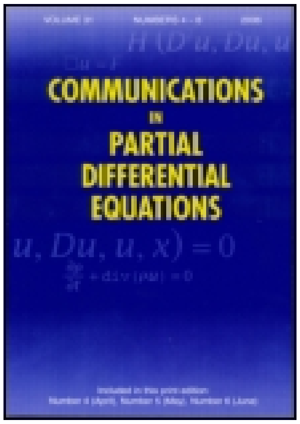

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An Improved Strichartz Estimate for Systems with Divergence Free Data

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An Improved Strichartz Estimate for Systems with Divergence Free Data

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Using the div-curl inequalities of Bourgain and Brezis [1] and van Schaftingen [9], we prove an improved Strichartz estimate for systems of inhomogeneous wave and Schrodinger equations, for which the inhomogeneity is a divergence-free vector field at each given time. The novelty of the result is that one can allow L^1_x norms of the inhomogeneity in the right hand side of the estimate.

Keywords Div-curl; Divergence-free; Strichartz estimates.

Mathematics Subject Classification 35L05; 35L10.

In this paper we are interested in improved Strichartz estimates for systems of inhomogeneous wave and Schrodinger equations, when the inhomogeneity is a divergence free vector field at any given time. The starting point is the following simple observation:

Proposition 1. *Suppose $u : \mathbb{R}^{1+2} \to \mathbb{R}^2$ is a (weak) solution of the following system of wave equations*

$$\begin{cases} \Box u = f \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

where $f = (f_1, f_2) : \mathbb{R}^{1+2} \to \mathbb{R}^2$ is a divergence free vector field at each given time t, i.e.

$$\partial_{x_1} f_1 + \partial_{x_2} f_2 = 0$$

for each t. Then

$$\|u\|_{C^0_t L^2_x} + \|\partial_t u\|_{C^0_t \dot{H}^{-1}_x} \leq C \left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|f\|_{L^1_t L^1_x} \right).$$

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Here $\Box = -\partial_t^2 + \Delta$ is the d'Alembertian acting componentwise on u, and $\dot{H}^s$ is the homogeneous Sobolev space $\dot{W}^{s,2}$.

A remarkable feature in our estimate is that on the right hand side we only need the L^1_x norm of f, which is usually not possible in the classical energy (or Strichartz) inequalities. Our estimate is only possible because we have the additional structural assumption that f is a divergence free vector field at each time t. In fact if one tries to prove the Proposition using Gagliardo-Nirenberg inequality naively without using this divergence free assumption, say when $u_0 = u_1 = 0$, then one would estimate, at any time t,

$$\begin{aligned}\|u\|_{L^2_x} &= \left\| \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} f(s,x)\,ds \right\|_{L^2_x} \\ &\leq \int_0^t \left\| \frac{1}{\sqrt{-\Delta}} f(s,x) \right\|_{L^2_x} ds \\ &\leq \int_0^t \|R_1 f(s,x)\|_{L^1_x} + \|R_2 f(s,x)\|_{L^1_x}\, ds\end{aligned}$$

where R_j are the Riesz transforms on $\mathbb{R}^2$, which are unfortunately not bounded on L^1.

Before we state our more general results, we first give a short proof of Proposition 1. The proof relies on the following simple observation that we first learned from Bourgain and Brezis (c.f. Remark 5 in [1]):

Lemma 1 (Bourgain-Brezis). *For each divergence free vector field $F = (F_1, F_2)$ on $\mathbb{R}^2$ with $F \in L^1$, there exists $G \in L^2$ such that $F_1 = \partial_{x_2} G$ and $F_2 = -\partial_{x_1} G$ with $\|G\|_{L^2} \leq C\|F\|_{L^1}$.*

Proof. The assumption that $\operatorname{div} F = 0$ allows one to find G such that $F_1 = \partial_{x_2} G$ and $F_2 = -\partial_{x_1} G$, and $G \in L^2$ by Gagliardo-Nirenberg inequality because $\nabla G = (-F_2, F_1) \in L^1$. □

In fact in [1] and the subsequent work [2, 9], Bourgain and Brezis and van Schaftingen obtained some far-reaching generalizations of this simple lemma, and the latter is what we shall exploit in our more general result in this paper.

