KEY

QUIZ 2, Math. 423

1. (a) (10) Given the initial-Boundary value problem for the heat equation:

$$u_t = ku_{xx},\ u(0,t) = u(1,t) = 0,\ u(x,0) = x^3(1-x).$$

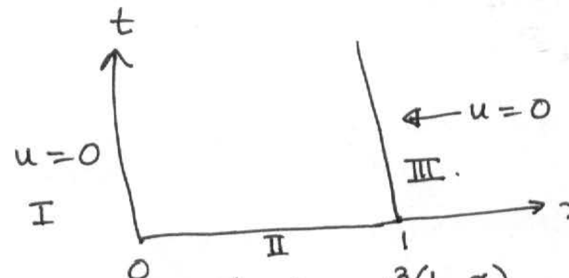
Find the best values M_1, M_2 such that for all (x,t) in the set

$$S = \{(x,t), 0 < x < 1, t > 0\},$$

$$M_1 < u(x,t) < M_2.$$

(b)(5) Are there points (x,t) in the set $0 \le x \le 1, t \ge 0$ where $u(x,t)$ attains the values M_1, M_2. If so, what are the coordinates of the points?

(a).



$u(x,0) = x^3(1-x)$.

note $\min_{\mathrm{I}} u = 0 = \min_{\mathrm{III}} u = \min_{\mathrm{II}} u$.

So by the strong minimum principle $u(x,t) > 0$ on S.

Pick $M_1 = 0$. Now consider $u(x,0) = x^3(1-x) = f(x)$ on $[0,1]$.

$\max u = \max\{0, \max_{\text{in } (0,1)} x^3(1-x)\}$. By elementary calculus.

$f'(x) = [x^3 - x^4]' = 3x^2 - 4x^3 = x^2(3-4x) = 0$

$x = \frac{3}{4}$. $f(3/4) = \left(\frac{3}{4}\right)^3\left(1-\frac{3}{4}\right) = \frac{1}{4}\left(\frac{3}{4}\right)^3$.

By the strong maximum principle $u(x,t) < \frac{1}{4}\left(\frac{3}{4}\right)^3$.

$M_2 = \frac{1}{4}(3/4)^3$.

(b). Minimum is achieved on $(0,t)$, $t \ge 0$ and on $(1,t)$, $t \ge 0$. Maximum is achieved at $\left(\frac{3}{4}, 0\right)$.