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Remarks on Commutators of Pseudo-differential Operators

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To F. Treves on his Sixty-fifth birthday

§1. Introduction

To state our theorem we recall a few definitions. We will denote by Q a cube in $\mathbb{R}^n$. Let b be a real-valued function in $L^1_{\text{loc}}(\mathbb{R}^n)$. We will say $b \in \text{BMO}$ if there exists a constant $C > 0$ such that

$$\sup_Q \frac{1}{|Q|} \int_Q |b - b_Q| \le C, \quad \text{where,} \quad b_Q = \frac{1}{|Q|} \int_Q b\, dx.$$

The norm in BMO of the function b is defined as,

$$\|b\|_\star = \sup_Q \frac{1}{|Q|} \int_Q |b - b_Q|\, dx. \tag{1.1}$$

We now consider a smooth function $p(x,\xi)$ where $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, satisfying the assumption,

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C(\alpha,\beta,n)(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \tag{1.2}$$

Let $f \in C_0^\infty(\mathbb{R}^n)$. We define the operator $Tf(x)$ to be,

$$Tf(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)} p(x,\xi) \hat{f}(\xi)\, d\xi, \tag{1.3}$$

where as usual $\hat{f}(\xi)$ is the Fourier transform of f. The operator T is said to be a pseudo-differential operator with symbol $p(x,\xi)$ in the class $S^m_{\rho,\delta}$. These classes were introduced by Hörmander [H]. See also Treves [T]. In this note we are interested in the L^2 boundedness properties of commutator operators of the type $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$ where b is a BMO function. If b were smooth then it is a simple matter to deduce refined boundedness properties of $[b,T]$ via the symbol calculus, see [T]. For "rough" functions b it is also not that difficult and our proof is based on one of the results of [CT]. We now state our result.

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THEOREM. *Let $b \in \mathrm{BMO}$. Let $\epsilon > 0$. Let T be an operator defined as in* (1.3) *with symbol in the class $S^{-\epsilon}_{\rho,\delta}$ with $0 \leq \delta < \rho < 1$. Then the commutator operator $[b,T]$ is a bounded operator on $L^2(\mathbb{R}^n)$ with operator norm bounded by $C\|b\|_\star$.*

We have nothing to add in the borderline case when $\delta = \rho = 1/2$, except to refer to the article by Coifman and Meyer [CM] which considers among others the situation when $\delta = \rho = 1/2$, $b \in L^\infty$ and $\epsilon = 1/2$, see Theorem 36, page 169. We remark that following the proof given below one may take $\epsilon = 0$ in the case $\delta = 0$ and $\rho = 1$. In the sequel, therefore, we will always reason under the hypothesis, $0 \leq \delta < \rho < 1$.

§2. Proof of the theorem

We now wish to recall a few facts that we will need in our proof of the theorem.

THEOREM (2.1). (C. FEFFERMAN [F]). *Let $\kappa > 0$ and $\sigma(x,\xi) \in S^{-\kappa}_{\rho,\delta}$, $0 \leq \delta < \rho < 1$. Then there exists $p_1 = p_1(\kappa,\rho)$, such that for $2 \leq p \leq p_1$, the operator,*

$$Tf(x) = \int_{\mathbb{R}^n} e^{i(x,\xi)}\sigma(x,\xi)\hat{f}(\xi)\,d\xi,$$

is a bounded operator on $L^p(\mathbb{R}^n)$.

We now need to recall the concept of a weight function $w(x)$ to be in the class A_p.

DEFINITION. *Let $w > 0$ and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We say $w \in A_p$ if and only if, for all cubes Q,*

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\Big)^{(p-1)} \leq C < \infty.$$

We will also use the notation,

$$\|f\|_{p,w} = \Big(\int_{\mathbb{R}^n} |f|^p w(x)\,dx\Big)^{1/p}.$$

When, $w \equiv 1$ we simply write $\|f\|_p$.

We next recall a result from [CT].

THEOREM (2.2). *Let $2 < p < \infty$. Let $w \in A_{\frac{p}{2}}$. Then if T is a pseudo-differential operator with symbol in $S^{-n(1-\rho)/2}_{\rho,\delta}$, we have,*

$$\|Tf\|_{p,w} \leq C\|f\|_{p,w}.$$

REMARK (2.3). *In both theorems (2.1) and (2.2) it is enough to hypothesize that* (1.2) *holds for $|\alpha| + |\beta| \leq 10n$.*

LEMMA (2.4). *Let $b \in \mathrm{BMO}$, with $\|b\|_\star = 1$. Then for $2 < p < \infty$, there exists $\tau_0 = \tau_0(p) > 0$ such that for all τ, $-\tau_0 \leq \tau \leq \tau_0$, $e^{\tau b} \in A_{\frac{p}{2}}$.*

PROOF. Fix a cube Q and consider,

$$\left(\frac{1}{|Q|}\int_Q e^{\tau b}\right)\left(\frac{1}{|Q|}\int_Q e^{-2\tau b/(p-2)}\right)^{(p-2)/2}. \tag{2.1}$$