Proof of Proposition 1. Let f be as in the Proposition. Applying the lemma to $f(t,\cdot)$ at each time t, we obtain a function $g(t,\cdot)$ such that $f_1 = \partial_{x_2} g$, $f_2 = -\partial_{x_1} g$, and $\|g\|_{L^2_x} \leq C\|f\|_{L^1_x}$ at each time t. Now the classical energy estimate says that

$$\|u\|_{C^0_t L^2_x} + \|\partial_t u\|_{C^0_t \dot{H}^{-1}_x} \leq C\left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \|(-\Delta)^{-\frac{1}{2}} f\|_{L^1_t L^2_x}\right).$$

Since for each fixed t,

$$\|(-\Delta)^{-\frac{1}{2}} f\|_{L^2_x} = \|(-\Delta)^{-\frac{1}{2}} \nabla g\|_{L^2_x} \leq C\|g\|_{L^2_x} \leq C\|f\|_{L^1_x}$$

where the last inequality follows by Gagliardo–Nirenberg inequality. Our result follows. □

The key observation in proving Proposition 1 is that the coefficients of f are in $\dot{H}^{-1}(\mathbb{R}^2)$ for all t under the given conditions. We remark that there are other situations under which the inhomogeneity of the wave equation lies in $\dot{H}^{-1}(\mathbb{R}^2)$; one instance is given in the appendix.

In what follows, we derive improved Strichartz inequalities similar to Proposition 1, using generalizations of Lemma 1 by van Schaftingen.

1. Strichartz Estimates for the Wave Equation

In the sequel we shall consider vector fields $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$. Our main result for the wave equation is the following:

Theorem 1. *Suppose $n \geq 2$, and let $u : \mathbb{R}^{1+n} \to \mathbb{R}^n$ be a (weak) solution of the system*

$$\begin{cases} \Box u = f \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

where $f(t, x) : \mathbb{R}^{1+n} \to \mathbb{R}^n$ is a divergence free vector field for all t. Suppose $s, k \in \mathbb{R}$, $2 \leq q, \tilde{q} \leq \infty$, $2 \leq r < \infty$, and we assume further that $\tilde{q} > \frac{4}{n-1}$ if $n = 2$ or 3. Suppose (q, r) satisfies the wave admissibility condition

$$\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4},$$

and the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + n - 2 - k.$$

Then

$$\|u\|_{L_t^q L_x^r} + \|u\|_{C_t^0 \dot{H}_x^s} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{s-1}} \leq C\left(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{\frac{k}{2}} f\|_{L_t^{\tilde{q}'} L_x^1}\right).$$

To prove this, the starting point is the following result of van Schaftingen [9]:

Theorem 2 (van Schaftingen). *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a divergence free vector field with components in L^1. Then for any $0 < \alpha < n$,*

$$\|F\|_{\dot{W}^{-\alpha, \frac{n}{n-\alpha}}} \leq C\|F\|_{L^1}.$$

The theorem was stated in [9] only for $0 < \alpha \leq 1$, but the rest of the theorem follows easily from Sobolev embedding of $\dot{W}^{-\alpha, \frac{n}{n-\alpha}}$ into $\dot{W}^{-\beta, \frac{n}{n-\beta}}$ in $\mathbb{R}^n$ if $0 < \alpha \leq \beta < n$.

We also need the following version of the Strichartz estimate for the scalar equation. It is stated in Proposition 3.1 of Ginibre and Velo [6] for the non-endpoint case (where both $(q, r), (\tilde{q}, \tilde{r}) \neq (2, \frac{2(n-1)}{n-3})$), and the endpoint case can be proved using the technology of Keel and Tao [7].

