) = 0$. From the integral formula we get

$$\beta_{,\overline{11}} \equiv 0; \qquad \beta_{,\overline{1}1} \equiv 0; \qquad P\beta \equiv 0.$$

The first two relations imply $|\beta_{,\overline{1}}|^2 \equiv \text{const.} = \frac{1}{2}$, from the behavior of β at infinity. We also have then $W \equiv 0$.

$P\beta = 0$ also implies $A_{11,\overline{1}} \equiv 0$. Let us show that indeed $A_{11} \equiv 0$.

Consider the flow φ_s generated by the Reeb v.f. T of N, and let

$$J_{(s)} = \varphi_s^* J \qquad (\dot{J} = L_T J = 2A_{J,\theta}).$$

For s small, it is possible to solve for

$$-4\Delta_b u_s + R_{J_{(s)},\theta} u_s = 0 \qquad \text{on } N,$$

so we get $m(J, u_s^2\theta) \geq 0$. Differentiating the mass in s one finds

$$\frac{d}{ds}|_{s=0} m(J, u_s^2\theta) = \frac{3}{2}\int_N |A_{11}|^2 \theta \wedge d\theta \qquad \Longrightarrow \qquad A_{11} \equiv 0.$$

Congruence to the Heisenberg group

To show that N coincides with $\mathbb{H}^1$, first we define a map from a (simply connected) nb. of infinity $\mathcal{U}$ in N to a neighborhood of infinity $\mathcal{V}$ in $\mathbb{H}^1$. From the above equations one finds

$$d(\beta_{,\overline{1}}\theta^{\overline{1}}) = \beta_{,\overline{1}1}\theta^1 \wedge \theta^{\overline{1}} + \beta_{,\overline{1}0}\theta \wedge \theta^{\overline{1}} = 0,$$

which implies that $d\beta = \beta_{,\overline{1}}\theta^{\overline{1}}$. Taking $z = \overline{\beta}$ we have

$$d(\theta - izd\overline{z} + i\overline{z}dz) = i\theta^1 \wedge \theta^{\overline{1}} - 2idz \wedge d\overline{z} = 0.$$

Hence there exists a function $\tilde{t}$ such that

$$d\tilde{t} = \theta - izd\overline{z} + i\overline{z}dz.$$

So we get a pseudohermitian isomorphism between $\mathcal{U}$ and its image in $\mathbb{H}_1$, $\mathcal{V}$, if we send $q \in N$ into

$$q \mapsto (z(q), t(q)) = \left(\overline{\beta}(q), \int_{q_0}^{q} d\tilde{t}\right).$$

where we are taking curves connecting q_0 to q inside $\mathcal{U}$.

We call $\Psi : \mathcal{V} \to \mathcal{U}$ (sets which we can assume to be connected by arcs) the inverse of this map: next, we want to extend Ψ globally on $\mathbb{H}^1$. Taking $q_0 \in \mathcal{V}$ and $q \in \mathbb{H}^1$ arbitrary, we can find a curve

$$\Gamma : [0,1] \to \mathbb{H}^1, \qquad \Gamma(0) = q_0, \quad \Gamma(1) = q.$$

We show that this procedure defines a map $\tilde{\Psi} : \mathbb{H}^1 \to N$, showing that $\tilde{\Gamma}(1)$ is independent of the choice of Γ, by patching local pseudohermitian isometries.

The CR Yamabe problem

Yamabe's problem consists in finding conformal metrics with constant scalar curvature. Solved in [Aubin, '76] and [Schoen, '84] in complementary cases.

In the CR case one looks for constant Webster curvature under a conformal change of contact form. If $\hat{\theta} = u^2\theta$, then

$$-4\Delta_b u + Wu = \hat{W}u^3.$$

If one wants to solve for constant $\hat{W}$, it is possible to do it looking for solutions of the following extremization problem

$$\mathcal{Y}(M,J) := \inf_{\hat{\theta}} \frac{\int_M W_{J,\hat{\theta}}\,\hat{\theta}\wedge d\hat{\theta}}{\left(\int_M \hat{\theta}\wedge d\hat{\theta}\right)^{\frac{1}{2}}} = \inf_{u\neq 0} \frac{\int_M \left(2|\nabla_b u|^2 + \frac{1}{2}W_{J,\theta}\,u^2\right)\theta\wedge d\theta}{\left(\int_M u^4\theta\wedge d\theta\right)^{\frac{1}{2}}}.$$

The cases $\mathcal{Y}(M,J) < 0$ and $\mathcal{Y}(M,J) = 0$ are easy, while the positive case is difficult since the embedding $\mathcal{S}^{1,2} \hookrightarrow L^4$ is critical.