Clearly (2.1) can be re-written as,

$$\left(\frac{1}{|Q|}\int_Q e^{\tau(b-b_Q)}\right)\left(\frac{1}{|Q|}\int_Q e^{-2\tau(b-b_Q)/(p-2)}\right)^{(p-2)/2}.$$

The above expression can be dominated by,

$$\left(\frac{1}{|Q|}\int_Q e^{|\tau||b-b_Q|}\right)\left(\frac{1}{|Q|}\int_Q e^{2|\tau||b-b_Q|/(p-2)}\right)^{(p-2)/2}. \tag{2.2}$$

Since $||b||_\star = 1$, by the basic result of John and Nirenberg [JN], regarding the exponential integrability of BMO functions, we see right away that (2.2) for $|\tau| \le \tau_0$ is bounded by a constant $C = C(n,\tau_0)$. This finishes the proof.

It is clear that we may without loss of generality assume, $||b||_\star = 1$, and prove our theorem under this normalization. In the rest of the proof we will reason under this normalization.

LEMMA (2.5). *Let $\epsilon > 0$. There exists p, $2 < p < 3$, and θ_0, $0 < \theta_0 < 1$, such that for any $w \in A_{\frac{p}{2}}$, and any pseudo-differential operator T with symbol in $S^{-\epsilon}_{\rho,\delta}$, one has for all θ, $0 \le \theta \le \theta_0$ and with a constant C independent of f the inequality,*

$$\int_{\mathbb{R}^n} |Tf|^p\, w^\theta\, dx \le C\int_{\mathbb{R}^n} |f|^p\, w^\theta\, dx.$$

PROOF. Fix $\kappa > 0$ and $0 < \kappa < \epsilon$. Let $p(x,\xi)$ be the symbol for T. Now consider the symbol $\sigma_z(x,\xi) = (1+|\xi|^2)^{z/2}p(x,\xi)$. We shall confine the complex variable z to lie in the strip $\mathcal{S}$, whose boundaries are the two vertical lines $-\kappa + \epsilon + i\tau$, $\tau \in \mathbb{R}$ and $-\frac{n}{2}(1-\rho) + \epsilon + i\tau$, $\tau \in \mathbb{R}$. It is easily verified that there exists a constant C_1 such that for $|\alpha| + |\beta| \le 100n$, we have,

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_z(x,\xi)| \le C(1+|\mathrm{Im}\, z|)^{C_1 n}(1+|\xi|)^\mu, \tag{2.3}$$

where $\mu = \mathrm{Re}\, z - \epsilon + \delta|\beta| - \rho|\alpha|$.

We shall denote by T_z the pseudo-differential operator with symbol σ_z. We shall also use the notation $\nu_0 = -\kappa+\epsilon$ and $\nu_1 = -\frac{n}{2}(1-\rho)+\epsilon$. For our choice of $\kappa > 0$ we can select a p such that $p < 3$ and $2 < p \le p_1$, for which the conclusion of Theorem (2.1) holds. We will establish Lemma (2.5) for this choice of p. For $f \in C_0^\infty(\mathbb{R}^n)$ we define the operator,

$$G_z f(x) = w^{(z-\nu_0)/p(\nu_1-\nu_0)}T_z\left(f w^{-(z-\nu_0)/p(\nu_1-\nu_0)}\right). \tag{2.4}$$

Note that $w \in A_{\frac{p}{2}}$, and since $p < 3$, $w \in A_2$ and hence $w^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Thus $w^{-1/p}$ and $w^{1/p}$ are both in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Keeping this in mind it is routine to establish that for $z \in \mathcal{S}$, G_z is an analytic family of operators in the sense of Stein, [SW].

In view of the bounds (2.3), Remark (2.3) and Theorem (2.1) we have for $\mathrm{Re}\, z_0 = \nu_0$,

$$\int_{\mathbb{R}^n} |G_{z_0}f|^p\, dx \le C(1+|\mathrm{Im}\, z_0|)^{C_1 n}\int_{\mathbb{R}^n}|f|^p\, dx. \tag{2.5}$$

On the other hand for Re $z_1 = \nu_1$, the bounds (2.3), Remark (2.3) and Theorem (2.2) gives,

$$\int_{\mathbb{R}^n} |G_{z_1} f|^p \, dx \le C(1 + |\mathrm{Im}\, z_1|)^{C_1 n} \int_{\mathbb{R}^n} |f|^p \, dx. \tag{2.6}$$

We then apply Stein's theorem on complex interpolation of operators [SW], to deduce that, by interpolating the estimates (2.5) and (2.6) we have for $z = 0$ and $\theta_0 = \nu_0/(\nu_0 - \nu_1)$, the estimate,

$$\int_{\mathbb{R}^n} |Tf|^p \, w^{\theta_0} \, dx \le C \int_{\mathbb{R}^n} |f|^p \, w^{\theta_0} \, dx. \tag{2.7}$$

Since $\kappa < \epsilon$, theorem (2.1) also gives,

$$\int_{\mathbb{R}^n} |Tf|^p \, dx \le C \int_{\mathbb{R}^n} |f|^p \, dx. \tag{2.8}$$

Using the main result of [SW] once again and interpolating between the estimates (2.7) and (2.8) we get our lemma.