Lemma 2. *Suppose $n \geq 2$, and let $u : \mathbb{R}^{1+n} \to \mathbb{R}$ be a (weak) solution of*

$$\begin{cases} \Box u = h \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

Suppose $s, \gamma \in \mathbb{R}$, $2 \leq q, \tilde{q} \leq \infty$, $2 \leq r, \tilde{r} < \infty$, (q, r) and $(\tilde{q}, \tilde{r})$ satisfy the wave admissibility conditions

$$\frac{1}{q} + \frac{n-1}{2r} \leq \frac{n-1}{4}, \quad \frac{1}{\tilde{q}} + \frac{n-1}{2\tilde{r}} \leq \frac{n-1}{4},$$

and the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \gamma.$$

Then

$$\|u\|_{L^q_t L^r_x} + \|u\|_{C^0_t \dot{H}^s_x} + \|\partial_t u\|_{C^0_t \dot{H}^{s-1}_x} \leq C\left(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{\frac{\gamma}{2}} h\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}\right).$$

For the convenience of the reader, we pause to outline a proof of the end-point case of Lemma 2:

Proof of the End-Point Case of Lemma 2. The desired estimate of $\|u\|_{C^0_t \dot{H}^s_x} + \|\partial_t u\|_{C^0_t \dot{H}^{s-1}_x}$ follows from the statement of Corollary 1.3 of [7]. To prove

$$\|u\|_{L^q_t L^r_x} \leq C\|(-\Delta)^{\frac{\gamma}{2}} h\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

one observes that since $2 \leq q, \tilde{q} \leq \infty$, $2 \leq r, \tilde{r} < \infty$, one can restrict attention to the situation where the frequency support of $h(t, \cdot)$ is contained in an annulus of size 2^j by using the Littlewood-Paley square function. By scale invariance we can take $j = 0$. In that case $(-\Delta)^{\frac{\gamma}{2}}$ on the right hand side can be dropped, and the result follows from Theorem 1.2 of [7]. □

Theorem 1 can be seen as the limiting case of Lemma 2 when $\tilde{r} = \infty$ except when $(n, \tilde{q}, \tilde{r}) = (2, 4, \infty)$ or $(3, 2, \infty)$. It says one still has the Strichartz inequality if in addition f is a vector field at each time t, and $f(t, x)$ is divergence free for all t.

Proof of Theorem 1. Assume n, q, $\tilde{q}$, r, k and s be as given in the statement of the Theorem. Then when $n \geq 4$, from $2 \leq \tilde{q} \leq \infty$ one automatically has

$$\frac{n}{2} - \frac{2n}{(n-1)\tilde{q}} > 0,$$

and the same inequality holds when $n = 2$ or 3 because then we assumed $\tilde{q} > \frac{4}{n-1}$. As a result, one can pick some $\alpha \in (0, \frac{n}{2} - \frac{2n}{(n-1)\tilde{q}}]$. Now let $\tilde{r} = \frac{n}{\alpha}$, and $\gamma = k - \alpha$. Then $\tilde{r} < \infty$, $\frac{1}{\tilde{q}} + \frac{n-1}{2\tilde{r}} \leq \frac{n-1}{4}$, which in particular implies that $\tilde{r} \geq 2$. The scale invariance condition in Lemma 2 is also verified. Hence

$$\|u\|_{L^q_t L^r_x} + \|u\|_{C^0_t \dot{H}^s_x} + \|\partial_t u\|_{C^0_t \dot{H}^{s-1}_x} \leq C\left(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}\right).$$

Now invoking Theorem 2 and the divergence free condition on f for each time t, we get

$$\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_x^{\tilde{r}'}} \leq C\|(-\Delta)^{\frac{k}{2}} f\|_{L_x^1},$$

from which the desired inequality follows. Note this is possible because $\alpha \in (0, n)$ automatically by our choice of α. □

We remark that under the conditions of Theorem 1, we necessarily have $s \geq 0$, and when $n \geq 3$ we necessarily have $k > 0$. In fact $k \geq \frac{n-3}{2}$ when $n \geq 3$, and $k = 0$ is impossible when $n = 3$ because we assumed that $\tilde{q} > 2$ when $n = 3$.