The positive case

In the *positive case*, one has always $\mathcal{Y}(M,J) \leq \mathcal{Y}(S^3, J_0)$. To see this, one can exploit the conformality of the map $\varpi : S^3 \setminus p \to \mathbb{H}^1$, $p = (0,1) \in \mathbb{C}^2$, defined as

$$\varpi(z_1, z_2) = \left(\frac{z_1}{1+z_2}, \text{Re } \left(i \frac{1-z_2}{1+z_2} \right) \right),$$

where (z_1, z_2) are standard coordinates in $\mathbb{C}^2$. Composing with a dilation, the conformal factor of the inverse map is given by

$$\omega_\lambda(z,t) = \frac{1}{\lambda} \left(t^2 + |z|^4 + \frac{2}{\lambda^2}|z|^2 + \frac{1}{\lambda^4} \right)^{-\frac{1}{2}}; \quad \lambda > 0, \quad (z,t) \in \mathbb{H}^1.$$

In [Jerison-Lee, '77] these functions were classified as extremals for the Sobolev-type ratio in $\mathbb{H}^1$, equal to $\mathcal{Y}(S^3, J_0)$.

Localizing these functions on any manifold with λ large, in the quotient one can get arbitrarily close to $\mathcal{Y}(S^3, J_0)$.

Compactness recovery

Compactness of minimizing sequences indeed holds provided one has the strict inequality $\mathcal{Y}(M, J) < \mathcal{Y}(S^3, J_0)$ (there would not be enough energy for blow-up).

This condition was proved in [Jerison-Lee, '84] in dimension greater or equal to 7 and non locally spherical manifolds using local expansions of the energy.

In low dimension the decay of extremals is slower, so a global argument is needed. Here the positive mass enters.

Following the argument in [Schoen, '84], one can glue a highly peaked ω_λ at $p \in M$ to a scaled Green's function.

More precisely, we set

$$u(z,t) \simeq \begin{cases} \omega_\lambda(z,t) & \text{in } \{\rho \le \rho_0\}; \varepsilon_0 \tilde{G}_p(z,t) \\ \text{in } M \setminus \{\rho \le \rho_0\}, \end{cases}$$

where

$$\varepsilon_0 = \frac{1}{\lambda(1 + 2\pi A \rho_0^2)}.$$

Then one finds

$$\frac{\int_M \left(2|\nabla_b u|^2 + \frac{1}{2} W_{J,\theta}\, u^2\right) \theta \wedge d\theta}{\left(\int_M u^4 \theta \wedge d\theta\right)^{\frac{1}{2}}} \le \mathcal{Y}(S^3, J_0) - \frac{c_0 A}{\lambda^2 \rho_0^2},$$

which implies strict inequality and compactness.

An example with positive W and negative mass

Let $J(s)$ be a perturbation of the standard structure on $\mathbb{H}^1$, fast decaying at infinity. As noticed before, one can solve for

$$-4\Delta_b u_s + W_{J(s),\theta} u_s = 0 \qquad \text{on } \mathbb{H}^1.$$

Using some asymptotic analysis one finds

$$u_s = 1 - \frac{1}{32\pi\rho^2} \int_N W_{J_{(s)},\theta} u_s \theta \wedge d\theta + O(\rho^{-3}) \qquad \text{at infinity,}$$

Recalling that $m(J,\theta) = 48\pi^2 A$ (2nd order term in $\mathcal{G}$), we get

$$m(J_{(s)}, u_s^2\theta) = -\frac{3}{4} \int_N W_{J(s),\theta} u_s \theta \wedge d\theta.$$

We choose a deformation $J(s)$ so that E_{11} is a CR function, i.e. $E_{11,\overline{1}} = 0$. We can take for example

$$E_{11}(z,\overline{z},t) = \left(t + i(|z|^2+1)\right)^{-k},$$

with k large (to have decay).

For this choice of E_{11} one has

$$\dot{W} = i\left(E_{11,\overline{11}} - E_{\overline{11},11}\right) - \left(A_{11}E_{\overline{11}} + A_{\overline{11}}E_{11}\right) = 0,$$

and moreover

$$\ddot{W} = -2|E_{11,1}|^2 - 2E_{11}E_{\overline{11},\bar{1}1} - 2E_{\overline{11}}E_{11,1\bar{1}}.$$

Since $W_0 \equiv 0$ and $u_0 \equiv 1$ we have that

$$\ddot{m}(J_{(s)}, u_s^2\theta)|_{s=0} = -\frac{3}{4}\int_N \ddot{W}\theta \wedge d\theta.$$

so integrating by parts we get negative mass since

$$\ddot{m}(J,\theta) = -\frac{3}{2}\int_{\mathbb{H}^1} |E_{11,1}|^2 \overset{\circ}{\theta} \wedge d\overset{\circ}{\theta} < 0.$$

We can transport the latter example on S^3 using

Open problems

On *spin manifolds*

Thanks for your attention