REMARK (2.6). *We emphasize that the interpolation between* (2.7) *and* (2.8) *produces a constant C in Lemma(2.5), that in the parameter θ only depends on θ_0.*

PROOF OF THE THEOREM. Let $f, h \in C_0^\infty(\mathbb{R}^n)$ with $\|f\|_p = \|h\|_q = 1$, where, $p^{-1} + q^{-1} = 1$ and where p has the same value as in Lemma (2.5).

Given the BMO function b we define the function $F(z)$ by,

$$F(z) = \int_{\mathbb{R}^n} e^{zb} T(e^{-zb} f)(x) h(x) \, dx. \tag{2.9}$$

The function $F(z)$ is seen to be holomorphic in the disk $|z| < 1$. Next we denote by γ the circle centered at $(0,0)$ and having radius, $\theta_0 \tau_0/4$ (see Lemma (2.4) for the choice of τ_0, and Lemma (2.5) for the choice of θ_0.). We claim that for $z \in \gamma$, we have a uniform bound,

$$|F(z)| \le C, \tag{2.10}$$

with C independent of f, h and b.

Applying Hölder's inequality to (2.9) we see that,

$$|F(z)|^p \le \int_{\mathbb{R}^n} |e^{pzb}| |T(e^{-zb} f)|^p \, dx. \tag{2.11}$$

But, $|e^{pzb}| = e^{\tau b}$, with $-\frac{\theta_0 \tau_0 p}{4} \le \tau \le \frac{\theta_0 \tau_0 p}{4}$. We write $e^{\tau b} = w^\theta$, $0 \le \theta \le \theta_0$, with $w = e^{\tau_0 p b/4}$ if $\tau \ge 0$ and $w = e^{-\tau_0 p b/4}$, if $\tau \le 0$. By lemma (2.4), since $p < 4$, we have $w \in A_{\frac{p}{2}}$. We may apply Lemma (2.5) to (2.11), to get,

$$|F(z)|^p \le C \int_{\mathbb{R}^n} |f|^p |e^{pzb}| |e^{-pzb}| \, dx \le C.$$

This proves (2.10). The uniformity of the estimate above and hence that of (2.10) follows from Remark (2.6).

From the fact that F is holomorphic in the disk bounded by γ and the estimate (2.10), it follows from the Cauchy integral formula that,

$$|F'(0)| \le C. \tag{2.12}$$

From (2.9) it is a simple matter to see that,

$$F'(0) = \int_{\mathbb{R}^n} [b,T]f(x)h(x)\,dx. \tag{2.13}$$

We shall now use the notation $\|[b,T]\|_p$ for the operator norm of $[b,T]$ on $L^p(\mathbb{R}^n)$. It is now a simple matter to observe that by combining (2.12) and (2.13) we easily have,

$$\|[b,T]\|_p \leq C. \tag{2.14}$$

Since the adjoint operator T^* is also a pseudo-differential operator with symbol in the class $S^{-\epsilon}_{\rho,\delta}$, see Treves [T], we also have,

$$\|[b,T^*]\|_p \leq C. \tag{2.15}$$

By duality from (2.15), we have that for $p^{-1} + q^{-1} = 1$,

$$\|[b,T]\|_q \leq C. \tag{2.16}$$

Interpolation between the estimates (2.14) and (2.16) easily gives us the boundedness on $L^2(\mathbb{R}^n)$ of $[b,T]$.

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References

[CT] S. Chanillo and A. Torchinsky, Sharp functions and weighted L^p estimates for a class of pseudo-differential operators, Arkiv för Matematik, **24**, No. 1 (1986), 1-25.

[CM] R. Coifman and Y. Meyer, Au-delà des opérateurs pseudo-differentiels, Asterisque, **57**, (1978).

[F] C. Fefferman, L^p bounds for pseudo-differential operators, Israel J. Math., **14**, (1973), 413-417.

[H] L. Hörmander, Pseudo-differential operators and hypoelliptic equations, Proc. Sympos. in Pure Math., **10**, (1967), 138-183.

[JN] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. in Pure and Appl. Math., **14**, (1961), 415-426.

[RW] R. Rochberg and G. Weiss, Derivatives of analytic families of Banach spaces, Annals of Math., **118**, (1983), 315-347.

[SW] E. Stein and G. Weiss, Interpolation of operators with change of measures, Trans. Amer. Math. Soc., **87**, (1958), 159-172.

[T] F. Treves, Introduction to pseudo-differential and Fourier Integral operators, Vol. 1, Plenum Press, (1980).

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