We also remark that in Theorem 1, when the initial conditions u_0 and u_1 are zero, one can actually obtain a wider range of exponents for which the desired inequality holds. This can be thought of as a limiting case of an inhomogeneous Strichartz estimate of Taggart [8], whose origin goes back to the work of Foschi [5]. To illustrate this, we state the following Theorem in 3 space dimensions.

Theorem 3. *Suppose $n = 3$, and let $u : \mathbb{R}^{1+3} \to \mathbb{R}^3$ be a (weak) solution of the system*

$$\begin{cases} \Box u = f \\ u|_{t=0} = 0 \\ \partial_t u|_{t=0} = 0 \end{cases}$$

where $f(t, x) : \mathbb{R}^{1+3} \to \mathbb{R}^3$ is a divergence free vector field for all t. Suppose $k \in \mathbb{R}$, $1 < q, \tilde{q} \leq \infty$, $2 \leq r < \infty$, and

$$\frac{1}{q} + \frac{1}{\tilde{q}} < \min\left\{1, \frac{k+1}{2}\right\}.$$

Suppose further that (q, r) satisfies the wave acceptability condition

$$\frac{1}{q} + \frac{2}{r} < 1 \quad or \quad (q, r) = (\infty, 2),$$

and that the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{3}{r} = 2 - k - \frac{1}{\tilde{q}}.$$

Then

$$\|u\|_{L_t^q L_x^r} \leq C\|(-\Delta)^{\frac{k}{2}} f\|_{L_t^{\tilde{q}'} L_x^1}.$$

To prove this, we need the following scalar inhomogeneous Strichartz estimate, which is a consequence of Corollary 8.7 of Taggart [8] in 3 space dimensions:

Theorem 4 (Taggart). *Suppose $n = 3$, and let $u : \mathbb{R}^{1+3} \to \mathbb{R}$ be a (weak) solution of*

$$\begin{cases} \Box u = h \\ u|_{t=0} = 0 \\ \partial_t u|_{t=0} = 0. \end{cases}$$

Suppose $\gamma \in \mathbb{R}$, $1 < q, \tilde{q} \leq \infty$, $2 \leq r, \tilde{r} < \infty$,

$$\frac{1}{q} + \frac{1}{\tilde{q}} < 1, \quad \textit{and} \quad \frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{\gamma + 1}{2}.$$

Suppose further that the exponents satisfy the wave acceptability condition

$$\frac{1}{q} + \frac{2}{r} < 1 \quad \textit{or} \quad (q, r) = (\infty, 2),$$
$$\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}} < 1 \quad \textit{or} \quad (\tilde{q}, \tilde{r}) = (\infty, 2),$$

and that the following scale invariance condition is verified:

$$\frac{1}{q} + \frac{3}{r} = 2 - \gamma - \frac{1}{\tilde{q}} - \frac{3}{\tilde{r}}.$$

Then

$$\|u\|_{L^q_t L^r_x} \leq C \|(-\Delta)^{\frac{\gamma}{2}} h\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}.$$

Proof of Theorem 4. Under the conditions of Theorem 4, one has

$$\frac{1}{q} + \frac{2}{r} < 1 \quad \text{or} \quad (q, r) = (\infty, 2),$$
$$\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}} < 1 \quad \text{or} \quad (\tilde{q}, \tilde{r}) = (\infty, 2),$$

and

$$\frac{1}{r} + \frac{1}{\tilde{r}} \leq 1 - \frac{1}{q} - \frac{1}{\tilde{q}},$$

the last inequality following from the condition $\frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{\gamma+1}{2}$ and the scale invariance condition. Thus one can find $r_1 \leq r$, $\tilde{r}_1 \leq \tilde{r}$ such that the wave acceptability conditions

$$\frac{1}{q} + \frac{2}{r_1} < 1 \quad \text{or} \quad (q, r_1) = (\infty, 2)$$

and

$$\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}_1} < 1 \quad \text{or} \quad (\tilde{q}, \tilde{r}_1) = (\infty, 2),$$

are satisfied, with

$$\frac{1}{r_1} + \frac{1}{\tilde{r}_1} = 1 - \frac{1}{q} - \frac{1}{\tilde{q}}.$$

Clearly $r_1, \tilde{r}_1 \in [2, \infty)$. As a result, Corollary 8.7 of Taggart [8] applies, yielding Theorem 4. □

Proof of Theorem 3. Assume q, $\tilde{q}$, r and k be as given in the statement of the Theorem. Then since

$$\frac{1}{q} + \frac{1}{\tilde{q}} < \frac{k+1}{2} \quad \text{and} \quad \frac{1}{\tilde{q}} < 1,$$

one can pick a small $\alpha > 0$ such that

$$\frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{(k-\alpha)+1}{2} \quad \text{and} \quad \frac{1}{\tilde{q}} + \frac{2\alpha}{3} < 1.$$

Now let $\tilde{r} = \frac{3}{\alpha}$, and $\gamma = k - \alpha$. Then $\tilde{r} < \infty$, $\frac{1}{q} + \frac{1}{\tilde{q}} \leq \frac{\gamma+1}{2}$, $\frac{1}{\tilde{q}} + \frac{2}{\tilde{r}} < 1$, which in particular implies that $\tilde{r} > 2$. The scale invariance condition in Theorem 4 is also verified. Hence

$$\|u\|_{L_t^q L_x^r} \leq C\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Now invoking Theorem 2 and the divergence free condition on f for each time t, we get

$$\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_x^{\tilde{r}'}} \leq C\|(-\Delta)^{\frac{k}{2}} f\|_{L_x^1},$$

from which the desired inequality follows. Note this is possible because $\alpha \in (0, 3)$ automatically by our choice of α; in fact $\alpha < \frac{3}{2}$ since $\frac{1}{\tilde{q}} + \frac{2\alpha}{3} < 1$. □

2. Strichartz Estimates for the Schrodinger Equation

Again, we consider vector fields $f : \mathbb{R}^{1+n} \to \mathbb{R}^n$. The main result is the following.

Theorem 5. *Suppose $n \geq 2$, and $u : \mathbb{R}^{1+n} \to \mathbb{R}^n$ is a (weak) solution of the system of Schrodinger equations*

$$\begin{cases} i\partial_t u + \Delta u = f \\ u|_{t=0} = u_0, \end{cases}$$

where $f(t, x) : \mathbb{R}^{1+n} \to \mathbb{R}^n$ is a divergence free vector field for all t. Suppose $2 \leq q, \tilde{q} \leq \infty$, $2 \leq r < \infty$, $s \geq 0$, $k > s$, and the following scale invariance conditions are satisfied:

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s, \quad \frac{2}{\tilde{q}} = \frac{n}{2} - k + s.$$

Then

$$\|u\|_{C_t^0\dot{H}_x^s} + \|u\|_{L_t^q L_x^r} \leq C\left(\|u_0\|_{\dot{H}^s} + \|(-\Delta)^{\frac{k}{2}} f\|_{L_t^{\tilde{q}'} L_x^1}\right).$$

In its proof we need Theorem 2 in the previous Section, as well as the following Strichartz inequality for the scalar Schrodinger equation (which follows from Corollary 1.4 of Keel and Tao [7] and the Sobolev inequality):

Lemma 3. *Suppose* $n \geq 2$, *and* $u : \mathbb{R}^{1+n} \to \mathbb{R}$ *is a* (*weak*) *solution of*

$$\begin{cases} i\partial_t u + \Delta u = h \\ u|_{t=0} = u_0. \end{cases}$$

Suppose $2 \leq q, \tilde{q} \leq \infty$, $2 \leq r, \tilde{r} < \infty$, $s \geq 0$, $\gamma > s$, *and the following scale invariance conditions are satisfied*:

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} - s, \quad \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} - \gamma + s.$$

Then

$$\|u\|_{C_t^0\dot{H}_x^s} + \|u\|_{L_t^q L_x^r} \leq C\left(\|u_0\|_{\dot{H}^s} + \|(-\Delta)^{\frac{\gamma}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}\right).$$

Theorem 5 can be thought of as the limiting case of the above lemma when $\tilde{r} = \infty$, which only works because we assumed that the inhomogeneity $f(t, x)$ is a divergence free vector field at each time t.

Proof of Theorem 5. Assume n, q, $\tilde{q}$, r, k and s be as given in the statement of the Theorem. Then $k - s > 0$, one can pick some $\alpha \in (0, \min\{k - s, \frac{n}{2}\}]$. Now let $\tilde{r} = \frac{n}{\alpha}$, and $\gamma = k - \alpha$. Then $2 \leq \tilde{r} < \infty$, $\gamma > s$, and

$$\frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2} - \gamma + s.$$

Hence

$$\|u\|_{C_t^0\dot{H}_x^s} + \|u\|_{L_t^q L_x^r} \leq C\left(\|u_0\|_{\dot{H}^s} + \|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}\right).$$

Now we invoke Theorem 2 and the divergence free condition on f at each time t; this is possible because $\alpha \in (0, n)$ automatically by our choice of α. Thus we get

$$\|(-\Delta)^{\frac{k-\alpha}{2}} f\|_{L_x^{\tilde{r}'}} \leq C\|(-\Delta)^{\frac{k}{2}} f\|_{L_x^1},$$

from which the desired inequality follows. □

3. Appendix

In this appendix we prove another improved Strichartz inequality for the wave equation in $\mathbb{R}^{1+2}$. Here we only need to work with scalar equations.

Proposition 2. *Suppose* $u : \mathbb{R}^{1+2} \to \mathbb{R}$ *satisfies*

$$\begin{cases} \Box u = \det(\nabla_x F) \\ u|_{t=0} = u_0 \\ \partial_t u|_{t=0} = u_1 \end{cases}$$

where F *is a map from* $\mathbb{R}^{1+2}$ *to* $\mathbb{R}^2$, *and* $\det(\nabla_x F)$ *denotes its Jacobian determinant in the* x *variable. Then*

$$\|u\|_{C_t^0 L_x^2} + \|\partial_t u\|_{C_t^0 \dot{H}_x^{-1}} \leq C \left(\|u_0\|_{L^2} + \|u_1\|_{\dot{H}^{-1}} + \int \|\nabla_x F\|_{L_x^2}^2 dt \right).$$

It is clear that $\|\nabla_x F\|_{L_x^2}^2$ controls the L_x^1 norm of $\det(\nabla_x F)$, but unfortunately this is not enough if one wants to prove the Proposition. On the other hand, we claim

$$\|(-\Delta)^{-\frac{1}{2}} \det(\nabla_x F)\|_{L_x^2} \leq C \|\nabla_x F\|_{L_x^2}^2.$$

This follows from Wente's inequality; see e.g., Theorem 0.2 of Chanillo and Li [3]. Alternatively, since we are in 2 space dimensions, by compensation compactness (see Coifman et al. [4]), $\|\nabla_x F\|_{L_x^2}^2$ controls the Hardy $\mathscr{H}_x^1$ norm of $\det(\nabla_x F)$, which in turn controls the negative Sobolev $\dot{H}_x^{-1}$ norm of $\det(\nabla_x F)$, from which our claim follows. Arguing using the classical energy estimate as in the proof of Proposition 1, the desired estimate follows.

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